Pseudo-Riemannian manifolds modelled on symmetric spaces

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joint work with Zdeněk Dušek

Conference in Geometry and Global Analysis
Celebrating P. Gilkey's 65th Birthday
Santiago de Compostela, December 13-17, 2010

Basic definitions

- A pseudo-Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for any pair of points p and q in M, there exists a linear isometry $F : T_pM \to T_qM$ such that $g_q(R(FX, FY)FZ, FW) = g_p(R(X, Y)Z, W)$ for any $X, Y, Z, W \in T_pM$.
- (M, g) *has the same curvature tensor* as a homogeneous manifold (M_0, g^0) if, for any points $p \in M$ and $q \in M_0$, there is a linear isometry $F : T_pM \to T_qM_0$ such that $g^0_q(R(FX, FY)FZ, FW) = g_p(R(X, Y)Z, W)$.
- (M, g) is said to be *semi-symmetric* if $R_{(X,Y)} \cdot R = 0$, where the curvature operator acts by the dot as a derivation on the tensor algebra.
- A space is semi-symmetric if, and only if, it has the same curvature tensor as a symmetric space.

Cahen at al:

- Lorentzian manifold (M, g) with the same curvature tensor as an irreducible Lorentzian symmetric space (M_0, g^0) has constant sectional curvature;
- a class of complete 3-dimensional Lorentzian manifolds with the same curvature tensor as an indecomposable (but not irreducible) Lorentzian symmetric space.
  - The Ricci tensor is always of rank 1,
  - the scalar curvature vanishes.

  This construction can be generalized to higher dimensions.

Riemannian geometry - "generalized Sekigawa examples"

- the only irreducible Riemannian manifolds which are locally non-homogeneous with the same curvature tensor as a Riemannian symmetric space;
- the model space must be a direct product of a 2-dimensional space form $M_2(c)(c \neq 0)$ and an Euclidean n-space;
- the Ricci tensor has rank 2 and the scalar curvature is nonzero.

  We generalize these examples to the pseudo-Riemannian case.

Generalized Sekigawa examples

$\mathbb{R}^{n+1}[w, x^1, \ldots, x^n]$, open set $U \subset \mathbb{R}^2[w, x^1]$,
$f\colon U \to \mathbb{R}$ smooth non-vanishing function,

$A(w) = (A^i_j(w)) \ldots$ skew-symmetric smooth
$(n \times n)$-matrix function of one variable

$\omega^0 = f(w, x^1)dw$,
$\omega^i = dx^i + \sum_{j=1}^n A^i_j(w)x^j dw, \quad i = 1, \ldots, n$

Metric of generalized Sekigawa example

- We define a metric $g_{f,A(w)}$ as follows:

  $g_{f,A(w)} = \sum_{j=0}^{n+1} \omega^j \otimes \omega^j$ on open $\tilde{U} \subset \mathrm{R}^{n+1}$.

- The components of the curvature tensor are:

  $R_{0101} = R_{1010} = -R_{1001} = -R_{0110} = -f^{-1} f''_{x^1 x^1}$,
  all other components R_{ijkl} vanish.

- $g_{f,A(w)}$ is **nonflat and curvature homogeneous** if and only if $f^{-1} f''_{x^1 x^1} = \lambda$ where $\lambda \neq 0$ is a constant.

Non-flat curvature homogeneous examples

It means:

$$f(w,x^1) = a(w)\exp(\sqrt{\lambda}x^1) + b(w)\exp(-\sqrt{\lambda}x^1) \text{ if } \lambda > 0,$$

or

$$f(w,x^1) = a(w)\cos(\sqrt{-\lambda}x^1) + b(w)\sin(\sqrt{-\lambda}x^1) \text{ if } \lambda < 0,$$

- $a(w)$ and $b(w)$ differentiable functions such that $f(w,x^1) > 0$ in U,
- U can be the whole plane in the case $\lambda < 0$ and an open strip in the plane for $\lambda > 0$.

Pseudo-Riemannian modification of generalized Sekigawa examples

Euclidean space $\mathbb{R}^{n+1}$, Cartesian coordinates $(w, x^1, \dots, x^n)$, signature $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i = \pm 1$.
Let $A = A^i_j(w)$ be a smooth matrix function such that $A^i_i(w) = 0$, $A^i_j(w)$ are arbitrary for $1 \leq j < i \leq n$,

$$A^j_i(w) \;=\; -\varepsilon_i \varepsilon_j A^i_j(w), \qquad 1 \leq j < i \leq n. \tag{1}$$

We define the 1-forms

$$\begin{aligned} \omega^0 &= f(w, x^1) dw, \\ \omega^i &= dx^i + \sum_{j=1}^{n} A^i_j(w) x^j dw, \quad i = 1, \dots, n. \end{aligned} \tag{2}$$

Pseudo-Riemannian metric g such that $\{\omega^0, \dots, \omega^n\}$ is a pseudo-orthonormal coframe with the above signature.

