Geometric Realization of Curvature

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Abstract

A central area of study in differential geometry is the examination of the relationship between purely algebraic properties of the curvature tensor and the underlying geometric properties of the manifold.

Many authors have worked in this area in recent years.

Nevertheless, many fundamental questions remain unanswered. When dealing with a geometric problem, it is frequently convenient to work first purely algebraically and pass later to the geometric setting.

For this reason, many questions in differential geometry are often phrased as problems involving the geometric realization of curvature.

We suppose given a vector space V and a family of tensors $\{T_1, ..., T_k\}$ on V. The structure $(V, T_1, ..., T_k)$ is said to be *geometrically realizable* if there exists a manifold M, if there exists a point P of M, and if there exists an isomorphism $\phi : V \to T_P M$ such that $\phi^* L_i(P) = T_i$ where $\{L_1, ..., L_k\}$ is a corresponding geometric family of tensor fields on M.

Thus, for example, if $k = 1$ and if $T_1 = \langle \cdot, \cdot \rangle$ is a non-degenerate inner product on V, then a geometric realization of $(V, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian manifold (M, g), a point P of M, and an isomorphism $\phi : V \to T_P M$ so that $\phi^* g_P = \langle \cdot, \cdot \rangle$; this is of course, a trivial problem.

Many people have worked in this area or in a closely related area including:

D. Alekseevsky, N. Blažić, M. Brozos-Vázquez, E. Calvino-Louzao, L. Cordero, A. Derdzinski, C. Dunn, I. Dotti, E. García-Río, P. Gilkey, T. Hervella, H. Kang, O. Kowalski, M. Fernandez, Y. Matushita, Y. Nikolayevsky, B. Opzoda, JH. Park, K. Sekigawa, I. Stavrov, S. Nikčević, U. Simon, E. Vázquez-Abel, R. Vázquez-Lorenzo, L. Vanhecke, V. Videv, D. Westerman, and G. Weingart; it is not possible to mention everyone working in this field!

Geometric realizations of:

1. Riemannian algebraic curvature tensors by pseudo-Riemannian manifolds.

2. Affine curvature tensors by affine manifolds.

3. Weyl curvature tensors by Weyl manifolds.

4. Kähler affine curvature tensors by affine Kähler manifolds.

5. Kähler Riemannian curvature tensors by Kähler manifolds.

6. Hermitian Riemannian curvature tensors by Hermitian manifolds.

7. Covariant derivative Kähler tensors by almost pseudo-Hermitian manifolds.

In each instance, curvature decompositions of the appropriate space of tensors under a suitable structure group play a crucial role.

And it is important to write down the appropriate symmetries of the curvature tensors involved; for example the Gray identity plays an important role in the study of Hermitian geometry.

Furthermore, some problems are not geometrically realizable; for example, a Ricci-antisymmetric projectively flat affine curvature tensor R is geometrically realizable by a Ricci-antisymmetric projectively flat affine connection if and only if $R = 0$.

We discuss not only the positive definite setting but also higher signature geometry and para-complex geometry.

Affine Structures

An *affine manifold* is a pair (M, ∇) where M is a smooth manifold and where ∇ is a torsion free connection on the tangent bundle TM. We refer to [Gilkey, Nikčević, and Simon (2009); Simon, Schwenk-Schellschmidt, and Viesel (1991)] for further information concerning affine geometry.

The associated *curvature operator* $\mathcal{R} \in \otimes^2 T^*M \otimes \mathrm{End}(TM)$ is defined by setting:

$$\mathcal{R}(x,y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]} \, .$$

This tensor satisfies the following identities:

$$\begin{aligned} &\mathcal{R}(x,y) = -\mathcal{R}(y,x), \\ &\mathcal{R}(x,y)z + \mathcal{R}(y,z)x + \mathcal{R}(z,x)y = 0 \, . \end{aligned} \tag{1}$$

A $(1,3)$ tensor $\mathcal{A} \in \otimes^2 V^* \otimes \mathrm{End}(V)$ satisfying the symmetries given in the Equation 1 is called an *affine algebraic curvature operator*; let $\mathfrak{A} = \mathfrak{A}(V)$ be the subspace of all such operators. We summarize below the fundamental decomposition of the space of affine curvature operators under the action of the general linear group [Strichartz (1988)]:

Theorem

If $m \geq 3$, then $\mathfrak{A} \approx \{\mathfrak{A} \cap \ker(\rho)\} \oplus \Lambda^2 \oplus S^2$ as a GL module where $\{\mathfrak{A} \cap \ker \rho, \Lambda^2, S^2\}$ are inequivalent irreducible GL modules.

Theorem

$\dim\{\mathfrak{A}\} = \frac{1}{3}m^2(m^2-1)$	$\dim\{S^2\} = \frac{1}{2}m(m+1)$
$\dim\{\Lambda^2\} = \frac{1}{2}m(m-1)$	$\dim \ker(\rho) \cap \mathfrak{A} = \frac{1}{3}m^2(m^2-4)$

An affine curvature operator $\mathcal{A} \in \mathfrak{A}$ is said to be *geometrically realizable* if there exists an affine manifold (M, ∇), if there exists a point P of M (which is called the point of realization), and if there exists an isomorphism $\phi : V \to T_P M$ so that $\phi^* \mathcal{R}_P = \mathcal{A}$.

The decomposition of $\mathfrak{A}$ as a GL module has 3 components so there are 8 natural geometric realization questions which are GL equivariant.

The following result [Gilkey and Nikčević (2008), Gilkey, Nikčević, and Westerman (2009)] which shows, in particular, that the symmetries of Equation 1 generate the universal symmetries of the curvature operator of a torsion free connection:

Theorem

1. *Any affine algebraic curvature operator is geometrically realizable by an affine manifold.*
2. *Any Ricci symmetric affine algebraic curvature operator is geometrically realizable by a Ricci symmetric affine manifold.*
3. *Any Ricci anti-symmetric affine algebraic curvature operator is geometrically realizable by a Ricci anti-symmetric affine manifold.*
4. *Any Ricci flat affine algebraic curvature operator is geometrically realizable by a Ricci flat affine manifold.*

Theorem

5. *Any projectively flat affine algebraic curvature operator is geometrically realizable by a projectively flat affine manifold.*
6. *Any projectively flat Ricci symmetric affine algebraic curvature operator is geometrically realizable by a projectively flat Ricci symmetric affine manifold.*
7. *A projectively flat Ricci anti-symmetric affine algebraic curvature operator which is not flat is not geometrically realizable by a projectively flat Ricci anti-symmetric affine manifold.*
8. *If $\mathcal{A}$ is flat, then $\mathcal{A}$ is geometrically realizable by a flat affine manifold.*

$\ker(\rho)$	S^2	Λ^2		$\ker(\rho)$	S^2	Λ^2	
$\star$	$\star$	$\star$	yes	0	$\star$	$\star$	yes
$\star$	$\star$	0	yes	0	$\star$	0	yes
$\star$	0	$\star$	yes	0	0	$\star$	no
$\star$	0	0	yes	0	0	0	yes

In fact, a bit more is true. Given $\mathcal{A}$, we will construct the germ of a torsion free connection ∇ at 0 in V so that the matrix of the Ricci tensor is constant relative to the coordinate frame, i.e. one has that $\rho(\mathcal{R})(\partial_{x_i}, \partial_{x_j}) = \rho(\mathcal{A})(e_i, e_j)$; this settles other associated realization questions.

