MINIMAL VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

OLGA GIL-MEDRANO
Universidad de Valencia, Spain

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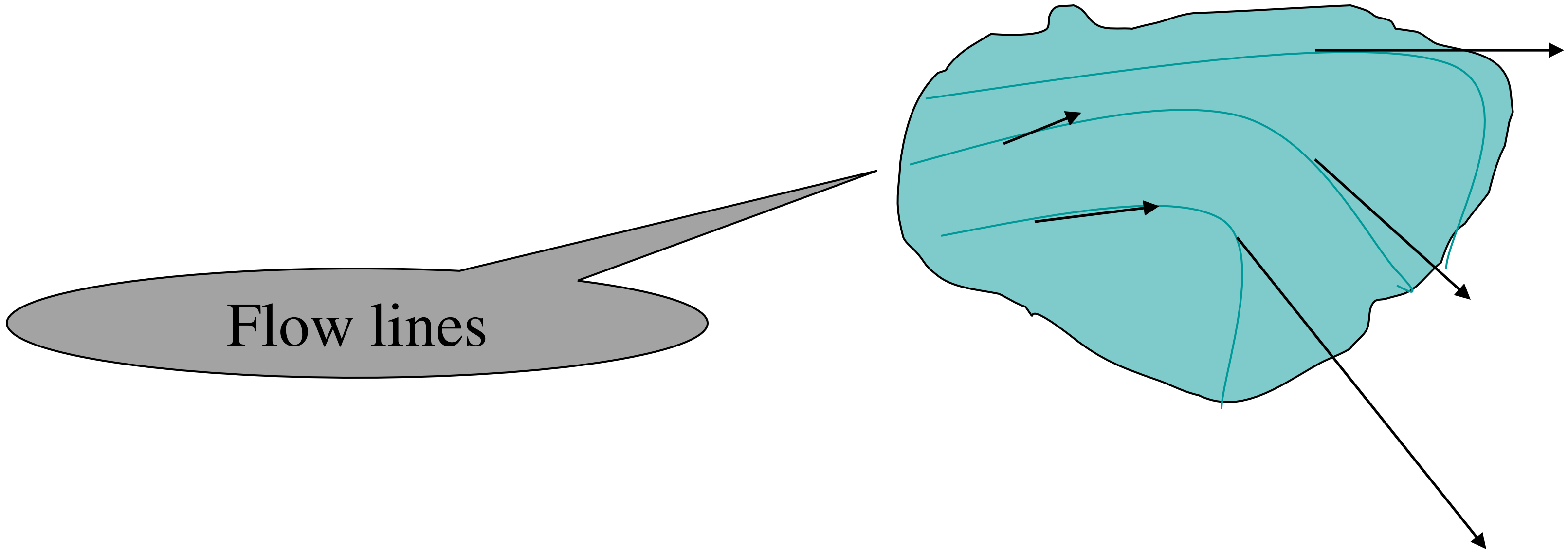
Conference Celebrating P. Gilkey's 65th Birthday

