Harmonic characteristic vector field on unit tangent sphere bundle

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(Joint work with S. H. Chun, and K. Sekigawa)

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Contact metric manifold

Definition

$(\bar{M}^{2n+1}, \eta)$: *contact manifold*

if $\exists$ *a global 1-form* η *s.t.* $\eta \wedge (d\eta)^n \neq 0$ *everywhere on* $\bar{M}$

Definition

For given contact form η,

$\exists !\ \xi$: *characteristic vector field s.t.* $\eta(\xi) = 1, \quad d\eta(\xi, \bar{X}) = 0$

$\exists\ \bar{g}$: *Riemannian metric and* ϕ : *(1,1)-tensor field s.t.*

$$\eta(\bar{X}) = \bar{g}(\bar{X}, \xi), \quad d\eta(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \phi\bar{Y}), \quad \phi^2\bar{X} = -\bar{X} + \eta(\bar{X})\xi$$

$\Rightarrow (\bar{M}; \eta, \bar{g}, \phi, \xi)$: *contact metric manifold*

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Harmonic vector field

A unit vector field V on Riemannian manifold (M, g) determines a map

$$V : (M, g) \to (T_1M, g') \subset TM$$

Here, $T_1M = \{(x, u) \in TM | |u| = 1\}$.

If M is compact and orientable, the energy of V (called total bending of V) is defined as:

$$E(V) = \frac{1}{2}\int_M |dV|^2 dv_g = \frac{n}{2} vol(M, g) + \frac{1}{2}\int_M |\nabla V|^2 dv_g.$$

V is said to be a harmonic vector field if it is a critical point for the energy functional E in the set of all unit vector fields of M.

Definition

A contact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an H-contact manifold.

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V : critical point for $E(V) \Longleftrightarrow \bar{\triangle}V // V$ ($\bar{\triangle}V$ is collinear to V.)

G. Wiegmink, *Total bending of vector fileds on Riemannian manifolds*, Math. Ann. **303** (1995) 325–344.

The rough Laplacian $\bar{\triangle}$ of a vector field $V \in \mathfrak{X}(M)$ is defined by

$$\bar{\triangle}V = -tr\nabla^2 V.$$

If $\{e_i\}$ is any local orthonormal frame field on M, we get

$$\bar{\triangle}V = \sum_{i=1}^{n}\{\nabla_{\nabla_{e_i}e_i}V - \nabla_{e_i}\nabla_{e_i}V\}.$$

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Theorem

Let $M = (M, g, \phi, \xi, \eta)$ be a $(2n+1)$-dimensional contact metric manifold. Then

$$\bar{\triangle}\xi = 4n\xi - \bar{Q}\xi.$$

Theorem

A contact metric manifold is an H-contact manifold if and only if the characteristic vector field ξ is an eigenvector of the Ricci operator $\bar{Q}$.

D. Perrone, Contact metric manifolds whose characteristic vector field is a harmonic vector field, *Differential Geom. Appl.*, **20** (2004), 367–378.

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Examples

- K-contact manifolds

$$\xi : \text{Killing vector field} \quad \Rightarrow \quad \bar{Q}\xi = 2n\xi. \quad (dim M = 2n+1)$$

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Examples

- η-Einstein manifolds[6]

$$\bar{\rho} = \alpha\bar{g} + \beta\eta\otimes\eta \quad \Rightarrow \quad \bar{Q}\xi = (\alpha+\beta)\xi$$

Tanno [2] showed $(T_1S^n(1), \bar{g}, \phi, \xi, \eta)$: η-Einstein with
$\bar{\rho} = 2(2n-3)\bar{g} + 2(2-n)\eta \otimes \eta$.
By setting

$$\eta^* = \frac{a}{2}\eta,$$
$$\xi^* = \frac{2}{a}\xi,$$
$$g^* = a\bar{g} + \frac{a(a-1)}{4}\eta \otimes \eta,$$
$$\phi^* = \phi, \qquad a = \frac{2n-4}{n+1}$$

Then $(T_1S^n(1), g^*, \phi^*, \xi^*, \eta^*)$: Einstein with $\rho^* = 2(n-1)g^*$ [1].

P. Boyer, K. Galicki and M. Krzysztof, On Eta Einstein Sasakian geometry, *Commun. Math. Phys.*, **262** (2006), 177–208.

S. Tanno, Geodesic flows on C_L-manifolds and Einstein metrics on $S^3 \times S^2$, in: *Minimal Submanifolds and Geodesics*, North-Holland,

Problem

The study of the relationship between the geometric properties of a Riemannian manifold and those of its unit tangent sphere bundle has been studied for decades by many authors and is still an active research area.