Let $\nabla\mathcal{R}(x,y;z)w$ be the *covariant derivative of the curvature operator*:

$$\begin{aligned}\nabla\mathcal{R}(x,y;z)w := {} & \nabla_z\mathcal{R}(x,y)w - \mathcal{R}(\nabla_z x,y)w \\ & -\mathcal{R}(x,\nabla_z y)w - \mathcal{R}(x,y)\nabla_z w\,.\end{aligned}$$

This has the symmetries:

$$\begin{aligned}&R_{ijk}{}^{l}{}_{;n} = -R_{jik}{}^{l}{}_{;n},\\ &R_{ijk}{}^{l}{}_{;n} + R_{jki}{}^{l}{}_{;n} + R_{kij}{}^{l}{}_{;n} = 0,\\ &R_{ijk}{}^{l}{}_{;n} + R_{jnk}{}^{l}{}_{;i} + R_{nik}{}^{l}{}_{;j} = 0\,.\end{aligned} \tag{2}$$

Let $\mathfrak{A}^1 \subset \otimes^3 V^* \otimes \mathrm{End}(V)$ be the subspace of all tensors (4,1) satisfying these relations. We will establish the following result:

Theorem

Let $\mathcal{A} \in \mathfrak{A}$ and let $\mathcal{A}^1 \in \mathfrak{A}^1$. Define a torsion free connection ∇ on TV by setting

$$\begin{aligned}
\Gamma_{uv}{}^{l} \quad := \quad & \tfrac{1}{3}(A_{wuv}{}^{l} + A_{wvu}{}^{l})x^w + \\
& \tfrac{5}{24}(A^1_{wuv}{}^{l}{}_{;n} + A^1_{wvu}{}^{l}{}_{;n})x^w x^n + \\
& \tfrac{1}{24}(A^1_{wun}{}^{l}{}_{;v} + A^1_{wvn}{}^{l}{}_{;u})x^w x^n \,.
\end{aligned}$$

Then $\mathcal{R}_{ijk}{}^{l}(0) = A_{ijk}{}^{l}\partial_{x_l}$ and $\mathcal{R}_{ijk}{}^{l}{}_{;n}(0) = A^1_{ijk}{}^{l}{}_{;n}$.

Mixed Structures

We now study an affine structure and a pseudo-Riemannian metric where the given affine connection is not necessarily the Levi-Civita connection of the pseudo-Riemannian metric; thus the two structures are decoupled. Let $\mathcal{A} \in \mathfrak{A}$. We use the metric to lower the final index and define $A \in \otimes^4 V^*$ by setting:

$$A(x, y, z, w) := \langle \mathcal{A}(x, y)z, w \rangle \,.$$

The symmetries of Equation 1 then become:

$$\begin{aligned} &A(x, y, z, w) = -A(y, x, z, w), \\ &A(x, y, z, w) + A(y, z, x, w) + A(z, x, y, w) = 0\,. \end{aligned}$$

Again, a curvature decomposition plays a central role. Set

$$W_6^{\mathcal{O}} := \{A \in \mathfrak{A} \cap \ker(\rho) : A_{ijkl} = -A_{ijlk}\},$$

$$W_7^{\mathcal{O}} := \{A \in \mathfrak{A} \cap \ker(\rho) : A_{ijkl} = A_{ijlk}\},$$

$$W_8^{\mathcal{O}} := \{A \in \otimes^4 V^* \cap \ker(\rho) : A_{ijkl} = -A_{jikl} = -A_{klij}\}.$$

Note that $W_6^{\mathcal{O}}$ and $W_7^{\mathcal{O}}$ are submodules of $\mathfrak{A}$ whereas $W_8^{\mathcal{O}} \not\subset \mathfrak{A}$. We will establish the following result [Bokan (1990]:

Theorem

Let $m \geq 4$. The $\mathcal{O}$ module decomposition of $\mathfrak{A}$ into irreducible and inequivalent $\mathcal{O}$ modules takes the form:

$$\mathfrak{A} \approx \mathbb{R} \oplus 2 \cdot S_0^2 \oplus 2 \cdot \Lambda^2 \oplus W_6^{\mathcal{O}} \oplus W_7^{\mathcal{O}} \oplus W_8^{\mathcal{O}} .$$

If $m = 3$, we set $W_6^{\mathcal{O}} = W_8^{\mathcal{O}} = 0$. If $m = 2$, then $\mathfrak{A} = \mathbb{R} \oplus S_0^2 \oplus \Lambda^2$.

Theorem

$\dim\{\mathfrak{R}\} = \frac{1}{12} m^2(m^2 - 1)$	$\dim\{\mathfrak{A}\} = \frac{1}{3} m^2(m^2 - 1)$
$\dim\{\mathbb{R}\} = 1$	$\dim\{S_0^2\} = \frac{1}{2} m(m+1) - 1$
$\dim\{\Lambda^2\} = \frac{1}{2} m(m-1)$	$\dim\{W_6^{\mathcal{O}}\} = \frac{m(m+1)(m-3)(m+2)}{12}$
$\dim\{W_7^{\mathcal{O}}\} = \frac{(m-1)(m-2)(m+1)(m+4)}{8}$	$\dim\{W_8^{\mathcal{O}}\} = \frac{m(m-1)(m-3)(m+2)}{8}$

Let τ be the scalar curvature and let ρ_0 be the trace free Ricci tensor. One has several geometric realization questions which are natural with respect to the structure group $\mathcal{O}$ and which can all be solved in the real analytic category. As our considerations are local, we take $M = V$ and $P = 0$. We establish the following result [Gilkey, Nikčević, and Westerman (2009)]:

Theorem

Let g be the germ at $0 \in V$ of a real analytic pseudo-Riemannian metric. Let $\mathcal{A} \in \mathfrak{A}$. There exists a the germ of a torsion free real analytic connection ∇ at $0 \in V$ so that:

1. $\mathcal{R}_0 = \mathcal{A}$.
2. ∇ *has constant scalar curvature.*
3. *If $\mathcal{A}$ is Ricci symmetric, then ∇ is Ricci symmetric.*
4. *If $\mathcal{A}$ is Ricci anti-symmetric, then ∇ is Ricci anti-symmetric.*
5. *If $\mathcal{A}$ is Ricci traceless, then ∇ is Ricci traceless.*

We note there are corresponding results in the C^k category.

Notational Conventions

Definition

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

1. We say that $J_- \in \mathrm{GL}$ is a *complex structure* on V if $J_-^2 = -Id$; if in addition $J_-^* \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$, then J_- is said to be a *pseudo-Hermitian complex structure* and the triple $(V, \langle\cdot,\cdot\rangle, J_-)$ is said to be a *pseudo-Hermitian vector space*. Such structures exist if and only if $(V, \langle\cdot,\cdot\rangle)$ has signature (p, q) where both p and q are even. The associated *Kähler form* is given by setting $\Omega_-(x, y) := \langle x, J_- y\rangle$. We shall often let $\Omega = \Omega_-$ when the context is clear.