- We will show that the scalar curvature is $\tau = -2\varepsilon_1 f''_{x^1x^1}/f$.
- $(\mathbb{R}^{n+1}, g)$ is curvature homogeneous iff $\tau = 2\lambda = const$.
- For $\lambda < 0$, or, $\lambda > 0$, respectively, we put

$$\begin{aligned} f(w,x^1) &= a(w)\exp(\sqrt{-\lambda}x^1) + b(w)\exp(-\sqrt{-\lambda}x^1), \\ f(w,x^1) &= a(w)\cos(\sqrt{\lambda}x^1) + b(w)\sin(\sqrt{\lambda}x^1). \end{aligned} \tag{3}$$

  Here $a(w), b(w)$ are non-negative smooth functions on $\mathbb{R}$ such that $a(w) + b(w) > 0$ and the definition domain $\mathcal{U}$ is a direct product $\mathcal{S} \times \mathbb{R}^{n-1}$, where $\mathcal{S}$ is an open strip in the plane $\mathbb{R}^2[w, x^1]$ so that $f(w, x^1) > 0$ on $\mathcal{S}$.
- In the Riemannian case,
  the curvature homogeneous metric associated with the first solution has the same curvature tensor as the direct product symmetric space $H^2(\lambda) \times \mathbb{R}^{n-1}$ and the metric associated with the second solution has the model $S^2(\lambda) \times \mathbb{R}^{n-1}$.

Curvature properties

The dual vector fields E_i to 1-forms ω^i are

$$\begin{aligned} E_0 &= \frac{1}{f(w,x^1)}\big(\partial_w - \sum_{i,j=1}^{n} A^i_j(w)x^j\partial_{x^i}\big), \\ E_i &= \partial_{x^i}, \qquad i=1,\ldots,n. \end{aligned}$$

For the commutators of these vector fields we obtain

$$\begin{aligned} [E_0,E_j] &= \frac{1}{f(w,x^1)}\big(\frac{\partial f(w,x^1)}{\partial x^j}E_0 + \sum_{i=1}^{n} A^i_j(w)E_i\big), \\ [E_i,E_j] &= 0. \end{aligned}$$

Because $\{E_i\}_{i=0}^{n}$ is a pseudo-orthonormal frame,
we calculate the covariant derivatives from the formula

$$2\langle \nabla_{E_i}E_j, E_k\rangle = -\langle E_i,[E_j,E_k]\rangle + \langle E_j,[E_k,E_i]\rangle + \langle E_k,[E_i,E_j]\rangle.$$

For any vector field $X = \sum_{i=0}^{n} x^i E_i$, the covariant derivatives are

$$\begin{aligned}
\nabla_X E_0 &= -\varepsilon_0\varepsilon_1 \frac{x^0}{f} f'_{x^1} E_1, \\
\nabla_X E_1 &= \frac{x^0}{f}\big(f'_{x^1} E_0 + \sum_{i=1}^{n} A_1^i(w) E_i\big), \\
\nabla_X E_j &= \frac{x^0}{f} \sum_{i=1}^{n} A_j^i(w) E_i, \quad j = 2, \dots, n.
\end{aligned} \tag{4}$$

The curvature operators:

$$R_{XY} Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z, \qquad X, Y, Z \in \mathcal{X}(M).$$

With respect to the frame $\{E_i\}_{i=0}^{n}$, the only nonzero operators are

$$\begin{aligned}
R_{E_0 E_1} E_0 &= -\varepsilon_0\varepsilon_1 \frac{f''_{x^1 x^1}}{f} \cdot E_1, \\
R_{E_0 E_1} E_1 &= \frac{f''_{x^1 x^1}}{f} \cdot E_0.
\end{aligned}$$

The Ricci curvature:

$$\mathrm{Ric}(X,Y)=\sum_{i=0}^{n}\varepsilon_i\langle R_{XE_i}Y,E_i\rangle,\qquad X,Y\in\mathcal{X}(M).$$

In the frame $\{E_i\}_{i=0}^{n}$, the nonzero principal Ricci curvatures are

$$\mathrm{Ric}(E_0,E_0)=-\varepsilon_0\varepsilon_1\frac{f''_{x^1x^1}}{f},\quad \mathrm{Ric}(E_1,E_1)=-\frac{f''_{x^1x^1}}{f}$$

and the scalar curvature is

$$\tau=-2\varepsilon_1\frac{f''_{x^1x^1}}{f}.$$

Lemma

Let (M,g), $\dim(M) > 2$, *be a proper pseudo-Riemannian manifold constructed according to the formulas* (1) *and* (2).