Problem. Determine the base manifold of the unit tangent sphere bundle whose characteristic vector field is harmonic.

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Main Theorem

An n-dimensional Einstein manifold $M = (M, g)$ is said to be 2-stein if M satisfies the following condition

$$\sum_{i,j=1}^{n} (R_{uiuj})^2 = \mu(p)|u|^4 \quad \text{for all } u \in T_pM,\ p \in M,$$

where μ is a real valued function on M.

Theorem

Let $M = (M, g)$ be an $n(\geq 3)$-dimensional Einstein manifold. Then the unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ is H-contact if and only if M is 2-stein.

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Motivation

Theorem

The unit tangent sphere bundle T_1M is Einstein ($\bar{\rho} = \alpha\bar{g}$) if and only if M is a surface of constant curvature 0 or 1.

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Theorem

Let T_1M be unit tangent sphere bundle of an n-dimensional Riemannian manifold $M = (M, g)$ equipped with the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$. Then $(T_1M, \bar{g}, \phi, \xi, \eta)$ is η-Einstein, if and only if M is a space of constant sectional curvature 1 or $n-2$.

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Definition

An Riemannian manifold M is called a 2-point homogeneous space if for every pair of points (p_1, q_1) and (p_2, q_2) with $d(p_1, q_1) = d(p_2, q_2)$, there is an isometry T of M such that $T(p_1) = p_2$ and $T(q_1) = q_2$.

Definition

An Riemannian manifold M is called a locally 2-point homogeneous space if for any points p_1, p_2 of M, there exists neighborhoods U_1, U_2 centered at p_1, p_2 respectively such that if for every pair of points (q_1, r_1) and (q_2, r_2) with $d(q_1, r_1) = d(q_2, r_2)$, there is an isometry T from U_1 onto U_2 satisfying $T(q_1) = q_2$ and $T(r_1) = r_2$.

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Theorem

A locally 2-point homogeneous space is locally symmetric ($\nabla R = 0$).

Z. I. Szabo, A short topological proof of for the symmetry of 2 point homogeneous spaces, *Invent. Math.*, **106** (1991), 61–64.

Proposition

A complete, simply connected locally 2-point homogeneous space is 2-point homogeneous space.

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A two-point homogeneous space is a Euclidean or a symmetric space of the rank 1.

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E. Boeckx and L. Vanhecke, Harmonic and minimal vector fields in tangent and unit tangent bundles, *Differential Geom. Appl.*, **13** (2000), 77–93.

Question

Calvaruso and Perrone [1] raised the following question, which was first asked in [1].

Question 1 *Are the (possibly locally) two-point homogeneous spaces the only Riemannian manifolds whose unit tangent sphere bundles are H-contact?*

G. Calvaruso and D. Perrone, H-contact unit tangent sphere bundles, *Rocky Mountain J. Math.*, **37** (2007), 1435–1458.

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Partial positive answers

Theorem

The unit tangent sphere bundle T_1M of a 2-dimensional or 3-dimensional Riemannian manifold M is H-contact if and only if the base manifold M has constant sectional curvature.

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Theorem

T_1M of the $n(\geq 4)$-dimensional conformally flat manifold M is H-contact if and only if M has constant sectional curvature.

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Theorem

If T_1M of a Kähler manifold is H-contact, then the base manifold is Kähler-Einstein, and further Kähler-Einstein locally symmetric when the base manifold is compact Kähler manifold with nonnegative sectional curvature.

Theorem

Let (M, J, g) be a four-dimensional Kähler manifold. Suppose that M satisfies one of the following properties:

(a) *M has either nonnegative or nonpositive sectional curvature, or*

(b) *M is not Ricci-flat.*

Then, T_1M is H-contact if and only if M has constant holomorphic sectional curvature.

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- Application
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Tangent bundle

Let $M = (M, g)$ be an n-dimensional Riemannian manifold and let TM denote its tangent bundle with the natural projection.

$$\pi : TM \longrightarrow M \; ; \; \pi(x, u) = x$$

Consider the tangent space $T_{(x,u)}TM$ to TM at $(x, u) \in TM$.

$$VTM_{(x,u)} = T_{(x,u)}(\pi^{-1}(x)) = Ker\pi_*|_{(x,u)} = \{X^v\}$$

is called the vertical subspace of $T_{(x,u)}TM$

Find a complementary subspace $HTM_{(x,u)}$ to $VTM_{(x,u)}$ in $T_{(x,u)}TM$.