$$V: M \longrightarrow TM = \cup\, T_p M$$

$$p \qquad\qquad V(p) = V_p$$

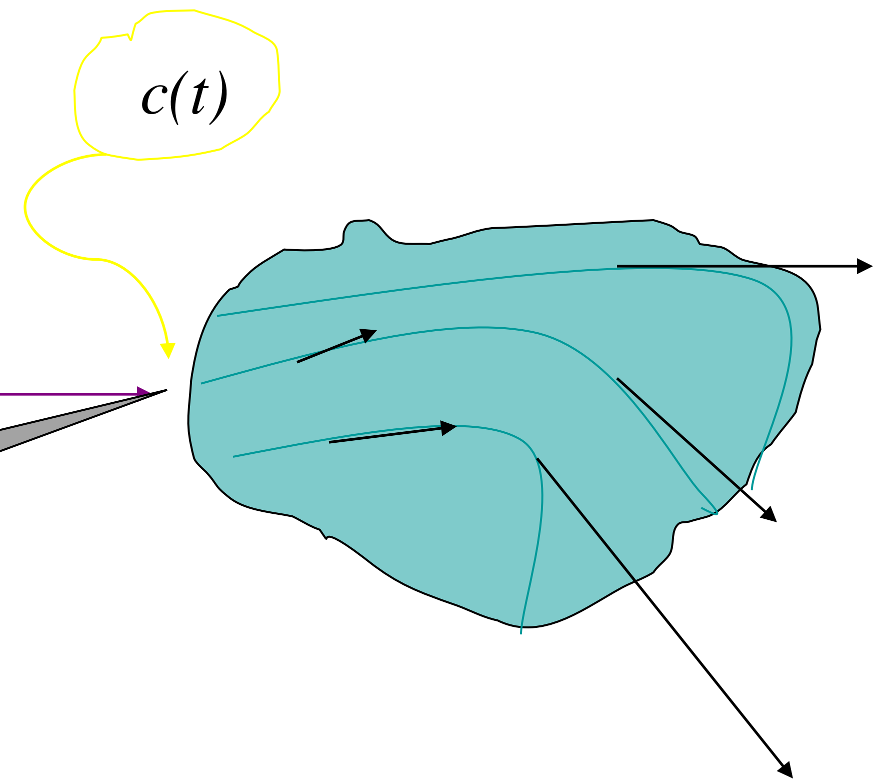


$$V: M \longrightarrow TM = \cup\, T_pM$$

$$p \qquad\qquad V(p) = V_p$$

$$c'(t) = V(c(t))$$

Flow lines

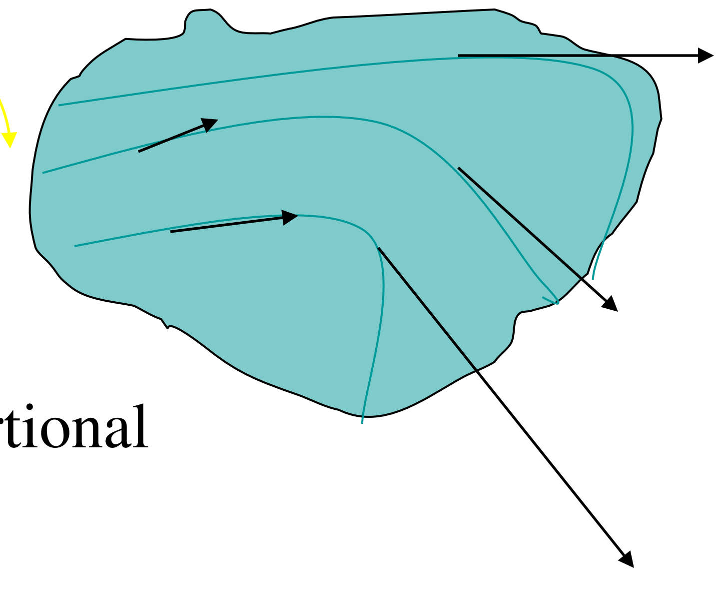


$$V: M \longrightarrow TM = \cup T_p M$$
$$p \qquad V(p) = V_p$$

If M has a metric

$c(t)$

$$c'(t) = V(c(t))$$



The energy of the curve at p is proportional to the norm of the aceleration

✓ Constant Norm

✓ Flow by geodesics

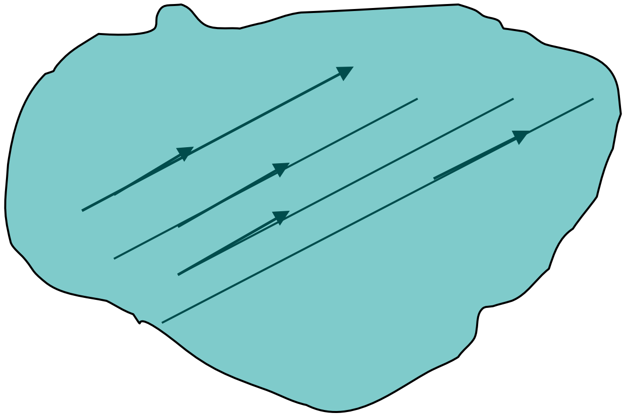
V is parallel

$\Updownarrow$

For all p,

$(\nabla V)_p$ *vanish*

$\Longleftrightarrow$



Endomorphism of T_pM determined by

$$(\nabla V)_p(\mathrm{w}) = \nabla_{\mathrm{w}} V$$

Gluck and Ziller problem.

