Spectral functions: Techniques and applications (The ugly business of heat kernel coefficients.)

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Baylor University

Conference in Geometry and Global Analysis
Celebrating Peter Gilkey's 65th birthday

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Outline

Introduction

- Partitioning problems:
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$$n(N) \sim \frac{1}{4\sqrt{3}N} e^{\pi\sqrt{\frac{2N}{3}}}$$

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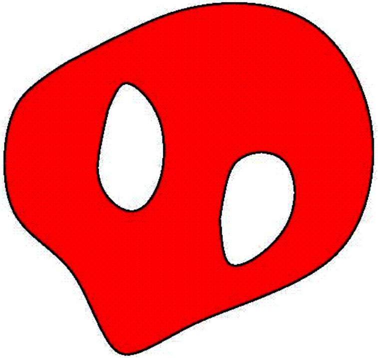
M. Coons and KK, J. Math. Phys. 50 (2009) 013517

Introduction



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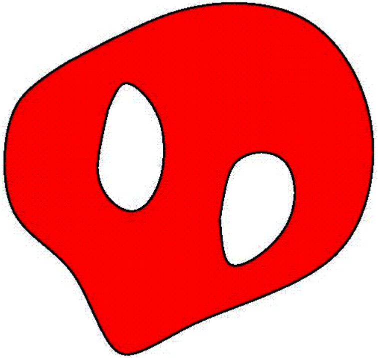
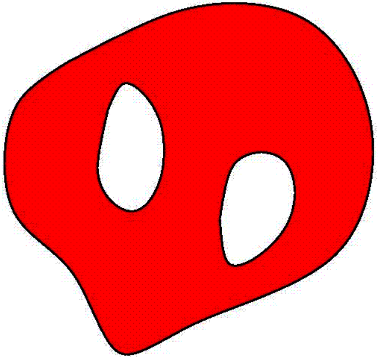


Can one hear the shape of a drum?

Introduction



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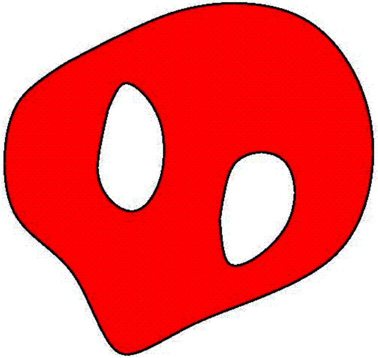

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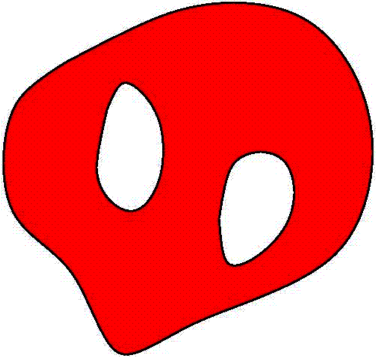

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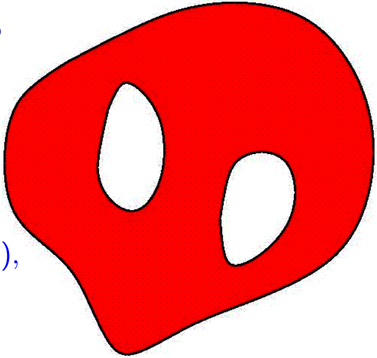

- Amplitude and fundamental tones of vibrations

$$\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\phi_k(x,y)=\lambda_k\phi_k(x,y),$$

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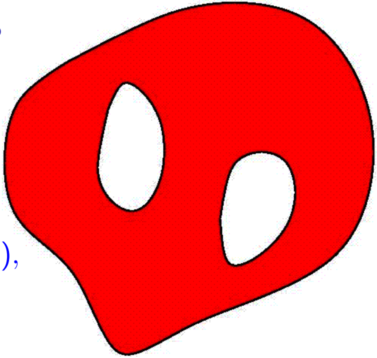

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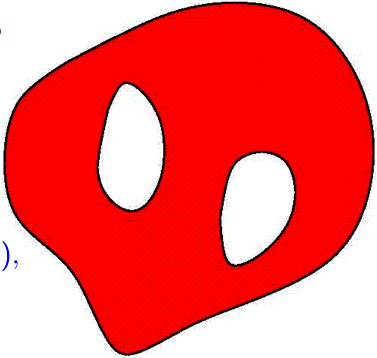

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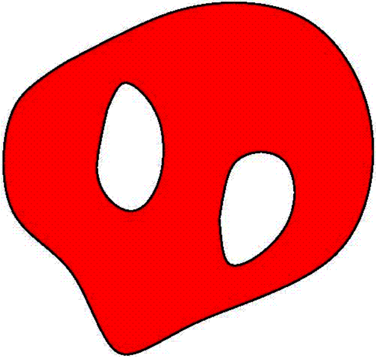
M. Kac, Am. Math. Monthly 73 (1966) 1
H.P. Mc Kean and I.M. Singer, J. Diff. Geom. 1 (1967) 43

Can a gas feel the shape of its container?