Definition

2. We say that $J_+ \in \mathrm{GL}$ is a *para-complex structure* if $J_+^2 = Id$ and if $\mathrm{Tr}(J_+) = 0$. This latter condition is automatic in the complex setting, but must be imposed in the para-complex setting. If $J_+^*\langle\cdot,\cdot\rangle = -\langle\cdot,\cdot\rangle$, then J_+ is said to be a *para-Hermitian complex structure* and the triple $(V, \langle\cdot,\cdot\rangle, J_+)$ is said to be a *para-Hermitian vector space*. Such structures exist only in the *neutral setting* $p = q$. The associated *para-Kähler form* is given by setting $\Omega_+(x, y) := \langle x, J_+ y\rangle$. Again, we shall often set $\Omega = \Omega_+$.

Representation Theory

We introduce the associated structure groups which will play an essential role throughout our work. Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian vector space. Let

$$\begin{aligned}
&\mathrm{GL}_\pm := \{T \in \mathrm{GL} : TJ_\pm = J_\pm T\}, \\
&\mathrm{GL}_\pm^\star := \{T \in \mathrm{GL} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}, \\
&\mathcal{U}_\pm := \{T \in \mathcal{O} : TJ_\pm = J_\pm T\}, \\
&\mathcal{U}_\pm^\star := \{T \in \mathcal{O} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}.
\end{aligned} \tag{3}$$

The group $\mathrm{GL}_\pm$ is the (para)-complex general group and the group $\mathcal{U}_\pm$ is the (para)-unitary group. The groups $\mathrm{GL}_\pm^\star$ and $\mathcal{U}_\pm^\star$ are $\mathbb{Z}_2$ extensions of $\mathrm{GL}_\pm$ and $\mathcal{U}_\pm$, respectively.

Theorem

Let $G \in \{\mathcal{O}, \mathcal{U}_-, \mathcal{U}_\pm^\star\}$. Then G acts naturally on the tensor algebra $\otimes^k V^*$ via pull-back.

1. *No non-trivial G-invariant subspace of $\otimes^k V^*$ is totally isotropic.*
2. *We may decompose any non-trivial G-submodule of $\otimes^k V^*$ as the orthogonal direct sum of irreducibles G-submodules $\xi = \sum_i n_i \xi_i$ where the multiplicities n_i are independent of the particular decomposition chosen.*
3. *If $\xi_1 = (V_1, \sigma_1)$ and $\xi_2 = (V_2, \sigma_2)$ are any two inequivalent irreducible G-submodules of $\otimes^k V^*$, then $V_1 \perp V_2$.*

Definition

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

1. Let $S_0^2 := \{\theta \in S^2 : \varepsilon^{ij}\theta_{ij} = 0\} = \{\theta \in S^2 : \theta \perp \langle\cdot,\cdot\rangle\}$; S_0^2 is a $\mathcal{O}$ module.
2. Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian vector space. Define $\mathcal{U}_\pm^\star$ modules:

$$
\begin{aligned}
S_+^{2,\mathcal{U}_\pm} &:= \{\theta \in S^2 : J_\pm^*\theta = +\theta\}, \qquad & \Lambda_+^{2,\mathcal{U}_\pm} &:= \{\theta \in \Lambda^2 : J_\pm^*\theta = +\theta\} \\
S_-^{2,\mathcal{U}_\pm} &:= \{\theta \in S^2 : J_\pm^*\theta = -\theta\}, \qquad & \Lambda_-^{2,\mathcal{U}_\pm} &:= \{\theta \in \Lambda^2 : J_\pm^*\theta = -\theta\}
\end{aligned}
$$

Definition

There is no linkage between the two sets of signs in the above equation. We now link the signs and define

$$\begin{aligned} S_{0,\mp}^{2,\mathcal{U}_\pm} &:= \{\theta \in S_{\mp}^{2,\mathcal{U}_\pm} : \theta \perp \langle \cdot, \cdot \rangle\}, \\ \Lambda_{0,\mp}^{2,\mathcal{U}_\pm} &:= \{\theta \in \Lambda_{\mp}^{2,\mathcal{U}_\pm} : \theta \perp \Omega_\pm\}\,. \end{aligned}$$

3. If $T \in \mathcal{U}_\pm^\star$ or if $T \in \mathrm{GL}_\pm^\star$, then $TJ_\pm = \chi(T) J_\pm T$ where χ defines a non-trivial representation of $\mathcal{U}_\pm^\star$ and of $\mathrm{GL}_\pm^\star$ into $\mathbb{Z}_2$.

Lemma

1. *S_0^2 and Λ^2 are irreducible inequivalent $\mathcal{O}$ modules.*
2. *$\{S_{0,+}^{2,\mathcal{U}_-},\ S_-^{2,\mathcal{U}_-},\ \Lambda_-^{2,\mathcal{U}_-}\}$ are irreducible inequivalent $\mathcal{U}_-$ modules.*
3. *$\Lambda_{0,+}^{2,\mathcal{U}_-}$ is isomorphic to $S_{0,+}^{2,\mathcal{U}_-}$ as a $\mathcal{U}_-$ module.*
4. *$\{S_{0,\mp}^{2,\mathcal{U}_\pm}, S_\pm^{2,\mathcal{U}_\pm}, \Lambda_{0,\mp}^{2,\mathcal{U}_\pm}, \Lambda_\pm^{2,\mathcal{U}_\pm}\}$ are irreducible inequivalent $\mathcal{U}_\pm^\star$ modules.*
5. *$\Lambda_{0,\mp}^{2,\mathcal{U}_\pm}$ is isomorphic to $S_{0,\mp}^{2,\mathcal{U}_\pm} \otimes \chi$ as a $\mathcal{U}_\pm^\star$ module.*

We are primarily concerned with local theory. Let P_i be points of metric spaces X_i. We say that f is the *germ* of a map from (X_1, P_1) to (X_2, P_2) if f is a continuous map from some neighborhood of P_1 in X_1 to X_2 with $f(P_1) = P_2$. We agree to identify two such maps if they agree on some (possibly) smaller neighborhood of P_1. In a similar fashion, we can talk about the germ of a pseudo-Riemannian manifold, the germ on a connection.

Affine Kähler Structures

The results in the complex setting they arise from work of [Brozos-Vázquez (2010),Brozos-Vázquez,Gilkey, and Nikčević(2010)] whereas in the para-complex setting, they are new. Let $J_\pm$ be a (para)-complex structure on V. Set:

$$\begin{aligned}
&\mathfrak{K}_\pm^{\mathfrak{A}} := \{\mathcal{A} \in \mathfrak{A} : \mathcal{A}(v_1, v_2)J_\pm = J_\pm\mathcal{A}(v_1, v_2) \ \forall\ v_1, v_2\},\\
&\mathfrak{K}_{\pm;+}^{\mathfrak{A}} := \{\mathcal{A} \in \mathfrak{K}_\pm^{\mathfrak{A}} : \mathcal{A}(J_\pm v_1, J_\pm v_2) = +\mathcal{A}(v_1, v_2) \ \forall\ v_1, v_2\},\\
&\mathfrak{K}_{\pm;-}^{\mathfrak{A}} := \{\mathcal{A} \in \mathfrak{K}_\pm^{\mathfrak{A}} : \mathcal{A}(J_\pm v_1, J_\pm v_2) = -\mathcal{A}(v_1, v_2) \ \forall\ v_1, v_2\}.
\end{aligned}$$