Which unit vector fields on (M,g) have less volume?

$V(M) \subset (T^1 M, g^S)$ *Sasaki metric*

$$\mathrm{Vol}\,(V) = \mathrm{vol}(M, V^* g^S)$$

$$\Updownarrow$$

$$\int \sqrt{\det L_V}\, dv.$$

$$(V^* g^S)(X,Y) = g(L_V(X),Y)$$

$$\Updownarrow$$

$$g(X,Y) + g(\nabla_X V, \nabla_Y V)$$

$$L_V = \mathrm{Id} + (\nabla V)^t(\nabla V)$$

Gluck and Ziller problem.

Which unit vector fields on odd-dimensional spheres are most efficient?

$V(S^n) \subset (T^1 S^n, g^S)$ *Sasaki metric*

$$\mathrm{Vol}\,(V) = \mathrm{vol}(S^n,\, V^*g^S)$$

$$\Updownarrow$$

$$\int_{S^n} \sqrt{\det L_V}\, dv.$$

$$(V^*g^S)(X,Y) = g(L_V(X),Y)$$

$$\Updownarrow$$

$$g(X,Y) + g(\nabla_X V, \nabla_Y V)$$

$$L_V = \mathrm{Id} + (\nabla V)^t(\nabla V)$$

Gluck and Ziller problem.

Which unit vector fields on odd-dimensional spheres are most efficient?

$$V: S^n \longrightarrow T^1 S^n \qquad n=2m+1$$

$$\Updownarrow$$

$$V_{2,n+1}$$

$$\Updownarrow$$

$$O(n+1)/O(n-1)$$

Stiefel manifold of R^{n+1}
(orthonormal 2- frames)

$$\text{Vol}\,(V) := \text{vol}\,(\,V(S^n)\,)$$

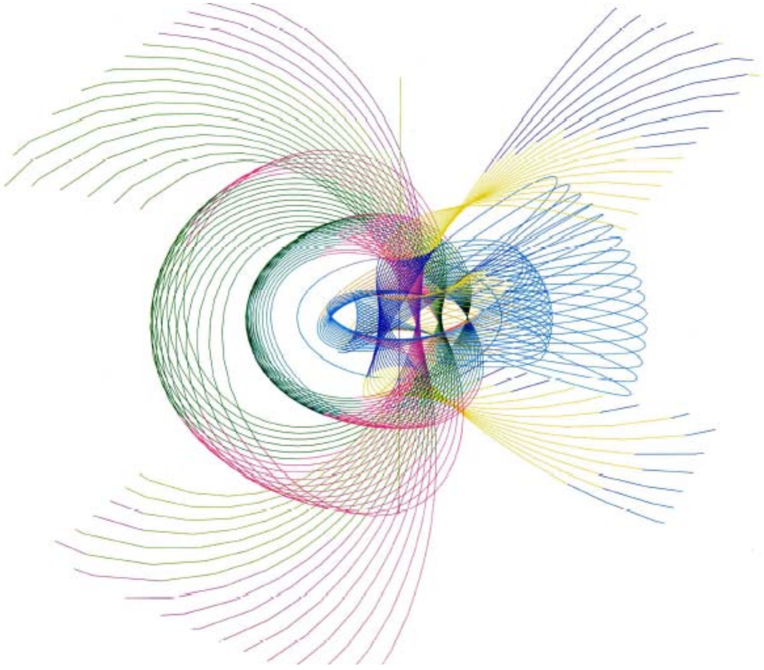
efficient = with less volume*

Hopf vector fields

$$\pi: S^n \longrightarrow \boldsymbol{CP}^m \qquad \pi^{-1}(\pi(p)) = \{e^{i\theta} p\}$$

$H(p) = i\,p = J(p)$ standard Hopf vector field

Hopf vector fields are exactly the unit Killing vector fields of spheres



Hopf vector fields

$\pi: S^n \longrightarrow \boldsymbol{C}P^m$ $\qquad$ $\pi^{-1}(\pi(p)) = \{e^{i\theta} p\}$

$H(p) = i\,p = J(p)$ standard Hopf vector field

$(H^*g^S)(H,H) = 1$ and $(H^*g^S) = 2g$ on $H^\perp$

$H(S^n)$ with the induced metric is a Berger sphere

$$\text{Vol}\,(H) = 2^m\,\text{vol}\,(S^n)$$

H. Gluck y W. Ziller, Comment. Math. Helv. 86

Theorem:

The unit vector fields of minimum volume on S^3 are precisely the Hopf vector fields, and no others.