- *If* $\varepsilon_0 = -\varepsilon_1$, *then* (M,g) *has the same curvature as the direct product* $L^2(c) \times \mathbb{R}_k^{n-1}$, *where* $L^2(c)$ *is the Lorentzian manifold with constant nonzero curvature and* $\mathbb{R}_k^{n-1}$ *is a flat pseudo-Riemannian space of signature* $(k, n-1-k)$, *where* k *is an integer* $0 \le k \le n-1$.
- *If* $\varepsilon_0 = \varepsilon_1$, *then* (M,g) *has the same curvature as the direct product* $S^2(c) \times \mathbb{R}_k^{n-1}$ *or* $H^2(c) \times \mathbb{R}_k^{n-1}$, *where* $\mathbb{R}_k^{n-1}$ *is a flat pseudo-Riemannian space of signature* $(k, n-1-k)$, *where* k *is an integer* $0 \le k < n-1$.

In both cases, (M,g) *has the same signature as the corresponding model space* $($*up to replacing* g *by* $-g)$.

Irreducibility

Irreducibility (for a special subclass of metrics):
Let $\varepsilon_i = 1$ for $i = 0, \ldots, p$ and $\varepsilon_i = -1$ for $i = p+1, \ldots, n$,

$$\begin{gathered} A_i^{i+1}(w) = \lambda_i(w) > 0, \quad i = 1, \ldots, n-1, \\ A_j^i(w) = 0, \quad \|i - j\| > 1. \end{gathered} \tag{5}$$

Clearly, $A_{p+1}^p(w) = A_p^{p+1}(w) = \lambda_p(w)$
and $A_{i+1}^i(w) = -A_i^{i+1}(w) = -\lambda_i(w)$ for $i \neq p$.

Formulas (4) in this special situation are:

$$
\begin{aligned}
\nabla_X E_0 &= -\frac{x^0}{f} f'_{x^1} E_1, \\
\nabla_X E_1 &= \frac{x^0}{f}\big(f'_{x^1} E_0 + \lambda_1 E_2\big), \\
\nabla_X E_j &= \frac{x^0}{f}\big(-\lambda_{j-1} E_{j-1} + \lambda_j E_{j+1}\big), \quad 2 < j < p+1, \\
\nabla_X E_{p+1} &= \frac{x^0}{f}\big(\lambda_p E_p + \lambda_{p+1} E_{p+2}\big), \\
\nabla_X E_j &= \frac{x^0}{f}\big(-\lambda_{j-1} E_{j-1} + \lambda_j E_{j+1}\big), \quad p+1 < j < n-1, \\
\nabla_X E_n &= -\frac{x^0}{f} \lambda_{n-1} E_{n-1}.
\end{aligned}
$$

Lemma

Let $f''_{x^1x^1} \neq 0$, $\varepsilon_0 = \varepsilon_1 = 1$ *and the matrix* $A^i_j(w)$ *is of the special form described above. Then, for any point* $p \in M$*, there is a neighbourhood* $\mathcal{V}$ *such that the local distribution* $TM|_{\mathcal{V}}$ *does not admit any proper parallel subdistribution.*

Because there is no proper parallel subdistribution H in $TM|_{\mathcal{V}}$, there is no proper subspace H_p in T_pM invariant with respect to the holonomy group $\Phi(\mathcal{V}, p)$. Hence, the full holonomy group $\Phi(M, p)$ acts irreducibly on T_pM. We obtain the following:

Theorem

Pseudo-Riemannian manifolds (M, g) *modelled on* $S^2(c) \times \mathbb{R}^{n-1}_k$ *or* $H^2(c) \times \mathbb{R}^{n-1}_k$ *and given by the formulas* (2), (3) *and* (5) *are irreducible.*

Concluding remarks

By the modification of the rather long computation as in [2], we are able to prove the following:

Theorem

Let (M^{n+1}, g) be locally non-homogeneous, locally irreducible, curvature homogeneous pseudo-Riemannian manifold modelled on $S^2(c) \times \mathbb{R}^{n-1}_k$ or $H^2(c) \times \mathbb{R}^{n-1}_k$, with the signature $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ such that $\varepsilon_0 = \varepsilon_1$. Then there exists a dense open subset $\mathcal{U}$ of M^{n+1} such that, in a neighbourhood of each point $p \in \mathcal{U}$, the metric g is expressed by the formulas (1) *and* (2).

- In [2], a stronger result was proved for the Riemannian case, namely that each irreducible and not locally symmetric space with a symmetric model must be a generalized Sekigawa example.
  In the pseudo-Riemannian case, the situation is more complicated (see examples in [3]).
- For the case $\varepsilon_0 = -\varepsilon_1$, i.e., for the pseudo-Riemannian manifolds modelled on $L^2(c) \times \mathbb{R}^{n-1}_k$,
  it is not so easy to prove the irreducibility.
- In [3], the authors proved the geodesic completeness of their example. The equations of geodesics are very simple there.
  Our equations of geodesics, are complicated.
  We leave the completeness problem open.