We may decompose

$$\mathfrak{K}_{\pm}^{\mathfrak{A}} = \mathfrak{K}_{\pm;+}^{\mathfrak{A}} \oplus \mathfrak{K}_{\pm;-}^{\mathfrak{A}} \,.$$

Suppose given an auxiliary pseudo-Hermitian inner product $\langle\cdot,\cdot\rangle$, not necessarily positive definite, which we use to lower indices and regard $\mathfrak{K}_{\pm}^{\mathfrak{A}}$, $\mathfrak{K}_{\pm;+}^{\mathfrak{A}}$, and $\mathfrak{K}_{\pm;-}^{\mathfrak{A}}$ as subspaces of $\otimes^4 V^*$. We may now express

$$\begin{aligned}
\mathfrak{K}_{\pm}^{\mathfrak{A}} &:= \{A \in \mathfrak{A} : A(v_1,v_2,v_3,v_4) = \mp A(v_1,v_2,J_\pm v_1,J_\pm v_2)\}, \quad (4)\\
\mathfrak{K}_{\pm;+}^{\mathfrak{A}} &:= \{A \in \mathfrak{K}_{\pm}^{\mathfrak{A}} : A(J_\pm v_1,J_\pm v_2,v_3,v_4) = A(v_1,v_2,v_3,v_4)\},\\
\mathfrak{K}_{\pm;-}^{\mathfrak{A}} &:= \{A \in \mathfrak{K}_{\pm}^{\mathfrak{A}} : A(J_\pm v_1,J_\pm v_2,v_3,v_4) = -A(v_1,v_2,v_3,v_4)\}\,.
\end{aligned}$$

The following decomposition [Brozos-Vázquez (2010)] generalizes Theorem to this setting.

Theorem

If $m \geq 4$, then we have the following isomorphisms decomposing $\mathfrak{K}_{\mathcal{U}_{\pm;\delta}\mathfrak{A}}$ as the direct sum of irreducible and inequivalent $\mathrm{GL}_{\pm}$ modules for $\delta = +$ and for $\delta = -$:

$$\mathfrak{K}^{\mathfrak{A}}_{\pm;\delta} \approx \{\mathfrak{K}^{\mathfrak{A}}_{\pm;\delta} \cap \ker \rho\} \oplus \Lambda^{2,\mathcal{U}_{\pm}}_{\delta} \oplus S^{2,\mathcal{U}_{\pm}}_{\delta} \,. \tag{5}$$

The decomposition of these spaces as $\mathcal{U}_-$ modules in the Hermitian setting is given by [Matzeu and Nikčević (1991), Nikčević (1989)] There are 4 submodules of $\mathfrak{K}_\pm^{\mathfrak{A}}$ which do not correspond to generalized Ricci tensors and which we must consider.

Definition

Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian vector space. Let

1. $W_{\pm,9}^{\mathfrak{A}} := \{A \in \mathfrak{K}_{\pm;\mp}^{\mathfrak{A}} : A(x,y,z,w) = -A(x,y,w,z)\} \cap \ker(\rho)$.
2. $W_{\pm,10}^{\mathfrak{A}} := \{A \in \mathfrak{K}_{\pm;\mp}^{\mathfrak{A}} : A(x,y,z,w) = A(x,y,w,z)\} \cap \ker(\rho)$.
3. $W_{\pm,11}^{\mathfrak{A}} := \mathfrak{K}_{\pm,\mp}^{\mathfrak{A}} \cap (W_{\pm,9}^{\mathfrak{A}})^\perp \cap (W_{\pm,10}^{\mathfrak{A}})^\perp \cap \ker(\rho_{13}) \cap \ker(\rho)$.
4. $W_{\pm,12}^{\mathfrak{A}} := \mathfrak{K}_{\pm;\mp}^{\mathfrak{A}} \cap \ker(\rho)$.
5. $\tau_\pm^{\mathfrak{A}} := \varepsilon^{il}\varepsilon^{jk} A(e_i, J_\pm e_j, e_k, e_l)$.

We may also express:

$$W^{\mathfrak{A}}_{\pm,9} = \mathfrak{K}^{\mathfrak{A}}_{\pm;\mp} \cap W^{\mathcal{O}}_{6}, \quad W^{\mathfrak{A}}_{\pm,10} = \mathfrak{K}^{\mathfrak{A}}_{\pm;\mp} \cap W^{\mathcal{O}}_{7}.$$

We will extend the curvature decomposition from positive definite signatures to more general signatures and also to the para-Hermitian setting to show:

Theorem

1. *We have the following isomorphisms decomposing* $\mathfrak{K}_-^{\mathfrak{A}}$ *as the direct sum of irreducible and inequivalent* $\mathcal{U}_-$ *modules:*
   1.1 *If* $\dim(V) = 4$,
   $\mathfrak{K}_-^{\mathfrak{A}} = 2 \cdot \mathbb{R} \oplus 4 \cdot S_{0,+}^{2,\mathcal{U}_-} \oplus \Lambda_-^{2,\mathcal{U}_-} \oplus S_-^{2,\mathcal{U}_-} \oplus 2 \cdot W_{-,9}^{\mathfrak{A}}$.
   1.2 *If* $\dim(V) \geq 6$,
   $\mathfrak{K}_-^{\mathfrak{A}} = 2 \cdot \mathbb{R} \oplus 4 \cdot S_{0,+}^{2,\mathcal{U}_-} \oplus \Lambda_-^{2,\mathcal{U}_-} \oplus S_-^{2,\mathcal{U}_-} \oplus 2 \cdot W_{-,9}^{\mathfrak{A}}$
   $\oplus W_{-,11}^{\mathfrak{A}} \oplus W_{-,12}^{\mathfrak{A}}$.
2. *We have the following isomorphisms decomposing* $\mathfrak{K}_\pm^{\mathfrak{A}}$ *as the direct sum of irreducible and inequivalent* $\mathcal{U}_\pm^\star$ *modules:*
   2.1 *If* $m = 4$,
   $\mathfrak{K}_\pm^{\mathfrak{A}} \approx \mathbb{R}\tau \oplus \mathbb{R}\tau_\pm^{\mathfrak{A}} \oplus 2 \cdot S_{0,\mp}^{2,\mathcal{U}_\pm} \oplus 2 \cdot \Lambda_{0,\mp}^{2,\mathcal{U}_\pm} \oplus \Lambda_\pm^{2,\mathcal{U}_\pm} \oplus S_\pm^{2,\mathcal{U}_\pm}$
   $\oplus W_{\pm,9}^{\mathfrak{A}} \oplus W_{\pm,10}^{\mathfrak{A}}$.
   2.2 *If* $m \geq 6$,
   $\mathfrak{K}_\pm^{\mathfrak{A}} \approx \mathbb{R}\tau \oplus \mathbb{R}\tau_\pm^{\mathfrak{A}} \oplus 2 \cdot S_{0,\mp}^{2,\mathcal{U}_\pm} \oplus 2 \cdot \Lambda_{0,\mp}^{2,\mathcal{U}_\pm} \oplus \Lambda_\pm^{2,\mathcal{U}_\pm} \oplus S_-^{2,\mathcal{U}_\pm}$
   $\oplus W_{\pm,9}^{\mathfrak{A}} \oplus W_{\pm,10}^{\mathfrak{A}} \oplus W_{\pm,11}^{\mathfrak{A}} \oplus W_{\pm,12}^{\mathfrak{A}}$.