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There are smooth unit vector fields on S^n ($n>3$) with less volume than Hopf vector fields.

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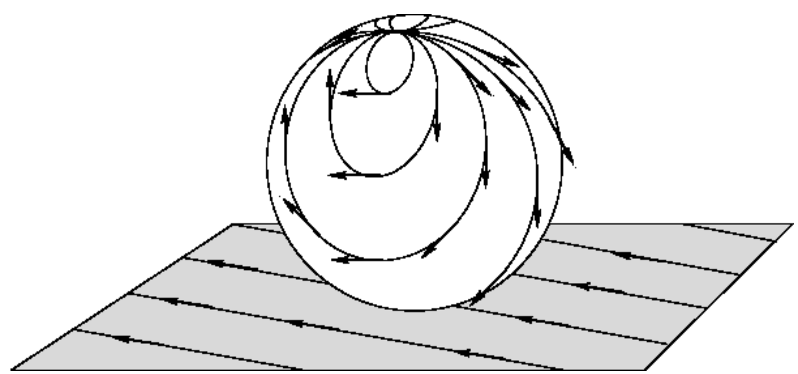
For the proof:

1) consider Pontryagin vector fields P, defined on S^n minus one point,

2) show that $Vol(P) < Vol(H)$ and

3) show a sequence of smooth unit vector fields on S^n, P_i such that $Vol(P_i)$ tends to $Vol(P)$.

Pontryagin vector fields



S. Pedersen, *Trans. Amer. Math. Soc. 93. Theorem: There are smooth unit vector fields on S^n ($n>3$) with less volume than Hopf vector fields.*

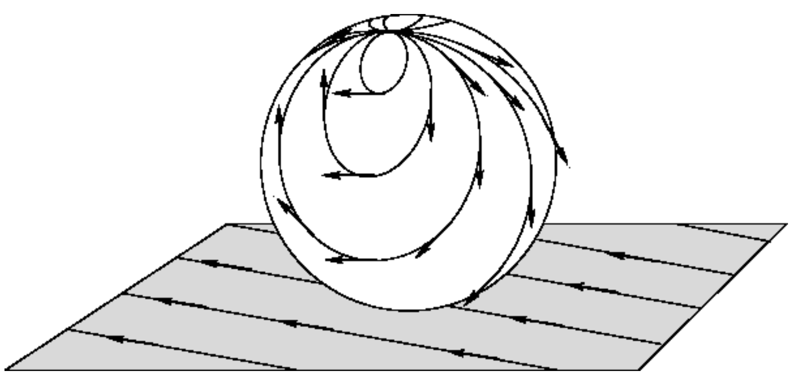
For the proof:

1) consider Pontryagin vector fields P, defined on S^n minus one point,

2) show that $Vol(P) < Vol(H)$ and

3) show a sequence of smooth unit vector fields on S^n, P_k such that $Vol(P_k)$ tends to $Vol(P)$.

A Pontryagin vector field (obtained by parallel transport of a fixed unit vector along radial geodesics) is a minimal immersion of the sphere minus one point, for any dimension.



S. Pedersen, *Trans. Amer. Math. Soc. 93. Theorem: There are smooth unit vector fields on* $S^n(1)$ *(* $n>3$ *) with less volume than Hopf vector fields.*

A Pontryagin vector field (obtained by parallel transport of a fixed unit vector along radial geodesics) is a minimal immersion of the sphere $S^n(1)$ minus one point, for any dimension.