$HTM_{(x,u)} = \{X^h\}$ is called the horizontal subspace of $T_{(x,u)}TM$. Then

$$T_{(x,u)}TM = VTM_{(x,u)} \oplus HTM_{(x,u)}$$

Take a local coordinate system $(x^1, \cdots, x^n)$ on an open subset U of M. On $\pi^{-1}(U)$, define coordinates $(\bar{x}^1, \cdots, \bar{x}^n; u^1, \cdots, u^n)$ as follows:

$$\bar{x}^i(x, u) = (x^i \circ \pi)(x, u) = x^i(x),$$
$$u^i(x, u) = dx^i(u) = ux^i$$

for $i = 1, \cdots, n$ and $(x, u) \in \pi^{-1}(U)$.

Then for $X = \sum_i X^i \frac{\partial}{\partial x^i}$, X^h and X^v are given by

$$X^h = \sum_i (X^i \circ \pi) \frac{\partial}{\partial \bar{x}^i} - \sum_{i,j,k} u^k ((X^j \Gamma^i_{jk}) \circ \pi) \frac{\partial}{\partial u^i},$$
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$$X^v = \sum_i (X^i \circ \pi) \frac{\partial}{\partial u^i}$$

Define a Riemannian metric $\tilde{g}$, the *Sasaki metric* on TM, in a natural way, by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M.

Define an almost complex structure tensor J of TM as the following:

$$JX^h = X^v, \quad JX^v = -X^h.$$

Also, an associated 2-form Ω is given by

$$\Omega(X^h, Y^h) = \Omega(X^v, Y^v) = 0, \quad \Omega(X^v, Y^h) = g(X, Y) \circ \pi.$$

Then $(TM; \tilde{g}, J)$ is an almost Kähler manifold.

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$$JX^h = X^v, \quad JX^v = -X^h.$$

Also, an associated 2-form Ω is given by

$$\Omega(X^h, Y^h) = \Omega(X^v, Y^v) = 0, \quad \Omega(X^v, Y^h) = g(X, Y) \circ \pi.$$

Then $(TM; \tilde{g}, J)$ is an almost Kähler manifold.

Unit tangent sphere bundle

Consider the **unit tangent sphere bundle** (T_1M, g'), which is an isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^v$.

Definition

For $X \in T_xM$, we define the tangential lift of X to $(x, u) \in T_1M$ by

$$X^t_{(x,u)} = X^v_{(x,u)} - g(X, u)N_{(x,u)}.$$

The tangent space $T_{(x,u)}T_1M$ is spanned by vectors of the form X^h and X^t.

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The Riemannian metric g' on T_1M is given by

$$g'(X^h, Y^h) = g(X, Y) \circ \pi,$$
$$g'(X^t, Y^t) = (g(X, Y) - g(X, u)g(Y, u)) \circ \pi,$$
$$g'(X^h, Y^t) = 0$$

for all vector fields X and Y on M.

We define the standard contact metric structure of the unit tangent sphere bundle T_1M of a Riemannian manifold (M, g). Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\bar{X}, \phi'\bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

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Let $\{e_1, \cdots, e_n = u\}$ be an orthonormal basis of T_xM. Then $\{2e_1^t, \cdots, 2e_{n-1}^t, 2e_1^h, \cdots, 2e_n^h = \xi\}$ is an orthonormal basis for $T_{(x,u)}T_1M$.

The Ricci tensor $\bar{\rho}$ and the scalar curvature $\bar{\tau}$ of T_1M are given by

$$\begin{aligned} \bar{\rho}(X^t, Y^t) &= (n-2)\big(g(X,Y) - g(X,u)g(Y,u)\big) \\ &\quad + \frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i), \\ \bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u\rho)(X,Y) - (\nabla_X\rho)(u,Y)), \\ \bar{\rho}(X^h, Y^h) &= \rho(X,Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y), \end{aligned} \tag{1}$$

Let $\{e_1, \cdots, e_n = u\}$ be an orthonormal basis of $T_x M$. Then $\{2e_1^t, \cdots, 2e_{n-1}^t, 2e_1^h, \cdots, 2e_n^h = \xi\}$ is an orthonormal basis for $T_{(x,u)} T_1 M$.