We say that $(V, J_\pm, \mathcal{A})$ is a *(para)-Kähler affine curvature model* if $J_\pm$ is a (para)-complex structure on V and if $\mathcal{A} \in \mathfrak{K}_\pm^{\mathfrak{A}}$. Similarly $(M, J_\pm, \nabla)$ is said to be a *(para)-Kähler affine manifold* if $J_\pm$ is a (para)-complex structure on M, if ∇ is a torsion free connection on M, and if $\nabla(J_\pm) = 0$. We say that a (para)-Kähler curvature model $(V, J_\pm, A)$ is geometrically realizable if there exists a (para)-Kähler manifold $(M, J_\pm, \nabla)$, a point P in M, and an isomorphism $\Xi : V \to T_P M$ so $\Xi^* \mathcal{R} = \mathcal{A}$ and $\Xi^* J_{\pm,P} = J_\pm$. One then has:

Theorem

Every (para)-Kähler affine curvature model is geometrically realizable by a (para)-Kähler affine manifold. If $A \in \mathfrak{K}_{\pm;\pm}^{\mathfrak{A}}$, the para/pseudo-Kähler manifold M can be chosen so that the curvature belongs to $\mathfrak{K}_{\pm;\pm}^{\mathfrak{A}}$ at every point.

The dimension of these modules is computed [Matzeu and Nikčević (1991), Nikčević (1989)] in the positive definite setting; the dimensions are the same in the para/pseudo-Hermitian settings.

Theorem

Let $m \geq 6$. Then:

$\dim\{\mathfrak{K}^{\mathfrak{A}}_{\pm}\} = \frac{2}{3}\bar{m}^2(\bar{m}+1)(5\bar{m}-2)$	$\dim\{\mathbb{R}\} = 1$
$\dim\{W^{\mathfrak{A}}_{\pm,9}\} = \frac{1}{4}\bar{m}^2(\bar{m}-1)(\bar{m}+3)$	$\dim\{S^{2,\mathcal{U}_{\pm}}_{0,\mp}\} = \bar{m}^2 - 1$
$\dim\{W^{\mathfrak{A}}_{\pm,10}\} = \frac{1}{4}\bar{m}^2(\bar{m}-1)(\bar{m}+3)$	$\dim(\Lambda^{2,\mathcal{U}_{\pm}}_{0,\mp}\} = \bar{m}^2 - 1$
$\dim\{W^{\mathfrak{A}}_{\pm,11}\} = \frac{1}{2}(\bar{m}-1)(\bar{m}+1)(\bar{m}-2)(\bar{m}+2)$	$\dim\{S^{2,\mathcal{U}_{\pm}}_{\pm}\} = \bar{m}^2 + \bar{m}$
$\dim\{W^{\mathfrak{A}}_{\pm,12}\} = \frac{2}{3}\bar{m}^2(\bar{m}-2)(\bar{m}+2)$	$\dim\{\Lambda^{2,\mathcal{U}_{\pm}}_{\pm}\} = \bar{m}^2 - \bar{m}$

Riemannian Structures

One says that $A \in \otimes^4 V^*$ is a *Riemannian algebraic curvature tensor* on V if A satisfies the symmetries of the Riemann curvature tensor:

$$\begin{aligned} A(x,y,z,w) &= -A(y,x,z,w) = A(z,w,x,y), \\ A(x,y,z,w) &+ A(y,z,x,w) + A(z,x,y,w) = 0\,. \end{aligned}$$

Let $\mathfrak{R} = \mathfrak{R}(V)$ be the space of all such 4 tensors. We have that $\mathfrak{R}$ is invariant under the action of $\mathcal{O}$. Thus $\langle \cdot, \cdot \rangle$ is a non-degenerate inner product on $\mathfrak{R}$. We say that $(V, \langle \cdot, \cdot \rangle, A)$ is a *curvature model* if $A \in \mathfrak{R}$.

Definition

1. Let $\phi \in S^2$ be a symmetric bilinear form. Set

$$A_\phi(x, y, z, w) := \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)\,.$$

These tensors arise in the study of hypersurface theory; if ϕ is the second fundamental form of a hypersurface in flat space, then the curvature tensor of the hypersurface is given by A_ϕ.

2. Let $\psi \in \Lambda^2$ be an anti-symmetric bilinear form. Set

$$A_\psi(x, y, z, w) := \psi(x, w)\psi(y, z) - \psi(x, z)\psi(y, w) - 2\psi(x, y)\psi(z, w)\,.$$

The study of the tensors A_ψ arose in the original instance from the Osserman conjecture and related matters [Garciá-Río, Kupeli,and Vázquez-Lorenzo (2002), Gilkey (2001)]

A result of [Fiedler (2003)] giving generators for $\mathfrak{R}$ and determine $\dim\{\mathfrak{R}\}$:

Theorem

1. $\mathfrak{R} = \mathrm{Span}_{\phi\in S^2}\{A_\phi\} = \mathrm{Span}_{\psi\in\Lambda^2}\{A_\psi\}$.
2. $\dim\{\mathfrak{R}\} = \frac{1}{12}m^2(m^2-1)$.

We note that $W_6^{\mathcal{O}} = \ker(\rho)\cap\mathfrak{R}$. The decomposition [Singer and Thorpe (1969]] of $\mathfrak{R}$ as an $\mathcal{O}$ module:

Theorem

Let $\dim(V)\geq 4$. The decomposition $\mathfrak{R} = W_6^{\mathcal{O}}\oplus S_0^2\oplus\mathbb{R}$ is an $\mathcal{O}$ module decomposition of $\mathfrak{R}$ into irreducible and inequivalent $\mathcal{O}$ modules.

If $m = 2$, then $\mathfrak{R} = \mathbb{R}$ and if $m = 3$, then $\mathfrak{R} = S_0^2 \oplus \mathbb{R}$.

Suppose given $A \in \mathfrak{R}$. We say that the curvature model $(V, \langle\cdot,\cdot\rangle, A)$ is *geometrically realizable* if there exists a pseudo-Riemannian manifold (M, g), if there exists a point P of M, and if there exists an isomorphism $\phi : V \to T_P M$ so that $\phi^* g_P = \langle\cdot,\cdot\rangle$ and $\phi^* R_P = A$.

The Weyl conformal curvature tensor W is the projection of A on $\ker(\rho)$; we say a curvature model or a pseudo-Riemannian manifold is *conformally flat* if and only if $W = 0$. We establish results [Brozos-Váquez, Gilkey, Kang,and Nikčević (2009)] dealing with geometric realizations by pseudo-Riemannian manifolds with constant scalar curvature:

Theorem

1. *Any curvature model is geometrically realizable by a pseudo-Riemannian manifold of constant scalar curvature.*
2. *Any conformally flat curvature model is geometrically realizable by a conformally flat pseudo-Riemannian manifold of constant scalar curvature.*

Weyl Geometry

Again, we consider a mixed structure. Consider a triple $\mathcal{W} := (M, g, \nabla)$ where g is a pseudo-Riemannian metric on a smooth m dimensional manifold M and where ∇ is a torsion free connection on TM. We say that $\mathcal{W}$ is a *Weyl manifold* if the following identity is satisfied:

$$\nabla g = -2\phi \otimes g \quad \text{for some} \quad \phi \in C^\infty(T^*M)\,. \tag{6}$$

This notion is conformally invariant. If $\mathcal{W} = (M, g, \nabla)$ is a Weyl manifold and if $f \in C^\infty(M)$, then $\tilde{\mathcal{W}} := (M, e^{2f}g, \nabla)$ is again a Weyl manifold where $\tilde{\phi} := \phi - df$. Let ∇^g be the Levi-Civita connection determined by the metric g. There exists a conformally equivalent metric $\tilde{g}$ locally so that $\nabla = \nabla^{\tilde{g}}$ if and only if $d\phi = 0$; if $d\phi = 0$, such a conformally equivalent metric exists globally if and only if $[\phi] = 0$ in de Rham cohomology.