Conjecture : The infimum of the volume of smooth unit vector fields on odd-dimensional spheres of radius 1 is Vol *(P)*

Minimal unit vector fields

Critical points of the functional Vol defined on $\Gamma^{\infty}(T^1M)$

$\Longleftrightarrow$

Vector fields such that the mean curvature vector field of the submanifold $V(M)$ into T^1M vanishes

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Vector fields such that the mean curvature vector field of the submanifold $V(M)$ into T^1M vanishes

—, Diff. Geom. Appl. 2001

Second fundamental form:

$$\nabla_X X - \widetilde{\nabla}_X X + R(\nabla_X V, V)X$$

vertical / horizontal proj.

$$\nabla_X \nabla_X V - (\nabla V)(\widetilde{\nabla}_X X)$$

Elements with tilde correspond to the metric V^*g^S

Minimal unit vector fields

Critical points of the functional Vol defined on $\Gamma^{\infty}(T^1M)$ ⟺ Vector fields such that the mean curvature vector field of the submanifold $V(M)$ into T^1M vanishes

Elements with tilde correspond to the metric V^*g^S

—, Diff. Geom. Appl. 2001

$$\sum \left(\nabla_{\widetilde{E}_i} \nabla_{\widetilde{E}_i} V - (\nabla V)(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) \right) \; col. \; V$$

↕

Mean curvature v. f. = 0

Minimal unit vector fields

Critical points of the functional Vol defined on $\Gamma^\infty(T^1M)$

Vector fields such that the mean curvature vector field of the submanifold $V(M)$ into T^1M vanishes

— and Llinares, *Tohoku, 2002*

Euler-Lagrange Equation:

$$\nabla^*(\sqrt{\det L_V}(\nabla V)L_V^{-1}) \ col.\ V$$

$$\int_M \sqrt{\det L_V}\,dv.$$

— and Llinares, *Math. Ann. 01*

Second variation of the Volume of unit vector fields of a Riemannian manifold, at a minimal v.f. W, Hessian in the direction of a v. f. A, orthogonal to W.

$$(Hess\,Vol)_W(A) = \int_M \Big(\qquad \Big) dv_g$$

$$\|A\|^2 g\Big(\nabla^*(\sqrt{\det L_W}\,\nabla W \circ L_W^{-1}), W\Big)$$
$$+2\sqrt{\det L_W}\,\sigma_2\Big(L_W^{-1} \circ (\nabla W)^t \circ \nabla A\Big)$$
$$+\sqrt{\det L_W}\,\operatorname{tr}\Big(L_W^{-1} \circ (\nabla A)^t \circ \nabla A\Big)$$
$$-\sqrt{\det L_W}\,\operatorname{tr}\Big(L_W^{-1} \circ (\nabla A)^t \circ \nabla W \circ L_W^{-1} \circ (\nabla W)^t \circ \nabla A\Big)$$

The tangent space at W of the space of unit vector fields is the space of v. f. that are orthogonal to W

Minimal unit vector fields on a Riemannian manifold M
(Minimal submanifolds of $T^1(M)$ that are "graphs")

Description of many examples obtained by several authors
(—, Survey, 2005)

— and Llinares, 2002

$$\left.\begin{array}{l} V \text{ unit Killing} \\ \\ M \text{ constant curvature } k \end{array}\right\} \; V \text{ minimal}$$

$$\mathrm{Vol}\,(V) = (1+k)^{(n-1)/2}\,\mathrm{vol}(M)$$

- *H^k the unit Hopf vector field of $S^n(r)$ is minimal* $(k = 1/r^2)$

- *The image is (up to homotheties) a Berger sphere*

- *The ambient manifold $T^1(S^n(r))$ has the Sasaki metric*

Up to now, they are the only examples of smooth unit minimal vector fields defined on the sphere.

A critical radius $\qquad r_0(m)^2 = 1/(2m-3) = 1/(n-4)$

$r_0(2) = 1 \qquad r_0(m) < 1$

Teorema (Borrelli and **----**, *Math. Ann.* 2006):

For $m>1$ the unit Hopf vector field of $S^n(r)$ is stable if and only if $r \le r_0(m)$.

If $r < r_0(m)$, the Hopf vector fields H^r are local minimizers of the volume

(Borrelli and **----**, *Math. Ann.* 2006):

If $m>1$ there is $r_2(m)$ such that if $r \leq r_2(m)$ then $\mathrm{Vol}(H^r)<\mathrm{Vol}(P^r)$

For $r \leq r_2(m)$ no unit smooth vector field is known to have less volume than Hopf vector fields.