- Boeckx, E., Kowalski, O. and Vanhecke, L.: Riemannian Manifolds of Conullity Two, World Scientific, 1996.
- Boeckx, E., Kowalski and Vanhecke, L.: *Non-homogeneous relatives of symmetric spaces*, Diff. Geom. and Appl.
- Cahen, M., Leroy, J., Parker, M., Tricerri, F. and Vanhecke, L.: *Lorentz manifolds modelled on a Lorentz symmetric space*, J. Geom. Phys.
- Dušek, Z., Kowalski, O.: *Pseudo-Riemannian spaces modelled on symmetric spaces*, preprint.
- Sekigawa, K.: *On some* 3*-dimensional Riemannian manifolds*, Hokkaido Math. J.

Curvature-homogeneous spaces of type (1,3)

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2010

Curvature homogeneity (classical)

(Pseudo-)Riemannian spaces whose curvature tensor of type (0,4) is "the same" at all points.

Trivial examples:

- connected locally homogeneous manifolds.

Curvature homogeneous spaces have been studied by many authors, starting from the basic paper

I.M. Singer: Infinitesimally homogeneous spaces, *Comm. Pure Appl. Math.* **13** (1960), 685-697.

First non-trivial Riemannian examples were given by

K. Sekigawa: On some 3-dimensional Riemannian manifolds, *Hokkaido Math. J.* **2** (1973), 259-270,

K. Sekigawa: On some 3-dimensional curvature homogeneous spaces, *Tensor* **31** (1977), 87-97,

and generalized later on. We will survey some older results (with E. Boeckx, F. Tricerri, L. Vanhecke) and some very fresh results (with Z. Dušek and A. Vanžurová).

For example, an older result says that every irreducible Riemannian manifold with the same curvature tensor as a symmetric space and NOT locally homogeneous (so-called "non-homogeneous relative of a symmetric space") is a "generalized Sekigawa example".

Generalized Sekigawa examples

$\mathbb{R}^{n+1}[w, x^1, \ldots, x^n]$, open $U \subset \mathbb{R}^2[w, x^1]$, $f\colon U \to \mathbb{R}$ (smooth) non-vanishing function,

$A(w) = (A^i_j(w))$ skew-symmetric (smooth) $(n \times n)$-matrix function of one variable

$\omega^0 = f(w, x^1)dw,$

$\omega^i = dx^i + \sum_{j=1}^n A^i_j(w)x^j dw, \quad i = 1, \ldots, n$

Metric

We define a metric $g_{f,A(w)}$ as follows:

$g_{f,A(w)} = \sum_{j=0}^{n+1} \omega^j \otimes \omega^j$ on open $\tilde{U} \subset \mathbb{R}^{n+1}$

curvature:

$R_{0101} = R_{1010} = -R_{1001} = -R_{0110} = -f^{-1} f''_{x^1 x^1}$,
all other components R_{ijkl} vanish.

$g_{f,A(w)}$ is **nonflat and curvature homogeneous** if and only if
$f^{-1} f''_{x^1 x^1} = k$ where $k \neq 0$ is a constant;

Non-flat curvature homogeneous examples

it means:

$f(w,x^1) = a(w)\exp(\sqrt{k}x^1) + b(w)\exp(-\sqrt{k}x^1)$ if $k > 0$, or

$f(w,x^1) = a(w)\cos(\sqrt{-k}x^1) + b(w)\sin(\sqrt{-k}x^1)$ if $k < 0$,

$a(w)$ and $b(w)$ differentiable functions such that
$f(w,x^1) > 0$ in U.

Here U can be the whole plane in the case $k > 0$ and an open strip in the plane for $k < 0$.

Recall that this class of spaces is remarkable:

it includes all irreducible curvature homogeneous spaces which are not locally homogeneous and whose curvature tensor $\mathcal{R}$ "is the same" as that of a Riemannian symmetric space (so-called "non-homogeneous relatives of symmetric spaces").

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- E. Boeckx, O. Kowalski, L. Vanhecke: *Riemannian Manifolds of Conullity two.* World Scientific, 1996.
- O.Kowalski, F.Tricerri, L.Vanhecke: New examples of non-homogeneous Riemannian manifolds whose curvature tensor is that of a Riemannian symmetric space. *C. R. Acad. Sci. Paris*, t. **311**, Série I, (1990), 355-360.
- L. Nicolodi and F. Tricerri: On two theorems of I.M. Singer about homogeneous spaces. *Ann. Global Anal. Geom.*, **8** (1990), 193-209.
- O. Kowalski, F. Tricerri, L. Vanhecke: Curvature homogeneous Riemannian manifolds. *J. Math. Pures Appl.*, **71** (1992) 471-501.

O. Kowalski, F. Tricerri, L. Vanhecke: Curvature homogeneous spaces with a solvable Lie group as homogeneous model. *J. Math. Soc. Japan* **44** (1992), 461-484.