Weyl geometry fits in between affine and Riemannian geometry. Let (M, g) be a pseudo-Riemannian manifold. Since ∇^g is torsion free and $\nabla g = 0$, the triple (M, g, ∇^g) is a Weyl manifold. There are, however, examples with $d\phi \neq 0$ so Weyl geometry is more general than Riemannian geometry or even conformal Riemannian geometry. Every Weyl manifold gives rise to an underlying affine and an underlying Riemannian manifold.
If (M, g, ∇) is a Weyl manifold, there is an extra curvature symmetry we shall establish in Theorem:

Theorem
If (N, g, ∇) is a Weyl manifold, then

$$R(x,y,z,w) + R(x,y,w,z) = \tfrac{2}{m}\{\rho(R)(y,x) - \rho(R)(x,y)\}g(z,w)\,.$$

The decomposition of $\mathfrak{M}$ as an $\mathcal{O}$ module is given by [Higa (1993, 1994)]:

Theorem

If $m \geq 4$, then $\mathfrak{W} \approx \mathfrak{R} \oplus \Lambda^2$ as $\mathcal{O}$ modules.

We say that a tensor $A \in \mathfrak{W}$ is *geometrically realizable* by a Weyl manifold $\mathcal{W} = (M, g, \nabla)$ if there exists a point $P \in M$ and an isomorphism $\phi : V \to T_P M$ so that $\phi^* g_P = \langle \cdot, \cdot \rangle$ and $\phi^* R_P = A$.

Theorem

Every $A \in \mathfrak{W}$ is geometrically realizable by a Weyl manifold with constant scalar curvature.

The following is an interesting illustration of the extent to which the geometric category is determined by the algebraic setting. The following useful result characterizes *trivial* Weyl manifolds:

Theorem

Let $\mathcal{W} = (M, g, \nabla)$ be a Weyl manifold with $H^1(M; \mathbb{R}) = 0$. The following assertions are equivalent and if any is satisfied, we say that $\mathcal{W}$ is trivial.

1. $d\phi = 0$.
2. $\nabla = \nabla^{\tilde{g}}$ *for some $\tilde{g}$ in the conformal class defined by g.*
3. $\nabla = \nabla^{\tilde{g}}$ *for some pseudo-Riemannian metric $\tilde{g}$.*
4. $R_P(\nabla) \in \mathfrak{R}$ *for every $P \in M$.*
5. ∇ *is Ricci symmetric.*

Remark 1.4. If (N, g, ∇) geometrically realizes A at a point $P \in N$, by considering a suitable conformal deformation $(N, e^{2f} g, \nabla)$, we can use the Cauchy-Kovalevskaya Theorem to construct a Weyl manifold where $f = O(|x - P|^3)$ which has constant scalar curvature and which realizes A at P. The argument is essentially the same as that we used to establish a similar fact in the pseudo-Riemannian setting.

M. Brozos-Vázquez, P. Gilkey, H. Kang, S. Nikčević, and G. Weingart, Geometric realizations of curvature models by manifolds with constant scalar curvature, *Differential Geometry and its Applications* **27** (2009), 696–701.

Almost Pseudo-Hermitian Geometry

We now discuss the decomposition of $\mathfrak{R}$ as a $\mathcal{U}_-$ and as a $\mathcal{U}_\pm^\star$ module. This result was given by [Tricerri and Vanhecke (1981)] in the positive definite setting; we will extend the decomposition to the remaining geometries.

$$\begin{aligned}\rho_{J_\pm}(x,y) &:= \mp\varepsilon^{il}A(e_i,x,J_\pm y,J_\pm e_l),\\ \tau_{J_\pm} &:= \mp\varepsilon^{il}\varepsilon^{jk}A(e_i,e_j,J_\pm e_k,J_\pm e_l)\,.\end{aligned} \tag{7}$$

We consider the following modules:

Definition

Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian vector space.

1. $\mathfrak{R}_+^{\mathcal{U}_\pm} := \{A \in \mathfrak{R} : A(J_\pm x, J_\pm y, J_\pm z, J_\pm w) = A(x, y, z, w)\}$.
2. $\mathfrak{R}_-^{\mathcal{U}_\pm} := \{A \in \mathfrak{R} : A(J_\pm x, J_\pm y, J_\pm z, J_\pm w) = -A(x, y, z, w)\}$.
3. $\mathfrak{G}_\pm := \{A \in \mathfrak{R} : 0 = A(x, y, z, w) + A(J_\pm x, J_\pm y, J_\pm z, J_\pm w)$

   $\pm A(J_\pm x, J_\pm y, z, w) \pm A(x, y, J_\pm z, J_\pm w) \pm A(J_\pm x, y, J_\pm z, w)$

   $\pm A(x, J_\pm y, z, J_\pm w) \pm A(J_\pm x, y, z, J_\pm w) \pm A(x, J_\pm y, J_\pm z, w)\}$.
4. $\mathfrak{K}_\pm^{\mathfrak{R}} := \{A \in \mathfrak{R} : A(x, y, z, w) = \mp A(J_\pm x, J_\pm y, z, w)\ \forall\ x, y, z, w\}$.

Definition

5. $W^{\mathfrak{R}}_{\pm,3} := \mathfrak{K}^{\mathfrak{R}}_{\pm} \cap \ker\{\rho\}$.
6. $W^{\mathfrak{R}}_{\pm,6} := \{\mathfrak{K}^{\mathfrak{R}}_{\pm}\}^{\perp} \cap \mathfrak{G}_{\pm} \cap \ker\{\rho \oplus \rho_{J_\pm}\}$.
7. $W^{\mathfrak{R}_\pm}_{7} := \{A \in \mathfrak{R} : A(J_\pm x, y, z, w) = A(x, y, J_\pm z, w)\ \forall\ x, y, z, w\}$.
8. $W^{\mathfrak{R}}_{\pm,10} := \mathfrak{R}^{\mathcal{U}_\pm}_{-} \cap \ker\{\rho \oplus \rho_{J_\pm}\}$.