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\bar{\rho}(X^t, Y^h) &= \frac{1}{2}((\nabla_u \rho)(X, Y) - (\nabla_X \rho)(u, Y)), \\
\bar{\rho}(X^h, Y^h) &= \rho(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g(R(u, e_i)X, R(u, e_i)Y),
\end{aligned} \tag{1}$$

From (1) and Theorem 2,

Theorem

T_1M is H-contact with respect to the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ if and only if the base manifold $M = (M, g)$ satisfies the following conditions:

(1) *the Ricci tensor ρ of (M, g) is Codazzi tensor, that is, $(\nabla_x \rho)(y, z) = (\nabla_y \rho)(x, z)$ for any $x, y, z \in T_pM$, and*

(2)
$$2\rho_{ux} = \sum_{i,j=1}^{n} R_{uixj} R_{uiuj} \tag{2}$$
for all $p \in M$, $x \perp u$, $|u| = |x| = 1$ and $\{e_i\}_{i=1}^{n}$ orthonormal basis of T_pM.

G. Calvaruso and D. Perrone, H-contact unit tangent sphere bundles, *Rocky Mountain J. Math.*, **37** (2007), 1435–1458.

We set

$$\begin{cases} u = cos\theta e_a + sin\theta e_b, \\ x = -sin\theta e_a + cos\theta e_b \qquad \text{for all } a \neq b. \end{cases} \tag{3}$$

Substituting (3) into the left hand side of (2), we get (using some standard trigonometric identities)

$$\begin{aligned} &2\rho(cos\theta e_a + sin\theta e_b, -sin\theta e_a + cos\theta e_b) \\ =&2\rho_{ab}cos(2\theta) + (\rho_{bb} - \rho_{aa})sin(2\theta). \end{aligned} \tag{4}$$

Similarly, substituting (3) into the right hand side of (2), we get

$$
\begin{aligned}
&\sum_{i,j=1}^{n} R(cos\theta e_a + sin\theta e_b, e_i, -sin\theta e_a + cos\theta e_b, e_j) \\
&\quad \times R(cos\theta e_a + sin\theta e_b, e_i, cos\theta e_a + sin\theta e_b, e_j) \\
=&2\rho_{ab} cos(2\theta) + \frac{1}{4}\Big\{ \sum_{i,j=1}^{n} (R_{bibj})^2 - \sum_{i,j=1}^{n} (R_{aiaj})^2 \Big\} sin(2\theta) \\
&+ \frac{1}{4} sin(4\theta)\Big\{ \sum_{i,j=1}^{n} (R_{aibj})^2 + \sum_{i,j=1}^{n} R_{aibj}R_{biaj} + \sum_{i,j=1}^{n} R_{aiaj}R_{bibj} \\
&\qquad - \frac{1}{2}\sum_{i,j=1}^{n} (R_{aiaj})^2 - \frac{1}{2}\sum_{i,j=1}^{n} (R_{bibj})^2 \Big\}.
\end{aligned} \tag{5}
$$

Then, comparing the finite Fourier series in (4) and (5), we obtain two equations:

$$4(\rho_{aa} - \rho_{bb}) = \sum_{i,j=1}^{n} (R_{aiaj})^2 - \sum_{i,j=1}^{n} (R_{bibj})^2, \tag{6}$$

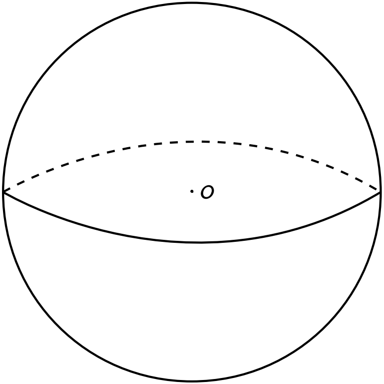
$$\begin{aligned} &2\Big\{ \sum_{i,j=1}^{n} (R_{aibj})^2 + \sum_{i,j=1}^{n} R_{aibj} R_{biaj} + \sum_{i,j=1}^{n} R_{aiaj} R_{bibj} \Big\} \\ &= \sum_{i,j=1}^{n} (R_{aiaj})^2 + \sum_{i,j=1}^{n} (R_{bibj})^2. \end{aligned} \tag{7}$$

Lemma

Let $S^n (n \geq 2)$ *be an n-dimensional unit sphere centered at the origin* 0 *in an* $(n+1)$*-dimensional Euclidean space* $\mathbb{E}^{n+1}$ *and* f *be a real-valued function on* S^n *satisfying the condition* $f(u) = f(v)$ *for any* $u, v \in S^n$ *such that* $u \perp v$. *Then,* f *is constant on* S^n.