E. Boeckx, O. Kowalski, L. Vanhecke: Non-homogeneous relatives of symmetric spaces. *Diff. Geom. and Appl.* **4** (1994), 45-69.

K. Sekigawa, H. Suga and L. Vanhecke: Four-dimensional curvature homogeneous spaces, *Comment. Math. Univ. Carolinae* **33** (1992), 261-268.

K. Sekigawa, H. Suga and L. Vanhecke: Curvature homogeneity for four-dimensional manifolds, *J. Korean Math. Soc.* **32** (1995), 93-101.

Lorentzian case

3-dimensional Lorentzian case, all relevant orders:

P. Bueken: On curvature homogeneous three-dimensional Lorentzian manifolds, *J. Geom. Phys.* **22** (1997), 349-362.

P. Bueken and M. Djorić: Three-dimensional Lorentz metrics and curvature homogeneity of order one, *Ann. Glob. Anal. Geom.* **18** (2000), 85-103.

Pseudo-Riemannian case

Pseudo-Riemannian curvature homogeneous spaces (arbitrary dimension, signature, order):

- P.J. Gilkey: *The Geometry of Curvature Homogeneous Pseudo-Riemannian Manifolds.* ICP Advanced Texts in Mathematics - Vol. 2. Imperial College Press, 2007.
- M. Brozos-Vásquez, P. Gilkey and S. Nikčević: Geometric realizations of curvature. *Imperial College Press* (in preparation).

Another topic:

Introducing a new concept of a curvature-homogeneous space, namely in the sense that **not** the curvature tensor of type (0,4) is preserved from point to point but the curvature tensor of type (1,3) is preserved from point to point, in some sense.

Each curvature-homogeneous space in the classical sense is also curvature-homogeneous in the modified sense. The original example by K. Sekigawa (which is curvature homogeneous just of order zero) is curvature homogeneous up to order one in the modified sense.

We give proper examples of the new spaces in all dimensions, and a complete classification of such spaces in dimension 3 and in generic case.

Fresh results:

- O. Kowalski, A. Vanžurová: On curvature homogeneous spaces of type (1,3) (to appear in Math. Nachr.)
- O. Kowalski, A. Vanžurová: On a generalization of curvature homogeneous spaces (submitted)
- O. Kowalski, Z. Dušek: Pseudo-Riemannian spaces modelled on symmetric spaces (to appear in Monatshefte Math.)

Notation

(M, g) a smooth Riemannian manifold with a (positive) metric g and the Riemannian (L.-C.) connection ∇,

R the type (1,3) curvature tensor,
$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$,
R_p is its value in $p \in M$,

$\mathcal{R}$ the type (0,4) curvature tensor
$\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, $\mathcal{R}_p$
$R(X,Y,Z,W) = g(R(X,Y)W,Z) = -\mathcal{R}(X,Y,Z,W)$, R_{ijkl}

Definitions

Definition 1
A smooth Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for any pair of points p and q in M, there exists a linear **isometry** $F\colon T_pM \to T_qM$ such that $F^*(\mathcal{R}_q) = \mathcal{R}_p$.

Definition 2
A Riemannian manifold (M, g) is said to be *(1,3)-curvature homogeneous* if for any pair of points p, q there is a curvature-preserving linear **homothety** $f\colon T_pM \to T_qM$, $f^*(R_q) = R_p$.

Proposition 1

If (M, g) is curvature homogeneous then it is also (1,3)-curvature homogeneous.

But NOT vice versa.

Examples show:

curvature homogeneous spaces of type (1,3) form a much bigger class.

Proposition 2

Let (M, g) be a smooth Riemannian manifold. Then the following two conditions are equivalent:

(i) For each $q \in M$, there is a linear homothety $f_q \colon T_pM \to T_qM$ such that $R_p = f_q^*(R_q)$, i.e. (M, g) is (1,3)-curvature homogeneous.

(ii) There is a smooth function φ on M such that $\varphi(p) = 0$ and for each $q \in M$, $\mathcal{R}_p = e^{2\varphi(q)} F_q^*(\mathcal{R}_q)$ where $F_q \colon T_pM \to T_qM$ is a linear isometry.

$\mathcal{R}$ of type (0,4), R of type (1,3).

Remarks

- The linear isometry F_q from part (i) of Proposition 2 is not always uniquely determined, but, it is also not arbitrary, in general. It must be just compatible with the property of the composed map f as given in part (ii).
- Eevery (1,3)-curvature homogeneous space of dimension n determines a uniquely determined smooth function of n variables. The converse remains an open problem but it is true in dimension 3.