Twisting of a vector field

$$\boxed{{}^{T}\nabla V \circ \nabla V}$$

$$\begin{aligned} Vol(V^k) &= \int_{\mathbb{S}^{2m+1}(1)} \sqrt{\det(\frac{1}{k}Id + M)}dvol \\ &= \int_{\mathbb{S}^{2m+1}(1)} \sqrt{\frac{1}{k^n} + \frac{1}{k^{n-1}}\sigma_1(M) + \ldots + \frac{1}{k}\sigma_{2m}(M) + \cancel{\sigma_{2m+1}(M)}}dvol \end{aligned}$$

$$\boxed{\int_{\mathbb{S}^{2m+1}(1)} \sqrt{\sigma_{2m}({}^{T}\nabla V \circ \nabla V)}dvol.}$$

Def:
Twisting of V

Twisting minimizers

Hopf vector fields minimize the twisting

(Borrelli and **----**, *Math. Ann.* 2006):

For any unit v. f. V of $S^n(1)$, $Tw(V) \geq Tw(H)$.

If $Tw(V) > Tw(H)$ there is a radius such that for all smaller radii the volume of V^r > the volume of H^r

$$\boxed{\int_{\mathbb{S}^{2m+1}(1)} \sqrt{\sigma_{2m}({}^T\nabla V \circ \nabla V)}\,dvol.}$$

$\Longleftarrow$ *Twisting of a unit v. f. V of $S^n(1)$*

(Borrelli and **----**, *Math. Ann.* 2006):

If $m>1$ there is $r_2(m)$ such that if $r \leq r_2(m)$ then $\mathrm{Vol}(H^r) < \mathrm{Vol}(P^r)$

For $r_2(m) < r \leq r_0(m)$ Hopf vector fields are minimal, stable but not minimizers.

Question:
is it natural to extend Pedersen's Conjecture to $r > r_2(m)$?

(Borrelli and **----**, *Math. Ann.* 2006):

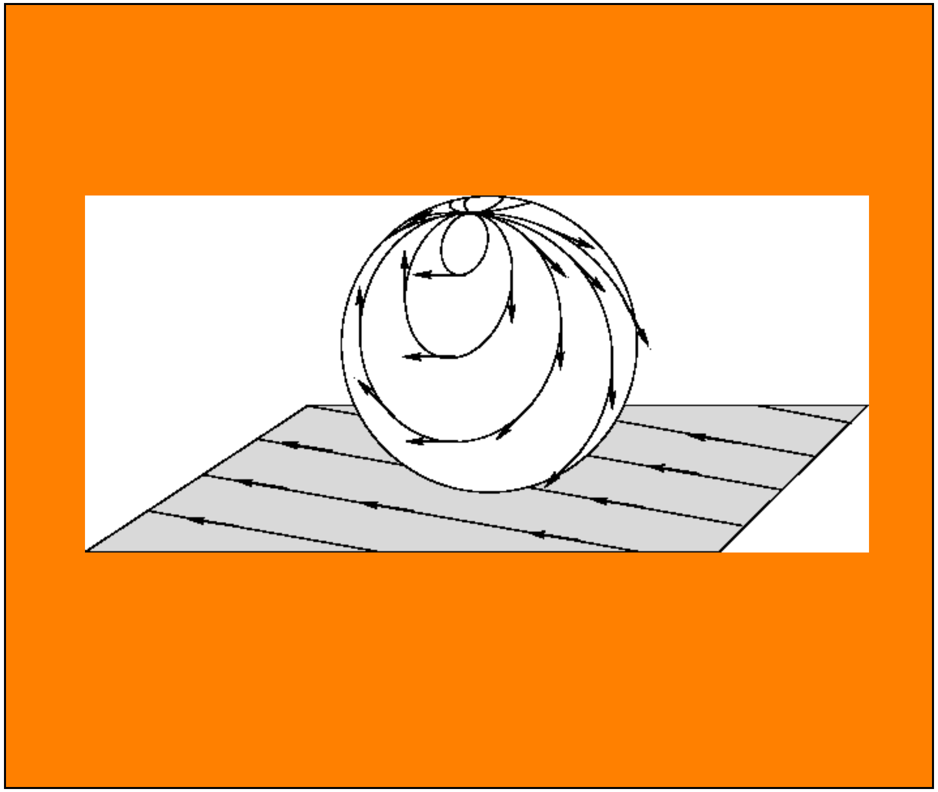
If $m>1$ there is $r_1(m)$ such that if $r > r_1(m)$ then vectors fields obtained by modifing a radial vector field have less volume than P^r