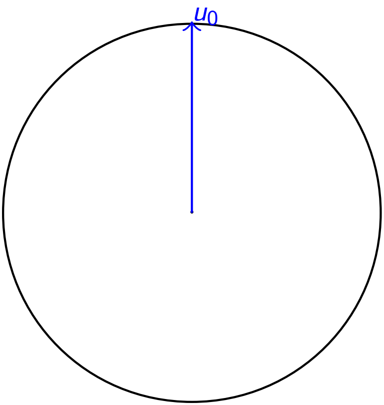
For any $u \in S^n$, we set $E(u) = \{v \in S^n | v \perp u\}$.
$\Rightarrow$ $E(u)$ is the intersection of S^n and the hyperplane in $\mathbb{E}^{n+1}$ through the origin 0 which is perpendicular to u.

$S^n \subset \mathbb{E}^{n+1}$



Proof of the Lemma

- Choose a point $u_o \in S^n$ and fix it.
- Let $w(\neq u_0) \in S^n$ and $S^1(w)$: the great circle
- $S^1(w)$ meets with $E(u_0)$ at a point $v_1 \in E(u_0)$
- $\exists\ v \in S^n$ s.t. $v \perp u_0,\ v \perp v_1$ $\Rightarrow\ v \perp w$
- $f(w) = f(v) = f(u_0)$ $\Rightarrow\ f(w) = f(u_0)$ for $w \neq u_0$ $\therefore\ f$ is constant



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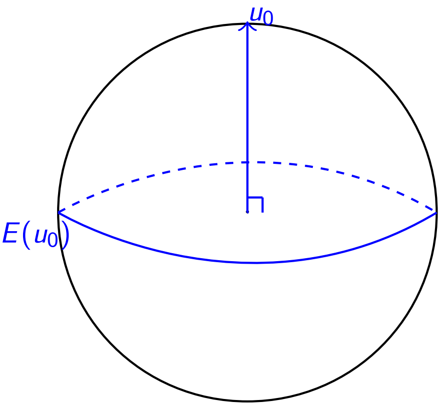

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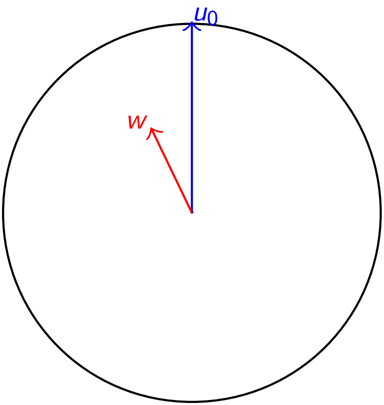

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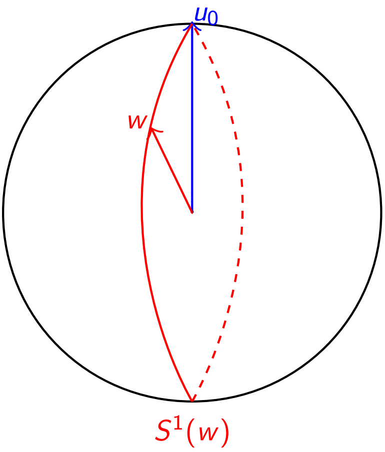

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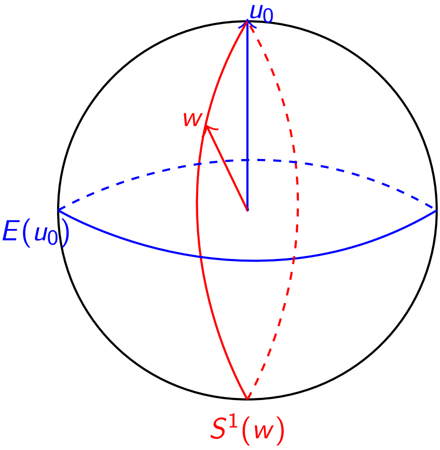

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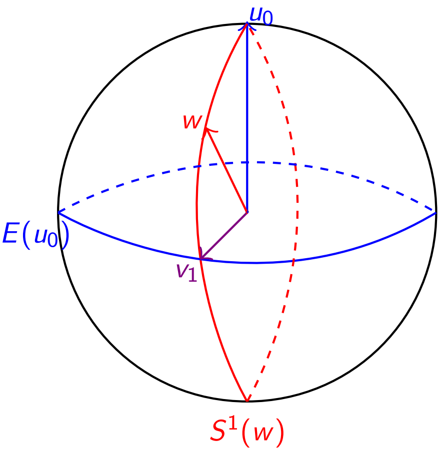

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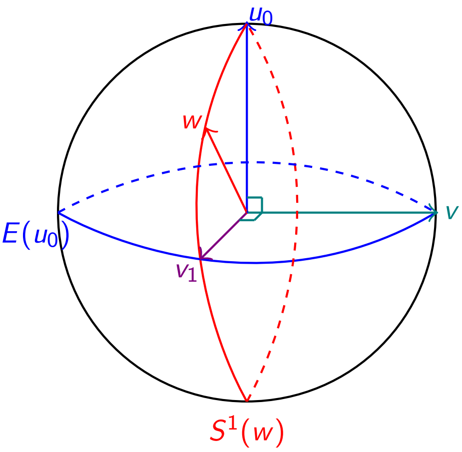