Lemma 1

Let (M, g) be a Riemannian manifold and let $\langle E_1, \dots, E_n \rangle$ be an orthonormal moving frame on a domain $U \subset M$. Fix a point $p \in U$. Suppose that, with respect to this moving frame, $R_{ijk\ell}(q) = \phi(q) R_{ijk\ell}(p)$ for each point $q \in U$ and for all choices of indices, where $\phi(q)$ is a smooth and positive function on U. Then there is a smooth function $\varphi(q)$ such that $\varphi(p) = 0$ and, for each point q, $\mathcal{R}_p = e^{2\varphi(q)} F_q^*(\mathcal{R}_q)$ where $F_q \colon T_pM \to T_qM$ is a linear isometry.

Generalized Sekigawa's examples:

$\mathbb{R}^{n+1}[w, x^1, \dots, x^n]$, open $U \subset \mathbb{R}^2[w, x^1]$, $f\colon U \to \mathbb{R}$ (smooth) non-vanishing function,

$A(w) = (A^i_j(w))$ skew-symmetric (smooth) $(n \times n)$-matrix function of one variable,

$\omega^0 = f(w, x^1)dw,$

$\omega^i = dx^i + \sum_{j=1}^n A^i_j(w)x^j dw, \quad i = 1, \dots, n$

$$g_{f,A(w)} = \sum_{j=0}^{n+1} \omega^j \otimes \omega^j$$

Construction

The space $(\mathbb{R}^{n+1}, \overline{g})$ which is (1,3)-curvature homogeneous but not (0,4)-curvature homogeneous:

f an arbitrary smooth function on $\mathbb{R}^2$ s.t. at all points, f and $f^{-1} f''_{x^1 x^1}$ are nonzero and $f''_{x^1 x^1}/f$ is never a constant in an open domain of $\mathbb{R}^2$.
The corresponding metric $\overline{g} = g_{f,A(w)}$ defined on $\mathbb{R}^{n+1}$ has the curvature components $\overline{R}_{ijkl}$ calculated as above, satisfying

$$\overline{R}_{ijkl}(q) = (f^{-1}(q) f''_{x^1 x^1}(q))/(f^{-1}(p) f''_{x^1 x^1}(p)) \overline{R}_{ijkl}(p)$$

for any pair of points $p, q \in \mathbb{R}^{n+1}$ and all indices i, j, k, l.

Let now the point p be fixed.

The assumptions of the Lemma are satisfied, where the corresponding function $\phi(q)$ is defined as

$$\phi(q) = f^{-1}(q) f''_{x^1 x^1}(q) / (f^{-1}(p) f''_{x^1 x^1}(p))$$

and hence positive.

From Proposition 2 and our special assumptions we deduce that the space $(\mathbb{R}^{n+1}, \overline{g})$ is (1,3)-curvature homogeneous but not (0,4)-curvature homogeneous.

The family of all such metrics depends, up to a local isometry, on 1 arbitrary function of 2 variables and a finite number of arbitrary functions of 1 variable. Moreover, if $f(w, x^1)$ is defined on the whole $\mathbb{R}^{n+1}$ and if there are two continuous functions $\alpha(x^1)$, $\beta(x^1)$ on $\mathbb{R}^{n+1}$ such that $0 < \alpha(x^1) \leq f(w, x^1) \leq \beta(x^1) < \infty$, then the space

$$(\mathbb{R}^{n+1}, g_{f,A(w)})$$

is a complete Riemannian manifold for an arbitrary skew-symmetric matrix function $A(w)$.

On the other hand, if $A(w)$ is a constant matrix, then the inequality $f(w, x^1) > \varepsilon > 0$ is sufficient for the completeness.

Dimension 3

In dimension 3, a Riemannian manifold (M, g) is curvature homogeneous if and only if the Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$ are constant at all points; the curvature tensor R is uniquely determined by the corresponding Ricci tensor ϱ and metric g,

$$\begin{aligned} R_{ijkl} &= \frac{1}{n-2}(g_{ik}\varrho_{jl} - g_{il}\varrho_{jk} + g_{jl}\varrho_{ik} - g_{jk}\varrho_{il}) \\ &+ \frac{\tau}{(n-1)(n-2)}(g_{il}g_{jk} - g_{ik}g_{jl}). \end{aligned} \tag{1}$$

(Metrics and functions - *real analytic*.)

Prescribed Ricci eigenvalues

The following results are known:

Theorem A

Any real analytic Riemannian manifold with the prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ depends (up to a local isometry) on two arbitrary functions of one variable.

O. Kowalski: A classification of Riemannian 3-manifolds with constant principal Ricci curvatures $\varrho_1 = \varrho_2 \neq \varrho_3$. *Nagoya Math. J.* **132** (1993), 1-36

Theorem B

Any real analytic Riemannian manifold with the prescribed distinct constant Ricci eigenvalues $\varrho_1 > \varrho_2 > \varrho_3$ depends (up to a local isometry) on 3 arbitrary functions of 2 variables.