Theorem

Let $\dim(V) \geq 8$. *Let* $(V, \langle\cdot,\cdot\rangle, J_\pm)$ *be a para/pseudo-Hermitian vector space. We have an orthogonal direct sum decomposition* $\mathfrak{R} = W^{\mathfrak{R}}_{\pm,1} \oplus \cdots \oplus W^{\mathfrak{R}}_{\pm,10}$ *into irreducible* $\mathcal{U}_-$ *and* $\mathcal{U}^\star_\pm$ *modules where*

$$W^{\mathfrak{R}}_{\pm,1} \approx W^{\mathfrak{R}}_{\pm,4} \approx \mathbb{R},\ W^{\mathfrak{R}}_{\pm,2} \approx W^{\mathfrak{R}}_{\pm,5} \approx S^{2,\mathcal{U}_\pm}_{0,+},\ W^{\mathfrak{R}}_{\pm,8} \approx S^{2,\mathcal{U}_\pm}_{\pm},\ W^{\mathfrak{R}}_{\pm,9} \approx \Lambda^{2,\mathcal{U}_\pm}_{\pm}$$

Except for the isomorphisms $W^{\mathfrak{R}}_{\pm,1} \approx W^{\mathfrak{R}}_{\pm,4}$ *and* $W^{\mathfrak{R}}_{\pm,2} \approx W^{\mathfrak{R}}_{\pm,5}$, *these are inequivalent* $\mathcal{U}_-$ *and* $\mathcal{U}^\star_\pm$ *modules.*

One says that $(V, \langle\cdot,\cdot\rangle, J_\pm, A)$ is an *almost para/pseudo-Hermitian curvature model* if $A \in \mathfrak{R}$, and if $(V, \langle\cdot,\cdot\rangle, J_\pm)$ is a para/pseudo-Hermitian vector space. The notion of geometric realizability in these contexts is defined similarly. We focus our attention on the scalar curvature.

Theorem

Let $m \geq 4$. Any almost para/pseudo-Hermitian curvature model is geometrically realizable by an almost para/pseudo-Hermitian manifold with τ and $\tau_{J_\pm}$ constant.

The Gray Identity

The curvature tensor of a para/pseudo-Hermitian manifold has an additional symmetry. It is quite striking that a geometric integrability condition imposes an additional algebraic symmetry on the curvature tensor.

We first extend a result in the positive definite case to more general signatures and to the para-Hermitian setting:

Theorem

If the model $\mathfrak{C} := (V, \langle\cdot,\cdot\rangle, J_\pm, A)$ is geometrically realizable by a para/pseudo-Hermitian manifold then $A \in \mathfrak{G}_\pm$, i. e.

$$\begin{aligned} 0 \;=\;& A(x,y,z,w) + A(J_\pm x, J_\pm y, J_\pm z, J_\pm w) \\ & \pm A(J_\pm x, J_\pm y, z, w) \pm A(x, y, J_\pm z, J_\pm w) \pm A(J_\pm x, y, J_\pm z, w) \\ & \pm A(x, J_\pm y, z, J_\pm w) \pm A(J_\pm x, y, z, J_\pm w) \pm A(x, J_\pm y, J_\pm z, w)\}\,. \end{aligned} \tag{8}$$

We say that a curvature model $(V, \langle\cdot,\cdot\rangle, J_\pm, A)$ is a *para/pseudo-Hermitian curvature model* if $A \in \mathfrak{G}_\pm$.

Theorem

Any para/pseudo-Hermitian curvature model is goemetrically realizable by a para/pseudo-Hermitiann manifold with τ and $\tau_{J_\pm}$ constant.

The relations of Equation (8) are called the *(para)-Gray identity*. The universal symmetries of the curvature tensor of a para/pseudo-Hermitian manifold are generated by the (para)-Gray identity and the usual curvature symmetries . This result emphasizes the difference between almost para/pseudo-Hermitian and para/pseudo-Hermitian manifolds.

The para/pseudo-Hermitian geometric realization in Theorem can be chosen so that $d\Omega_\pm(P) = 0$. Thus imposing the (para)-Kähler identity $d\Omega_\pm(P) = 0$ at a single point imposes no additional curvature restrictions. If $d\Omega_- = 0$ globally, then the manifold is said to be *almost Kähler*. This is a very rigid structure and there are additional curvature restrictions, also emphasizes the difference between $d\Omega_\pm$ vanishing at a single point and $d\Omega_\pm$ vanishing globally.

Riemannian Kähler Geometry

We will report on work of [Brozos-Vázquez, Gilkey, and Merino (2009)]. We begin with a well known result.

Theorem

Let $(M, g, J_\pm)$ be an almost para/pseudo-Hermitian manifold.

1. *The following assertions are equivalent and if either is satisfied, then $(M, g, J_\pm)$ is said to be a* (para-)Kähler manifold.
   1.1 $\nabla(J_\pm) = 0$.
   1.2 $J_\pm$ *is integrable and* $d\Omega = 0$.
2. *If* $\nabla\Omega = 0$, *then*
   2.1 $d\Omega = 0$ *and* $\delta\Omega = 0$.
   2.2 $J_\pm \mathcal{R}(x, y) = \mathcal{R}(x, y) J_\pm \;\forall\; x, y$.
   2.3 $R(J_\pm x, J_\pm y, z, w) = \mp R(x, y, z, w) \;\forall\; x, y, z, w$.

We say that $(V, \langle\cdot,\cdot\rangle, J_\pm, A)$ is a *(para)-Kähler model* if $(V, \langle\cdot,\cdot\rangle, J_\pm)$ is a para/pseudo-Hermitian vector space and if $A \in \mathfrak{K}_\pm^{\mathfrak{R}}$. The (para)-Gray identity is then necessarily satisfied. We have the following associated geometric realization theorem; note that $\tau = \tau_{J_\pm}$ in the Kähler setting.

Theorem

Any (para)-Kähler curvature model is geometrically realizable by a (para)-Kähler manifold of constant scalar curvature.

Theorems provide necessary and sufficient linear identities for a curvature model to be geometrically realizable by a Kähler manifold and for a curvature model to be geometrically realizable by a para-Kähler manifold. There are examples where one has relations rather than identities. For example, an almost Kähler manifold satisfies $\tau_{J_-} - \tau = \frac{1}{2}|\nabla J_-|^2$ and thus the curvature lies in the half-space defined by the relation $\tau_{J_-} \geq \tau$ if the metric in question is positive definite.

Curvature Kähler Weyl geometry

The results described here arise from work of [Gilkey, Nikčević, and Simon (2011)]. We suppose the dimension $m \geq 6$ as the 4 dimensional setting is very different. We say that the $(N, g, \nabla, J_{\pm})$ is a *(para)-complex Weyl manifold* if (N, g, ∇) is a Weyl manifold and if $J_{\pm}$ is an almost para/pseudo Hermitian almost complex structure on (N, g). If $\nabla(J_{\pm}) = 0$, the structure is said to be a *(para)-Kähler Weyl manifold*. Necessarily $J_{\pm}$ is integrable . Pedersen, Poon, and Swann (1993) used work of Vaisman (1982, 1983) to establish the following result in the Riemannian setting; the extension to the higher signature setting and to the para-Kähler setting is immediate.

Theorem

Let $H^1(M;\mathbb{R}) = 0$ and let $m \geq 6$. Any (para)-Kähler Weyl structure on M is trivial.