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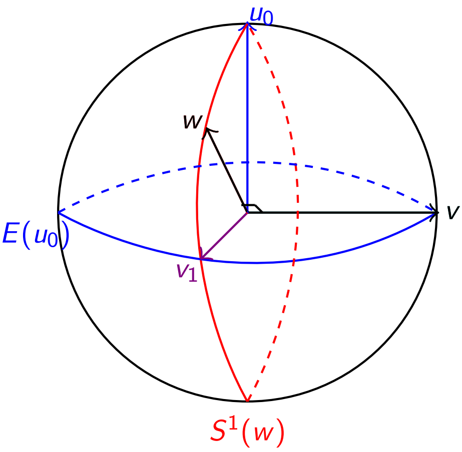

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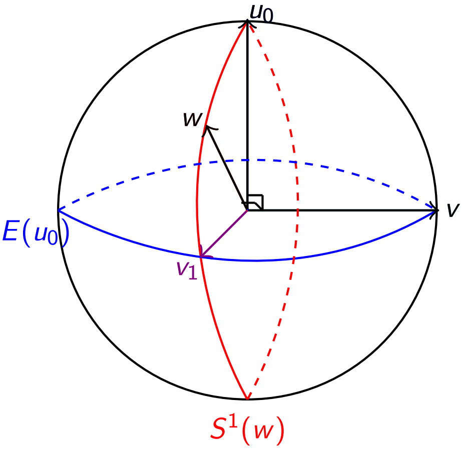

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Proof of the Main Theorem 3

($\Rightarrow$) Assume that $(T_1M, \bar{g}, \phi, \xi, \eta)$ is H-contact. For each point $p \in M$, we may regard

$$f(u) = \sum_{i,j=1}^{n} (R_{uiuj})^2, \quad u \in T_pM \; (|u| = 1)$$

as a function on an $(n-1)$-dimensional unit sphere S^{n-1}. Because M is Einstein(i.e., $\rho_{aa} = \rho_{bb}$), then (6) implies that $f(e_a) = f(e_b)$ for $a \neq b$ with respect to any fixed orthonormal basis $\{e_i\}_{i=1}^{n}$.

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$(\Leftarrow)$ Assume that M is 2-stein. Then, since M is Einstein, the Ricci tensor ρ is the Codazzi tensor. Further, from the definition of 2-stein manifold, we may obtain the following equality:

$$\sum_{i,j=1}^{n} R(cos\theta u + sin\theta v, e_i, cos\theta u + sin\theta v, e_j)^2 = \mu(p) \tag{8}$$

for any real number θ and any orthonormal pair (u, v) in T_pM.

Thus, from (8) and the hypothesis, we have easily

$$\rho(u, v) = 0, \tag{9}$$

$$\begin{aligned} 0 &= \frac{d}{d\theta}|_{\theta=0}\left(\sum_{i,j=1}^{n} R(cos\theta u + sin\theta v, e_i, cos\theta u + sin\theta v, e_j)^2\right) \\ &= 4\sum_{i,j=1}^{n} R_{uivj}R_{uiuj} \end{aligned} \tag{10}$$

for any orthonormal pair (u, v) in T_pM. From (9) and (10), we see that

$$2\rho_{uv} = \sum_{i,j=1}^{n} R_{uivj}R_{uiuj}.$$

Therefore, by Theorem 13, we see that $(T_1M, \bar{g}, \phi, \xi, \eta)$ is H-contact. □

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Harmonic spaces

$$H1 : \sum_{a=1}^{n} R_{auau} = \lambda_1 \|u\|^2,$$

$$H2 : \sum_{a,b=1}^{n} R_{aubu}^2 = \lambda_2 \|u\|^4,$$

$$H3 : 9 \sum_{a,b,c=1}^{n} R_{aubu} R_{bucu} R_{cuau} - 32 \sum_{a,b=1}^{n} (\nabla_u R_{aubu})^2 = \lambda_3 \|u\|^6,$$

$$\begin{aligned} H4 : 72 \sum_{a,b,c,d=1}^{n} R_{aubu} R_{bucu} R_{cudu} R_{duau} - 50 \sum_{a,b,c=1}^{n} \nabla_u R_{aubu} \nabla_u R_{bucu} R_{cuau} \\ + 8 \sum_{a,b,c=1}^{n} \nabla_{uu}^2 R_{aubu} R_{bucu} R_{cuau} + 3 \sum_{a,b=1}^{n} (\nabla_{uu}^2 R_{aubu})^2 = \lambda_4 \|u\|^8, \end{aligned} \tag{11}$$

for any tangent vector u to M. Here each λ_k $(k = 1, 2, \cdots)$ is a constant.

