Abelian connections on complex manifolds

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Outline

- Metric connections.
- Complex connections.
- General results.
- Complex connections with trivial holonomy.
- Complex parallelizable manifolds.
- Flat complex connections with $(1, 1)$-torsion.
- Complete complex connections with parallel torsion and trivial holonomy.
- Examples.

Metric connections

Given a pseudo-riemannian manifold (M, g) a classical problem was to study the existence of flat connections ∇ on M such that:

1. $\nabla g = 0$,
2. ∇ has the same geodesics as the Levi-Civita connections.

Condition 2 is equivalent to:

- the torsion T of ∇ is totally skew-symmetric.

Main references are:
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Theorem (J. Wolf, 1972)

Let (M, g, ∇), g a pseudo-riemannian metric and ∇ a connection on M with trivial holonomy. If moreover the connection is required to be complete with parallel torsion, the resulting manifolds are of the form $\Gamma\backslash G$ where

(i) *G is a simply connected Lie group and Γ a discrete subgroup of G,*

(ii) *the pseudo-Riemannian metric g is induced from a bi-invariant metric on G,*

(iii) *∇ is induced by the affine connection corresponding to the parallelism of left translation on G.*

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Complex connections

Aim: To investigate an analogue of the previous problem in the case of almost complex manifolds instead of pseudo-Riemannian manifolds.

Given an almost complex manifold (M, J), we will be interested in studying complex connections ∇ on M with trivial holonomy and such that:

1. $\nabla J = 0$,
2. the torsion T is either of type $(2, 0)$ or of type $(1, 1)$ with respect to J.

Motivation: Quotients of Lie groups $\Gamma \backslash G$, where J is induced by either a bi-invariant or an abelian complex structure have such connections.

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General results

Let ∇ be an affine connection on a manifold M with torsion tensor field T, where $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for all X, Y vector fields on M.

Given an almost complex structure J on M the torsion T of a connection ∇ on the almost complex manifold (M, J) is said to be:

- of type $(1, 1)$ if $T(JX, JY) = T(X, Y)$,
- of type $(2, 0)$ if $T(JX, Y) = JT(X, Y)$,
- of type $(2, 0) + (0, 2)$ if $T(JX, JY) = -T(X, Y)$,

for all vector fields X, Y on M.

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One says that ∇ is a complex connection when the tensor field J is parallel with respect to ∇ $(\nabla J = 0)$.

We denote by N the Nijenhuis tensor of J, defined by

$$N(X,Y) := [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y].$$

Newlander-Nirenberg (1957): J is integrable if and only if $N \equiv 0$.
Recalling that $(\nabla_X J)\,Y = \nabla_X\,(JY) - J\,(\nabla_X Y)$, we obtain the following identity:

$$\begin{aligned} N(X,Y) &= (\nabla_{JX} J)\,Y - (\nabla_{JY} J)\,X + (\nabla_X J)\,JY - (\nabla_Y J)\,JX \\ &\quad + T(X,Y) - T(JX,JY) + J\,(T(JX,Y) + T(X,JY)), \end{aligned}$$

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for all X, Y vector fields on M.

Lemma

Let (M, J) be an almost complex manifold with a complex connection ∇. Then J is integrable if and only if the torsion T of ∇ satisfies:

$$T(X,Y) - T(JX,JY) + J\left(T(JX,Y) + T(X,JY)\right) = 0,$$

for all vector fields X, Y on M.

Corollary

Let (M, J) be an almost complex manifold.