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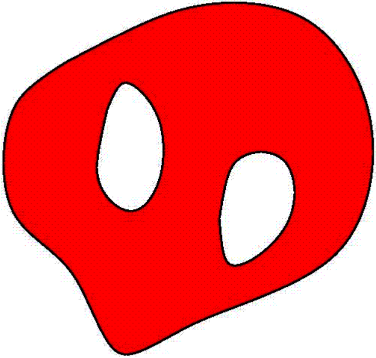

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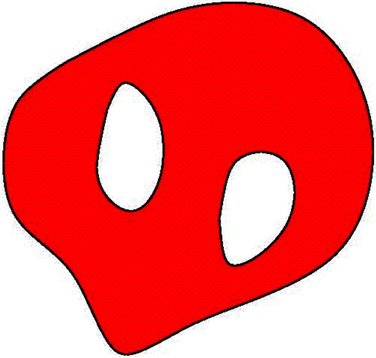

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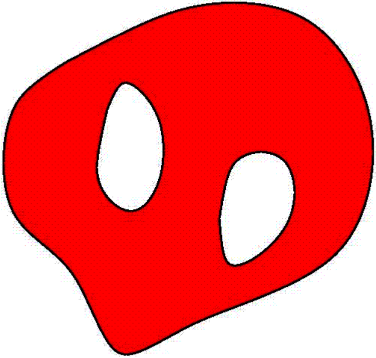

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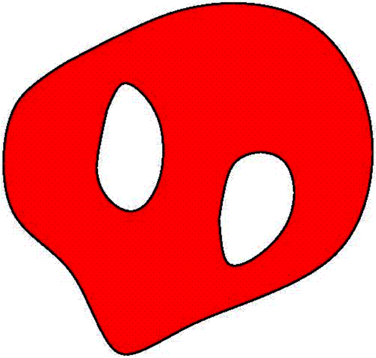
What does the Casimir effect know about a boundary?



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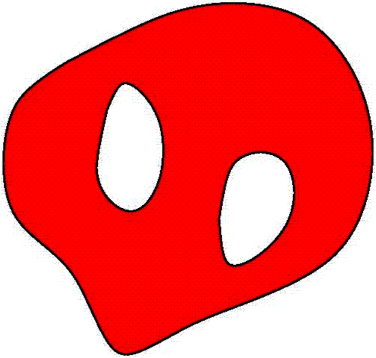

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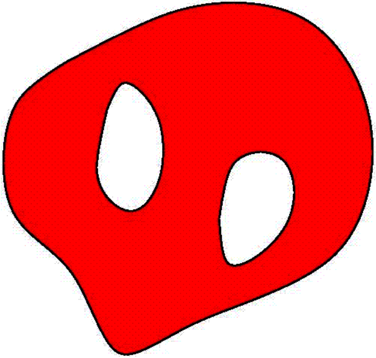

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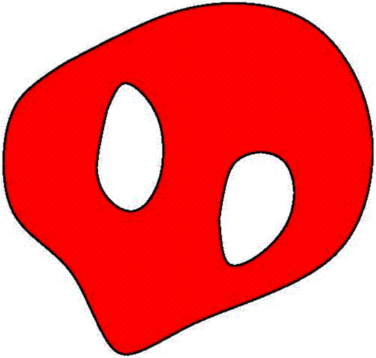

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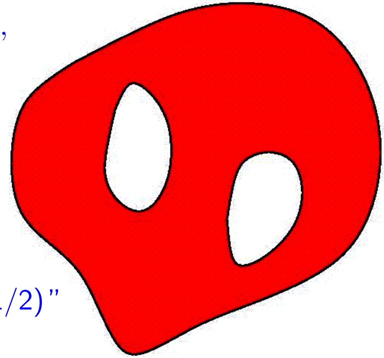

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What does the Casimir effect know about a boundary?