(V. Borrelli, -------- and D. Johnson, *In progress*)

The volume of P^r can't be the infimum of the volume of smooth unit vector fields for any r.

The case of the 2-dimensional spheres

In dimension 2, the image is a minimal surface of the (3-dimensional) unit tangent bundle.



Unit vector fields (with singularities) on the 2-dimensional spheres

$T^1S^2 = \mathbf{R}P^3$
Klingenber and Sasaki
1975

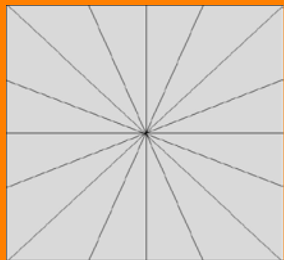
Berger and Fomenko
1972

The closure of the image of the Pontryagin field is
a totally geodesic $\mathbf{R}P^2$

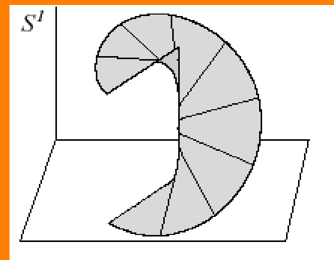
Unit vector fields (with singularities) on the 2-dimensional spheres

Theorem (Borrelli and ----, *Crelle's J. 2010)*
Among the unit vector fields (without boundary) of the radius 1 round 2-sphere those of least area are Pontryagin fields and no others.*

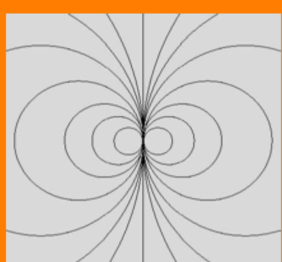
*Unit smooth vector fields defined on a dense open subset such that the closure of its image is a smooth submanifold without boundary. If such a v.f. has a finite number of singularities and the fiber at every singular point is included in the closure of its image then this submanifold is homeomorphic to the connected sum of a projective plane and a torus with holes.



Index 1



Submanifold with a fiber as boundary



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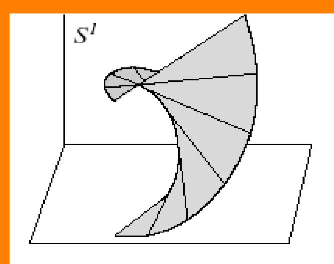

Submanifold without boundary

Radial vector field

Pontryagin vector field

2-spheres: What happens if $r \neq 1$?

$T^1S^2(r)$ is the projective space obtained by quotient of of S^3 endowed with a Berger metric.

Theorem (Borrelli and ----, *Crelle's J. 2010)*
The Pontryagin fields of $S^2(r)$ are minimal surfaces of $T^1S^2(r)$

The "great" spheres are minimal surfaces of the Berger sphere.

The "great" spheres provide an open book structure of the 3-dimensional Berger sphere with minimal leaves and with binding a fiber of the Hopf fibration.

2-spheres: What happens if $r \neq 1$?

*Proposition (*Borrelli and ----, *Crelle's J. 2010)*
The "great" spheres provide an open book structure of the 3-dimensional Berger sphere with minimal leaves and with binding a fiber of the Hopf fibration.

Used by Hardt and Rosenberg, Ann. Inst. Fourier 90, to study unicity of minimal submanifolds

*Theorem (*Borrelli and ----, *Crelle's J. 2010)*
The only minimal surfaces of $T^1S^2(r)$ homeomorphic to the projective plane and arising from unit vector fields without boundary of $S^2(r)$ are Pontryagin cycles.

Thanks for your attention!