Definition

The ℓ*th Ricci curvature* $\rho^{[\ell]}$ *of* M *is the symmetric covariant tensor field of degree* 2ℓ *given by*

$$\rho^{[\ell]}(u, \cdots, u) = \sum_{a_1, \cdots, a_\ell = 1}^{n} R_{a_1 u u a_2} R_{a_2 u u a_3} \cdots R_{a_l u u a_\ell}$$

for $u \in T_p(M)$ and $p \in M$. Here $\{1, \cdots, n\}$ is an arbitrary orthonormal basis of $T_p(M)$. $\rho^{[1]}$ is the ordinary Ricci curvature.

Definition

A Riemannian manifold $M = (M, g)$ *is called a k-stein manifold provided there are real-valued functions* μ_ℓ *on* M *such that* $\rho^{[\ell]}(u) = \mu_\ell \|u\|^{2\ell}$ *for all* $u \in T_p(M)$ *and* $p \in M$ *for* $1 \leq \ell \leq k$.

Definition

The ℓth Ricci curvature $\rho^{[\ell]}$ of M is the symmetric covariant tensor field of degree 2ℓ given by

$$\rho^{[\ell]}(u, \cdots, u) = \sum_{a_1, \cdots, a_\ell = 1}^{n} R_{a_1 u u a_2} R_{a_2 u u a_3} \cdots R_{a_l u u a_\ell}$$

for $u \in T_p(M)$ and $p \in M$. Here $\{1, \cdots, n\}$ is an arbitrary orthonormal basis of $T_p(M)$. $\rho^{[1]}$ is the ordinary Ricci curvature.

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Application

Theorem

Let $M = (M, g)$ be a simply connected irreducible symmetric space. Then the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ on the unit tangent sphere bundle T_1M of M is H-contact if and only if M is isometric to any of the spaces belonging to the following three series :

(1) *Simply connected rank one symmetric spaces,*

(2) *Lie groups SU(2), SU(3), Spin(8), G_2, F_4, E_6, E_7, E_8, and their noncompact duals,*

(3) *The compact symmetric spaces* $\frac{SU(3)}{SO(3)}$, $\frac{SO(8)}{SO(4)\times SO(4)}$, $\frac{SO(8)}{SO(5)\times SO(3)}$, $\frac{SO(8)}{SO(6)\times SO(2)}$, $\frac{SU(6)}{Sp(3)}$, $\frac{G_2}{SO(4)}$, $\frac{F_4}{(Sp(3)\times Sp(2))/\mathbb{Z}_2}$, $\frac{E_6}{F_4}$, $\frac{E_6}{(Spin(10)\times U(1))/\mathbb{Z}_4}$, $\frac{E_6}{(SU(6)\times Sp(1))/\mathbb{Z}_2}$, $\frac{E_6}{Sp(4)/\mathbb{Z}_2}$, $\frac{E_7}{(E_6\times U(1))/\mathbb{Z}_3}$, $\frac{E_7}{(Spin(12)\times SU(2))/\mathbb{Z}_2}$, $\frac{E_7}{SU(8)/\mathbb{Z}_2}$, $\frac{E_8}{(E_7\times SU(2))/\mathbb{Z}_2}$, $\frac{E_8}{SO(16)}$, *and their non-compact duals.*

Table I. Harmonic, 2-stein and 3-stein compact Lie groups and their noncompact dual spaces