O. Kowalski, Z. Vlášek: Classification of Riemannian 3-manifolds with distinct constant principal Ricci curvatures. *Bulletin of the Belgian Mathematical Society-Simon Stevin,* **5** (1998), 59-68.

The spaces (M, g) with prescribed constant Ricci eigenvalues are, with rare exceptions, *not locally homogeneous.*

The classification of all triplets of distinct real numbers which can be realized as Ricci eigenvalues on a 3-dimensional locally homogeneous space was made in

O. Kowalski, S.Ž. Nikčević: On Ricci eigenvalues of locally homogeneous Riemannan manifolds. *Geometriae Dedicata* **62** (1996), 65-72.

Generalized Yamato examples

On an open subset of $\mathbb{R}^3$, the prescribed triplets of constant Ricci eigenvalues can be realized *only* on spaces which are not locally homogeneous.

Explicit examples of this kind: the authors constructed so-called *generalized Yamato examples* which are explicit for each choice of the triplet $\varrho_1 > \varrho_2 > \varrho_3$ of prescribed Ricci eigenvalues, in

O. Kowalski: A classification of Riemannian 3-manifolds with constant ... *Nagoya Math. J.* **132** (1993), 1-36

O. Kowalski, F. Prüfer: On Riemannian 3-manifolds with distinct constant Ricci eigevalues. *Math. Ann.* **300** (1994), 17-28.

The original construction by K. Yamato - some restrictions are put on the triplets $\varrho_1 > \varrho_2 > \varrho_3$, all constructed metrics are complete.

K. Yamato: A characterization of locally homogeneous Riemannian manifolds of dimension 3. *Nagoya Math. J.* **123** (1993), 77-99.

Theorem B was later generalized in

Theorem C

All Riemannian metrics defined in a domain $U \subset \mathbb{R}^3[x, y, z]$ with the prescribed distinct real analytic Ricci eigenvalues $\varrho_1(x, y, z) > \varrho_2(x, y, z) > \varrho_3(x, y, z)$ depend, up to a local isometry, on three arbitrary (real analytic) functions of two variables. Every solution of the problem is defined at least locally, i.e. in a neighborhood $U' \subset U$ of a fixed $p \in U$.

O. Kowalski, Z. Vlášek: On 3D-manifolds with prescribed Ricci eigenvalues. In: *Complex, Contact and Symmetric Manifolds-In Honor of L. Vanhecke.* Progress in Mathematics, Vol. **234**, Birkhäuser Boston-Basel-Berlin, pp. 187-208 (2005).

In a domain $U \subset \mathbb{R}^3[x, y, z]$, fix a point p and choose a real analytic function $\varphi(x, y, z)$ on U vanishing at p. According to Theorem C, we can construct a (local) Riemannian metric around p such that their Ricci eigenvalues are of the form $\varrho_i = e^{2\varphi}\lambda_i$, $i = 1, 2, 3$ where $\lambda_1 > \lambda_2 > \lambda_3$ are nonzero constants. Denote by g such a local metric. Choose a Ricci adapted orthonormal moving frame $\langle E_1, E_2, E_3\rangle$ in a neighborhood of p. Then we get $\varrho_{ij} = \varrho_i\delta_{ij} = \varrho_j\delta_{ij} = e^{2\varphi}\lambda_i\delta_{ij} = e^{2\varphi}\lambda_j\delta_{ij}$ for $i, j = 1, 2, 3$, and the expression (1) for curvature components is reduced to

$$R_{ijkl} = \tfrac{e^{2\varphi}}{n-2}(\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \lambda_i(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il})$$

$$+\frac{e^{2\varphi}(\lambda_1+\lambda_2+\lambda_3)}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}),$$

$i, j, k, l = 1, 2, 3$. In particular, we get

$$R_{ijkl}(p) = \tfrac{1}{n-2}(\lambda_j(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \lambda_i(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il})$$

$$+\frac{(\lambda_1+\lambda_2+\lambda_3)}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}),$$

i.e., $\mathcal{R} = \exp(2\varphi)\mathcal{R}_p$.

Now, the assumption of Lemma is satisfied and hence Proposition 2 can be applied. Thus, the corresponding metric g is (1,3)-curvature homogeneous and not (0,4)-curvature homogeneous, in general.

Let us call "generic" those Riemannian manifolds for which the Ricci eigenvalues are distinct at all points. Due to Proposition 2, we see easily that all generic 3-dimenisonal (1,3)-curvature homogeneous Riemannian manifolds are constructed in the way described above. Hence we get the following

Theorem 1

In dimension 3, all generic real analytic (1,3)-curvature homogeneous spaces are locally parametrized, up to a local isometry, by one arbitrary real analytic function of 3 variables and three arbitrary real analytic functions of 2 variables.

Classical higher order curvature homogeneity (Singer)

A space (M, g) is said to be *curvature homogeneous up to order* r if it satisfies

$P(r)$: For every $p, q \in (M, g)$ there exists a linear isometry $F\colon T_pM \to T_qM$ such that $F^*((D^k\mathcal{R})_q) = (D^k\mathcal{R})_p$ for all $k = 0, 1, \ldots, r$.