(i) *If $\nabla J = 0$ and T is of type $(1,1)$ then J is integrable.*

(ii) *If $\nabla J = 0$ and T is of type $(2,0)$, then J is integrable.*

(iii) *If $\nabla J = 0$, T is of type $(2,0) + (0,2)$ and J is integrable, then T is of type $(2,0)$.*

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Examples of connections as above are given by ∇^1 and ∇^2 defined by

$$g\left(\nabla^1_X Y, Z\right) = g\left(\nabla^g_X Y, Z\right) + \frac{1}{4}\left(d\omega(X, JY, Z) + d\omega(X, Y, JZ)\right),$$

$$g\left(\nabla^2_X Y, Z\right) = g\left(\nabla^g_X Y, Z\right) - \frac{1}{2}\, d\omega(JX, Y, Z),$$

where $\omega(X, Y) := g(JX, Y)$ is the Kähler form corresponding to g and J. These connections satisfy

$$\nabla^1 g = 0, \quad \nabla^1 J = 0, \quad T^1 \text{ is of type } (1,1),$$

$$\nabla^2 g = 0, \quad \nabla^2 J = 0, \quad T^2 \text{ is of type } (2,0),$$

The connections ∇^1 and ∇^2 appearing above are known, respectively, as the first and second canonical connection associated to the Hermitian manifold (M, J, g).

The connection ∇^2 is also known as the *Chern* connection, and it is the unique connection on (M, J, g) satisfying

$$\nabla^2 g = 0, \quad \nabla^2 J = 0, \quad T^2 \text{ is of type } (2,0).$$

In the almost Hermitian case, the Chern connection is the unique complex metric connection whose torsion is of type $(2,0)+(0,2)$, equivalently, the $(1,1)$-component of the torsion vanishes.

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Complex connections with trivial holonomy

Let M be an n-dimensional connected manifold and ∇ an affine connection on M with trivial holonomy (hence flat). Then

- the space $\mathcal{P}^{\nabla}$ of parallel vector fields on M is an n-dimensional real vector space;
- $T(X, Y) = -[X, Y], \qquad$ for all $X, Y \in \mathcal{P}^{\nabla}$;
- $\mathcal{P}^{\nabla}$ es a Lie algebra if and only if T is parllel.

Lemma

Let (M, J), be a connected almost complex manifold with an affine connection ∇ on M, Hol (∇) trivial. Then the following conditions are equivalent:

(i) $\nabla J = 0$;

(ii) *the space $\mathcal{P}^{\nabla}$ of parallel vector fields is J-stable;*

(iii) *there exist parallel vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of M.*

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Complex parallelizable manifolds

We recall that a complex manifold (M, J) is called *complex parallelizable* when there exist n holomorphic vector fields $Z_1, \ldots, Z_n$, linearly independent at every point of M.

The following classical result, due to Wang, characterizes the compact complex parallelizable manifolds.

Theorem

Every compact complex parallelizable manifold may be written as a quotient space $\Gamma \backslash G$ of a complex Lie group by a discrete subgroup Γ.

We prove next a result which relates the notion of complex parallelizability with the existence of a flat complex connection with torsion of type $(2, 0)$.

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We prove next a result which relates the notion of complex parallelizability with the existence of a flat complex connection with torsion of type $(2, 0)$.

Proposition

Let M be a connected $2n$-dimensional manifold with a complex structure J. Then the following conditions are equivalent:

(i) *there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of M, such that*

$$[X_k, X_l] = -[JX_k, JX_l], \;\; k < l, \quad [JX_k, X_l] = J[X_k, X_l], \;\; k \leq l,$$

(ii) *there exist n holomorphic vector fields $Z_1, \ldots, Z_n$ which are linearly independent at every point of M (in other words, (M, J) is complex parallelizable);*

(iii) *there exist n linearly independent holomorphic $(1,0)$-forms $\alpha_1, \ldots, \alpha_n$ on M such that $d\alpha_i$ is a section of $\Lambda^{2,0}M$ for every i;*

(iv) *there exists a complex connection ∇ on M with trivial holonomy whose torsion tensor field T is of type $(2,0)$.*

Corollary

Let (M, J) be a complex manifold. The following conditions are equivalent:

(i) *(M, J) is complex parallelizable;*

(ii) *there exists a Hermitian metric g on M such that the Chern connection associated to (M, J, g) has trivial holonomy.*

Definition : An affine connection ∇ on a connected complex manifold (M, J) will be called a *Chern-type* connection if it satisfies condition (iv) of the previous Proposition.