- In more detail

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T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rep. 66 (1980) 213 (895 citations)

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What is the small-t asymptotics of the heat kernel?

Introduction

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$$K(t,F) \sim t^{-D/2} \sum_{n=0,1,2,\ldots}^{\infty} c_n(F,P) t^n$$

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$c_n(F,P)$ and $b_\ell(F,P,\mathcal{B})$ are the heat kernel coefficients depending on geometric invariants and the boundary condition.

Introduction

- General strategy: Write down general form using independent invariants

$$c_i I_i, \qquad i = 1, 2, ...$$

c_i are numerical multipliers and I_i are

$$R,\ R_{ij}R^{ij},\ \Delta R,\ ... \qquad L_a^a,\ L_{ab}L^{ab},\ ...$$

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  a.) special cases
  b.) functorial properties (product formula, conformal transformations)

- Advantages:
  a.) long calculation split into pieces
  b.) character of a riddle, quite enjoyable
  c.) provides many crosschecks due to overlapping information obtained by the conglomerate of methods

Boundary conditions and invariance theory

- Laplace type operator P

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- Heat trace

$$\begin{aligned} K(t,F) &= \mathrm{Tr}_{L^2(M)}\left(Fe^{-tP}\right) \\ &\sim t^{-D/2} \sum_{n=0,1/2,1,\dots}^{\infty} a_n(F,P,\mathcal{B})\, t^n \end{aligned}$$

Boundary conditions and invariance theory

- General form of the coefficients; dimensional consideration

$$\begin{aligned} P &: \text{length}^{-2} \\ e^{-tP} &: [t] = \text{length}^{2} \\ a_n(F, P, \mathcal{B}) &: \text{length}^{D-2n} \end{aligned}$$

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$$\begin{aligned} a_n(F,P,\mathcal{B}) &= \int_M dx\; c_n(x,F,P) + \int_{\partial M} dy\; b_n(y,F,P,\mathcal{B}) \\ c_n(x,F,P) &: \text{length}^{-2n} \\ b_n(y,F,P,\mathcal{B}) &: \text{length}^{1-2n} \end{aligned}$$

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- Building blocks

$$E,\ R,\ R_{ij},\ R_{ijkl} : \text{length}^{-2} \qquad L_{ab} : \text{length}^{-1}$$

and contractions, covariant derivatives thereof.

Boundary conditions and invariance theory

$$b_{1/2} \;=\; (4\pi)^{-(D-1)/2}\, \delta \; F \; [\partial M]$$

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b_2: 19 terms
$b_{5/2} \sim 75$ terms

Special case calculation on the ball

Example with non-vanishing extrinsic curvature is very useful

- Geometry of the ball $(D = d + 1)$

$$L_a^b = \delta_a^b, \quad L_a^a = d, \quad L_{ab}L^{ab} = d, \ldots$$

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- Localizing function

$$F(r) = f_0 + f_1 r^2 + f_2 r^4$$

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b_{3/2} &= (4\pi)^{-(D-1)/2} \delta 96^{-1} \left[F(7d^2 - 10d) + 30dF' + 24F''\right] [\partial M]
\end{aligned}
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Functorial properties

- Let $M = M_1 \times M_2$, $\partial M = M_1 \times \partial M_2$, $g_M = g_{M_1} + g_{M_2}$, then $R(M) = R(M_1) + R(M_2)$.

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- Furthermore let $P_M = P_{M_1} + P_{M_2}$, $F_M = F_{M_1} F_{M_2}$, then

$$\begin{aligned} e^{-tP_M} &= e^{-tP_{M_1}} \, e^{-tP_{M_2}}, \\ \mathrm{Tr}_{L^2(M)} \left(F_M e^{-tP_M} \right) &= \mathrm{Tr}_{L^2(M_1)} \left(F_{M_1} e^{-tP_{M_1}} \right) \, \mathrm{Tr}_{L^2(M_2)} \left(F_{M_2} e^{-tP_{M_2}} \right) \end{aligned}$$

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- In particular,

$$\begin{aligned} a_{3/2}(M, P, \mathcal{B}) = a_1(M_1, P_1) \; & b_{1/2}(M_2, P_2, \mathcal{B}) \\ & + a_0(M_1, P_1) \; b_{3/2}(M_2, P_2, \mathcal{B}) \\ \Longrightarrow c_0 = 96, \quad c_1 = 16. & \end{aligned}$$