All standard **first order** curvature homogeneous Riemannian manifolds of dimension 3 are automatically **locally homogeneous**.

Singer, for Riemannian space

Connected locally homogeneous spaces are curvature homogeneous of all orders.

There is always a finite number $s \leq n(n-1)/2$ ($n = \dim M$) such that, if (M, g) is curvature homogeneous up to order s, then it is automatically locally homogeneous.

New setting, higher order (1,3)-curvature homogeneous (M, g)

$Q(k)$: There exists a linear homothety $h\colon T_pM \to T_qM$ such that $h^*((D^kR)_q) = (D^kR)_p$ ($p, q \in (M, g)$, k separate).

A *curvature homogeneous up to order* r *and of type (1,3)*: satisfying $Q(0), \ldots, Q(r)$

- for different integers k, the linear homotheties above are completely independent; otherwise, the condition would be too restrictive.

Proposition 3 (K. Sekigawa; O. Kowalski)

Each 3-dimensional Riem. mfd. satisfying $P(1)$ is locally homogeneous.

The same in dimension 4:

- K. Sekigawa, H. Suga and L. Vanhecke: Four-dimensional curvature homogeneous spaces, *Comment. Math. Univ. Carolinae* **33** (1992), 261-268.
- K. Sekigawa, H. Suga and L. Vanhecke: Curvature homogeneity for four-dimensional manifolds, *J. Korean Math. Soc.* **32** (1995), 93-101.

• Up to now, a Riemannian manifold satisfying $P(1)$ and not locally homogeneous is not known.

• The situation is completely different in the pseudo-Riemannian case (P. Bueken, M. Djorić, P.J. Gilkey).

An analogy of Proposition 2 for higher order:

Proposition 4

Given a smooth Riemannian mfd (M, g), the conditions are equivalent:

(a): $P(k)$ holds, i.e., for every $p, q \in (M, g)$ there exists a linear homothety $h \colon T_pM \to T_qM$ s. t. $h^*((D^kR)_q) = (D^kR)_p$.

(b): There is $p \in M$ and a smooth function φ on M s. t. $\varphi(p) = 0$ and for each $q \in M$, $(D^k\mathcal{R})_p = e^{(k+2)\varphi(q)}F^*((D^k\mathcal{R})_q)$ where $F \colon T_pM \to T_qM$ is a linear isometry.

The Sekigawa's example:

The metric $g = \sum_{i=0}^{2} (\omega^i)^2$ on $\mathbb{R}^3[w, x, y]$,

$\omega^0 = f(x)dw$, $f(x) = ae^x + be^{-x}$, a, b positive numbers,

$\omega^1 = dx - ydw$, $\omega^2 = dy + xdw$.

The space $(\mathbb{R}^3, g)$ is simply connected, complete, irreducible, satisfies $P(0)$, is semi-symmetric $(R(X, Y) \cdot R = 0)$.

As a contrast to the above results, we show:

- a 3-dimensional manifold (M, g) of Sekigawa type satisfies $Q(0)$ and $Q(1)$
(thus it is curvature homogeneous up to order 1 and of type (1,3))
- but $Q(2)$ is not fulfilled,
- (M, g) is not locally homogeneous,
- does not satisfy the condition $P(1)$.

Proof

$\langle E_0, E_1, E_2\rangle$ - orthonormal moving frame dual to $\langle \omega^0, \omega^1, \omega^2\rangle$ above, standard evaluation:

$$\mathcal{R} = 4\,\omega^0 \wedge \omega^1 \otimes \omega^0 \wedge \omega^1, \quad f^{-1} f'' = 1.$$

$(D_{E_0}\mathcal{R})(E_0, E_1, E_2, E_0) = -f^{-1}$ and all other components of $D\mathcal{R}$ (up to natural permutations of the four inner arguments) vanish. Aassume $a + b = 1$,

define $\varphi(q)$ on $\mathbb{R}^3$ by $e^{3\varphi(q)} = f(x) = ae^x + be^{-x}$ where $x = x(q)$;

the condition (b) of Proposition 3 holds for the origin $p = [0, 0, 0]$ in the case $k = 1$;

$Q(1)$ is satisfied, as well as $P(0)$, and hence $Q(0)$.

$Q(2)$ is not satisfied:

$$(D^2_{E_0E_0}\mathcal{R})(E_0, E_1, E_1, E_0) = 2f^{-2},$$

$$(D^2_{E_0E_0}\mathcal{R})(E_0, E_1, E_2, E_0) = yf^{-3}f',$$

the last two components are never equal.

There is no function $\varphi(q)$ satisfying the condition (b) of Proposition 3 for $k = 2$. $\square$

Than you for your attention