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Let (M, J) *be a complex manifold. The following conditions are equivalent:*

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Corollary

Let (M, J) be a connected complex manifold and ∇ an affine connection with trivial holonomy. Then ∇ is a Chern-type connection on (M, J) if and only if the space $\mathcal{P}^\nabla$ of parallel vector fields is J-stable and J satisfies

$$J[X, Y] = [X, JY] \quad \textit{for any} \quad X, Y \in \mathcal{P}^\nabla.$$

Remark

Analogously to previous the Corollary, we have that the torsion T of a complex connection ∇ with trivial holonomy is of type $(2,0)+(0,2)$ if and only if J satisfies

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Flat complex connections with $(1,1)$-torsion

Proposition

Let M be a connected $2n$-dimensional manifold with an almost complex structure J. Then the following conditions are equivalent:

(i) *there exist vector fields $X_1, \ldots, X_n, JX_1, \ldots, JX_n$, linearly independent at every point of M, such that*

$$[X_k, X_l] = [JX_k, JX_l], \qquad [JX_k, X_l] = -[X_k, JX_l], \qquad k < l;$$

(ii) *there exist n commuting vector fields $Z_1, \ldots, Z_n$ which are linearly independent sections of $T^{1,0}M$ at every point of M;*

(iii) *there exist n linearly independent $(1,0)$-forms $\alpha_1, \ldots, \alpha_n$ on M such that $d\alpha_i$ is a section of $\Lambda^{1,1}M$ for every i;*

(iv) *there exists a complex connection ∇ on M with trivial holonomy whose torsion tensor field T is of type $(1,1)$.*

Moreover, any of the above conditions implies that J is integrable.

Definition : An affine connection ∇ on a connected almost complex manifold (M, J) will be called an *abelian* connection if it satisfies condition (iv) of previous Proposition.

Corollary

Let (M, J) be a connected complex manifold and ∇ an affine connection with trivial holonomy. Then ∇ is an abelian connection on (M, J) if and only if the space $\mathcal{P}^{\nabla}$ of parallel vector fields is J-stable and J satisfies

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Complete complex connections with parallel torsion and trivial holonomy

We exhibit next a large class of complex manifolds equipped with complex connections with trivial holonomy whose torsion tensors are of type $(2, 0)$ or $(1, 1)$

We consider:

1. A connected Lie group G,
2. a complex structure J on its Lie algebra $\mathfrak{g}$,
3. a $\mathfrak{g}$-valued bilinear form $\nabla : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$,
4. a compatibility, $\nabla(x, Jy) = J\nabla(x, y)$.

If $\Gamma \subset G$ is any discrete subgroup of G the induced J and ∇ on the quotient $\Gamma \backslash G$ will be denoted J_0 and ∇_0 respectively.

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If $\Gamma \subset G$ is any discrete subgroup of G the induced J and ∇ on the quotient $\Gamma\backslash G$ will be denoted J_0 and ∇_0 respectively.

The left invariant affine connection ∇ on G defined by $\nabla_X Y = 0$ for all X, Y left invariant vector fields on G is known as the $(-)$-connection. This connection satisfies:

1. Its torsion T is given by $T(X, Y) = -[X, Y]$ for all X, Y left invariant vector fields on G;
2. $\nabla T = 0$ and $\mathcal{P}^{\nabla} = \mathfrak{g} \subset \mathfrak{X}(G)$;
3. The holonomy group of ∇ is trivial, thus, ∇ is flat;
4. The geodesics of ∇ through the identity $e \in G$ are Lie group homomorphisms $\mathbb{R} \to G$, therefore, ∇ is complete;
5. The parallel transport along any curve joining $g \in G$ with $h \in G$ is given by the derivative of the left translation $(dL_{hg^{-1}})_g$.

If J is a left invariant complex structure on G, then J is parallel with respect to the $(-)$-connection ∇. Hence

$(\Gamma \backslash G, J_0)$ carries a complete complex connection with trivial holonomy and parallel torsion.

In the next result we prove that the converse also holds.

Theorem

The triple (M, J, ∇) where M is a connected manifold endowed with a complex structure J and a complex connection ∇ with trivial holonomy is equivalent to a triple $(\Gamma \backslash G, J_0, \nabla^0)$ as above if and only if ∇ is complete and its torsion is parallel.