If $(N, g, \nabla, J_\pm)$ is a (para)-Kähler Weyl manifold, then one has an additional curvature symmetry

$$\begin{array}{l} \mathcal{R}(x,y)J_\pm = J_\pm\mathcal{R}(x,y) \quad \forall \quad x,y, \\ R(x,y,z,w) = \mp R(x,y,J_\pm z,J_\pm w) \quad \forall \quad x,y,z,w\,. \end{array} \tag{9}$$

We say $(N, g, \nabla, J_\pm)$ is a *(para)-Kähler curvature Weyl manifold* if (N, g, ∇) is a Weyl manifold, if $(N, g, J_\pm)$ is an almost para/pseudo-Hermitian manifold, and Equation 8 is satisfied; we will show that there exist (para)-Kähler curvature Weyl manifolds which are not (para)-Kähler Weyl manifolds.

The following result gives a curvature condition in the complex and para-complex settings which ensures that the Weyl structure is trivial.

Theorem

Let $H^1(M;\mathbb{R}) = 0$ and let $m \geq 6$.

1. *Any curvature Kähler Weyl structure on M is trivial.*
2. *Any curvature para-Kähler Weyl structure on M is trivial.*

Theorem

Let $n \geq 6$, let $(V, \langle \cdot, \cdot \rangle, J_\pm)$ be a para/pseudo-Hermitian vector space, and let $A \in \mathfrak{W}$. If A satisfies the (para)-Kähler identity , then $A \in \mathfrak{A}$.

We remark that the previous result fails if $n = 4$ [Calderbank and Pedersen (2000)].
The proof of Theorem relies on curvature decompositions.
$\mathfrak{W} \approx \mathfrak{R} \oplus \Lambda^2$ as an $\mathcal{O}$ module. There is an orthogonal direct sum decomposition into inequivalent irreducibles

$$\Lambda^2 \approx \left\{ \begin{array}{ll} \mathbb{R} \cdot \Omega_- \oplus \Lambda^2_{0,+} \oplus \Lambda^2_- \quad \text{as} \quad \mathcal{U}_- \quad \text{and} \quad \mathcal{U}^\star_- \quad \text{modules} \\ \mathbb{R} \cdot \Omega_+ \oplus \Lambda^{2,\mathcal{U}_+}_{0,-} \oplus \Lambda^{2,\mathcal{U}_+}_+ \quad \text{as an} \quad \mathcal{U}^\star_+ \quad \text{module} \end{array} \right\} .$$

The following decompositions are then an immediate consequence of previous results.

Theorem

Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be a para/pseudo-Hermitian vector space of dimension $n \geq 8$. We have the following isomorphism decomposing $\mathfrak{W}$ as the direct sum of irreducible $\mathcal{U}_-$ and $\mathcal{U}_\pm^\star$ modules:

$$\mathfrak{W} = W_{\pm,1}^{\mathfrak{R}} \oplus \cdots \oplus W_{\pm,10}^{\mathfrak{R}} \oplus W_{\pm,11}^{\mathfrak{W}} \oplus W_{\pm,12}^{\mathfrak{W}} \oplus W_{\pm,13}^{\mathfrak{W}}, \quad \textit{where}$$
$$W_{\pm,11}^{\mathfrak{W}} \approx \Omega_\pm \cdot \mathbb{R}, \quad W_{\pm,12}^{\mathfrak{W}} \approx \Lambda_{0,\mp}^{2,\mathcal{U}_\pm}, \quad W_{\pm,13}^{\mathfrak{W}} \approx \Lambda_{\pm}^{2,\mathcal{U}_\pm}.$$

Except for the isomorphisms $W_{\pm,1}^{\mathfrak{R}} \approx W_{\pm,4}^{\mathfrak{R}}$, $W_{\pm,2}^{\mathfrak{R}} \approx W_{\pm,5}^{\mathfrak{R}}$, $W_{\pm,9}^{\mathfrak{R}} \approx W_{\pm,13}^{\mathfrak{W}}$, these are inequivalent $\mathcal{U}_-^\star$ modules. As $\mathcal{U}_-$ modules, we also have

$$W_{-,1}^{\mathfrak{R}} \approx W_{-,4}^{\mathfrak{R}} \approx W_{-,11}^{\mathfrak{W}}, \quad W_{-,2}^{\mathfrak{R}} \approx W_{-,5}^{\mathfrak{R}} \approx W_{-,12}^{\mathfrak{W}}.$$

If $m = 4$, we set $W_{\pm,5}^{\mathfrak{R}} = W_{6,\pm}^{\mathfrak{R}} = W_{\pm,10}^{\mathfrak{R}} = \{0\}$ to obtain the corresponding decompositions. If $m = 6$, we set $W_{\pm,6}^{\mathfrak{R}} = \{0\}$ to derive the corresponding decompositions.

The Covariant Derivative of the Kähler Form

Let $\nabla\Omega_\pm$ be the covariant derivative of the Kähler form of $(M, g, J_\pm)$. The following symmetries are satisfied:

$$\nabla\Omega_\pm(x, y; z) = -\nabla\Omega_\pm(y, x; z) = \pm\nabla\Omega_\pm(J_\pm x, J_\pm y; z)\,.$$

We therefore define: $\mathfrak{H}_\pm := \Lambda_\pm^{2,\mathcal{U}_-} \otimes V^*$. We will establish the following geometric realization result:

Theorem

Let $H_\pm \in \mathfrak{H}_\pm$. There exists $(M, g, J_\pm)$, a point P of M, and an isomorphism $\phi : V \to T_P M$ so that $\phi^* g_P = \langle\cdot,\cdot\rangle$, so that $\phi^* J_\pm = J_\pm^0$, and so that $\phi^*\nabla\Omega_\pm(P) = H_\pm$.

We will prove this result by giving a decomposition of $\mathfrak{H}_{\pm}$ as a $\mathcal{U}_{\pm}^{\star}$ module. The corresponding decomposition of $\mathfrak{H}_{\pm}$ as a $\mathcal{U}_{\pm}$ module is:

Definition

1. If $H \in \otimes^3 V^*$, define $\tau_1(H) \in V^*$ by contracting the 2^{nd} and 3^{rd} indices:

   $(\tau_1 H)(x) := \varepsilon^{ij} H(x, e_i, e_j)$.

2. If $\kappa \in \mathrm{GL}$ and if $\phi \in V^*$, define $\sigma_\kappa(\phi) \in \otimes^3 V^*$ by setting:

   $\sigma_\kappa(\phi)(x, y; z) :=$
   $\phi(\kappa x)\langle y, z\rangle - \phi(\kappa y)\langle x, z\rangle + \phi(x)\langle \kappa y, z\rangle - \phi(y)\langle \kappa x, z\rangle$.

Definition

3. $W^{\mathfrak{H}}_{1,\pm} := \{H \in \mathfrak{H}_\pm : H(x,y;z) + H(x,z;y) = 0 \ \forall \ x,y,z \in V\}$.
4. $W^{\mathfrak{H}}_{2,\pm} := \{H \in \mathfrak{H}_\pm : H(x,y;z) + H(y,z;x) + H(z,x;y) = 0 \forall x,y,z \in V\}$.
5. $U^{\mathfrak{H}}_{3,\pm} := \{H \in \mathfrak{H}_\pm : H(x,y;z) \pm H(x, J_\pm y; J_\pm z) = 0 \ \forall \ x,y,z \in V\}$.
6. $W^{\mathfrak{H}}_{3,\pm} := U_{3,\pm} \cap \ker(\tau_1)$.
7. $W^{\mathfrak{H}}_{4,\pm} := \mathrm{Range}(\sigma_{J_\pm})$.