Functorial properties

- Idea: consider variations of the coefficients

$$\begin{aligned} P(\epsilon) &= e^{-2\epsilon F} P \\ &=: -(g^{ij}(\epsilon)\nabla_i(\epsilon)\nabla_j(\epsilon) + E(\epsilon)) \end{aligned}$$

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- Relation between the associated coefficients

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_n(1, P(\epsilon), \mathcal{B}(\epsilon)) = (D-2n) a_n(F, P, \mathcal{B})$$

Functorial properties

- For $n = 3/2$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} b_{3/2}(1, P(\epsilon), \mathcal{B}(\epsilon)) - (D-3) b_{3/2}(F, P, \mathcal{B}) = 0$$

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- Set the coefficient of $F_{;mm}$ equal zero

$$\frac{1}{2}(D-2)c_0 - 2(D-1)c_1 - (D-1)c_2 - (D-3)c_6 = 0$$
$$\Longrightarrow c_2 = -8$$

$$a_{5/2}(F,P,\mathcal{B}) = \mp 5760^{-1}(4\pi)^{-d/2}\{F\left\{g_1 E_{;mm} + g_2 E_{;m}S + g_3 E^2\right.$$

$$+g_4 E_{:a}{}^{a} + g_5 RE + j_1 \Omega_{ab}\Omega^{ab} + g_6 \Delta R + g_7 R^2 + g_8 R_{ij}R^{ij} + g_9 R_{ijkl}R^{ijkl}$$

$$+g_{10} R_{mm}E + g_{11} R_{mm}R + g_{12} RS^2 + j_2 \Omega_{am}\Omega^{a}{}_{m} + g_{13} R_{;mm} + g_{14} R_{mm:a}{}^{a}$$

$$+g_{15} R_{mm;mm} + g_{16} R_{;m}S + g_{17} R_{mm}S^2 + g_{18} SS_{:a}{}^{a} + g_{19} S_{:a}S_{:}^{a}$$

$$\left. +g_{20} R_{ammb}R^{ab} + g_{21} R_{mm}R_{mm} + g_{22} R_{ammb}R^{a}{}_{mm}{}^{b} + g_{23} ES^2 + g_{24} S^4\right\}$$

$$+F_{;m}\left\{g_{25} R_{;m} + g_{26} RS + g_{27} R_{mm}S + g_{28} S_{:a}{}^{a} + g_{29} E_{;m} + g_{30} ES + g_{31} S^3\right\}$$

$$+F_{;mm}\left\{g_{32} R + g_{33} R_{mm} + g_{34} E + g_{35} S^2\right\} + g_{36} SF_{;mmm} + g_{37} F_{;mmmm}$$

$$+F\left\{d_1 KE_{;m} + d_2 KR_{;m} + d_3 K^{ab}R_{ammb;m} + d_4 KS_{:b}{}^{b} + d_5 K_{ab}S_{:}^{ab}\right.$$

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T.P. Branson, P.B. Gilkey, KK and D.V. Vassilevich, Nucl. Phys. B563 (1999) 603

On the analysis of spectral functions

- Example: two dimensional ball
  Laplacian in polar coordinates:

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\phi_{m,n}(r,\varphi)=\lambda_{m,n}\phi_{m,n}(r,\varphi)$$

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- Impose boundary condition:

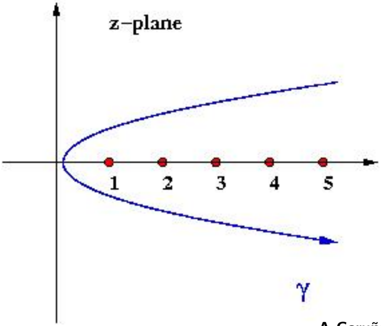
$$J_{|m|}\left(\sqrt{\lambda_{m,n}} \right) = 0$$

On the analysis of spectral functions

- Zeta function:

$$\zeta(s) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n}^{-s}$$

$$= \sum_{m=-\infty}^{\infty} \int_{\gamma} \frac{dz}{2\pi i} \, z^{-2s} \, \frac{\partial}{\partial z} \ln J_{|m|}(z)$$



On the analysis of spectral functions

- Techniques:
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- Results:
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Conclusions

- A combination of techniques allows for the very effective evaluation of heat kernel coefficients for a variety of boundary conditions in the smooth and singular setting. [Branson, Dowker, Park, Seeley, van den Berg, Vassilevich]

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- The approach is expected to work whenever the configuration allows for a separation of variables.