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Corollary

Let (M, J) *be a complex manifold with a Chern-type connection* ∇*. If the torsion tensor field* T *is parallel, then:*

(i) *the space* $\mathcal{P}^\nabla$ *of parallel vector fields on* M *is a complex Lie algebra and* J *is a bi-invariant complex structure on* $\mathcal{P}^\nabla$*;*

(ii) *if, furthermore,* ∇ *is complete, then* (M, J, ∇) *is equivalent to* $(\Gamma\backslash G, J_0, \nabla^0)$*, where* G *is a simply connected complex Lie group and* $\Gamma \subset G$ *is a discrete subgroup.*

Corollary

Let (M, J, g) *be a Hermitian manifold such that the associated Chern connection* ∇ *is complete, has trivial holonomy and parallel torsion. Then* (M, J, g) *is equivalent to a triple* $(\Gamma\backslash G, J_0, g_0)$*, where* G *is a simply connected complex Lie group and* g_0 *is induced by a left invariant Hermitian metric on* G*. Furthermore, the Chern connection on the quotient coincides with* ∇^0*.*

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Corollary

Let ∇ be an abelian connection on a connected complex manifold (M, J) such that the torsion tensor field T is parallel. Then:

(i) *the space $\mathcal{P}^\nabla$ of parallel vector fields on M is a Lie algebra and J is an abelian complex structure on $\mathcal{P}^\nabla$;*

(ii) *the Lie algebra $\mathcal{P}^\nabla$ is 2-step solvable.*

(iii) *if, furthermore, ∇ is complete, then (M, J, ∇) is equivalent to $(\Gamma\backslash G, J_0, \nabla^0)$, where G is a simply connected 2-step solvable Lie group equipped with a left invariant abelian complex structure and $\Gamma \subset G$ a discrete subgroup.*

Remark

We note that a complete classification of the Lie algebras admitting abelian complex structures is known up to dimension 6 and there are structure results for arbitrary dimensions.

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Examples

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Let N be the Heisenberg Lie group, Γ the subgroup of matrices in N with integer entries and M the 4-dimensional compact manifold $M = (\Gamma\backslash N) \times S^1 = (\Gamma \times \mathbb{Z})\backslash(N \times \mathbb{R})$. $N \times \mathbb{R}$ admits a left-invariant abelian complex structure and therefore M inherits a complex structure J admitting an abelian connection. On the other hand, M is not complex parallelizable since there are only two 2-dimensional simply connected complex Lie groups, namely $\mathbb{C}^2$ and $\widetilde{\mathrm{Aff}}(\mathbb{C})$, where $\widetilde{\mathrm{Aff}}(\mathbb{C})$ is the universal cover of the group of affine motions of $\mathbb{C}$ The group $\mathbb{C}^2$ gives rise to a torus and M would admit a Kähler structure, which is impossible and $\widetilde{\mathrm{Aff}}(\mathbb{C})$ is not unimodular.

Example

Let $M = \mathbb{R}^4$ *with canonical coordinates* (x_1, x_2, x_3, x_4) *and corresponding vector fields* $\partial_1, \ldots, \partial_4$. *Let* $f(x_1, x_2, x_3, x_4) = x_3 x_4$.

- *Let* ∇ *so that*

$$\mathcal{P}^{\nabla} = span_{\mathbb{R}}\{\partial_1, \partial_2, \partial_3, f\partial_2 + \partial_4\}.$$

- *Let* J

$$\begin{aligned} J\partial_1 &= \partial_2, & J\partial_3 &= f\partial_2 + \partial_4, \\ J\partial_2 &= -\partial_1, & J\partial_4 &= f\partial_1 - \partial_3. \end{aligned}$$

One finds that ∇ *is an abelian connection on* $(\mathbb{R}^4, J)$ *and* T *is not parallel. Hence*

$$(M, J, \nabla) \text{ is NOT equivalent to } (\Gamma\backslash G, J_0, \nabla^0).$$

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