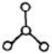
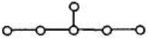
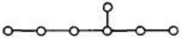
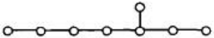
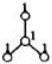
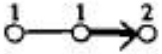
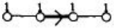
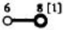
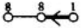
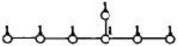
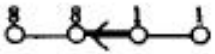
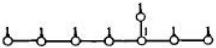
Compact	Noncompact	Dynkin diagram	Type	$\frac{(\rho^{[1]})^k}{\rho^{[k]}}$		Orders of the primitive invariants	Dimension
$SU(2)$	$\frac{SU(2,\mathbb{C})}{SU(2)}$		harmonic	2^{k-1}	$(k=1,2,\ldots)$	2	3
$SU(3)$	$\frac{SU(3,\mathbb{C})}{SU(3)}$		2-stein	4	$(k=2)$	2, 3	8
$SO(8)$	$\frac{SO(8,\mathbb{C})}{SO(8)}$		2-stein	12	$(k=2)$	2, 4, 4, 6	28
G_2	$\frac{G_2^{\mathbb{C}}}{G_2}$		2-stein	$\frac{32}{5}$	$(k=2)$	2, 6	14
F_4	$\frac{F_4^{\mathbb{C}}}{F_4}$		2-stein	$\frac{108}{5}$	$(k=2)$	2, 6, 8, 12	52
E_6	$\frac{E_6^{\mathbb{C}}}{E_6}$		2-stein	32	$(k=2)$	2, 5, 6, 8, 9, 12	78
E_7	$\frac{E_7^{\mathbb{C}}}{E_7}$		2-stein	54	$(k=2)$	2, 6, 8, 10, 12, 14, 18	133
E_8	$\frac{E_8^{\mathbb{C}}}{E_8}$		3-stein	100 7,200	$(k=2)$ $(k=3)$	2, 8, 12, 14, 18, 20, 24, 30	248

TABLE II. Simply connected symmetric harmonic spaces

Compact	Noncompact	Root system	Dynkin diagram	Dimension
—	$\mathbf{R}^n$	—	—	n
$S^n = \dfrac{SO(n+1)}{SO(n)}$	$H^n = \dfrac{SO(n,1)}{SO(n)}$	A_1	$n-1$ $\circ$	n
$\mathbf{C}P^n = \dfrac{U(n+1)}{U(n)\times U(1)}$	$\mathbf{C}H^n = \dfrac{U(n,1)}{U(n)\times U(1)}$	BC_1	$2(n-1)[1]$ $\circledcirc$	$2n$
$\mathbf{Q}P^n = \dfrac{Sp(n+1)}{Sp(n)\times Sp(1)}$	$\mathbf{Q}H^n = \dfrac{Sp(n,1)}{Sp(n)\times Sp(1)}$	BC_1	$4(n-1)[3]$ $\circledcirc$	$4n$
$CayP^2 = \dfrac{F_4}{Spin(9)}$	$CayH^2 = \dfrac{F_4^*}{Spin(9)}$	BC_1	$8[7]$ $\circledcirc$	16

Table III. 2-stein and 3-stein spaces of type I

Space	Root system	Dynkin diagram	$\frac{(\rho^{[1]})^k}{\rho^{[k]}}$		Type	Dimension
$A_2^R = \frac{SU(3)}{SO(3)}$	A_2		2	$(k=2)$	2-stein	5
$D_4^{R,4} = \frac{SO(8)}{SO(4)\times SO(4)}$	D_4		6	$(k=2)$	2-stein	16
$D_4^{R,3} = \frac{SO(8)}{SO(5)\times SO(3)}$	B_3		6	$(k=2)$	2-stein	15
$D_4^{R,2} = \frac{SO(8)}{SO(6)\times SO(2)}$	B_2		6	$(k=2)$	2-stein Hermitian Symmetric	12
$A_5^H = \frac{SU(6)}{Sp(3)}$	A_2		8	$(k=2)$	2-stein	14
$G_{2(2)} = \frac{G_2}{SO(4)}$	G_2		$\frac{16}{5}$	$(k=2)$	2-stein	8
$F_{4(4)} = \frac{F_4}{A_1\times C_3}$	F_4		$\frac{54}{5}$	$(k=2)$	2-stein	28
$E_{6(-26)} = \frac{E_6}{F_4}$	A_2		16	$(k=2)$	2-stein	26

$E_{6(-14)} = \dfrac{E_6}{D_5 \times SO(2)}$	BC_2		16	$(k=2)$	2-stein Hermitian Symmetric	32
$E_{6(2)} = \dfrac{E_6}{A_1 \times A_5}$	F_4		16	$(k=2)$	2-stein	40
$E_{6(6)} = \dfrac{E_6}{C_4}$	E_6		16	$(k=2)$	2-stein	42
$E_{7(-25)} = \dfrac{E_7}{E_6 \times SO(2)}$	C_3		27	$(k=2)$	2-stein Hermitian Symmetric	54
$E_{7(-5)} = \dfrac{E_7}{D_6 \times A_1}$	F_4		27	$(k=2)$	2-stein	64
$E_{7(7)} = \dfrac{E_7}{A_7}$	E_7		27	$(k=2)$	2-stein	70
$E_{8(-24)} = \dfrac{E_8}{E_7 \times A_1}$	F_4		50 1,800	$(k=2)$ $(k=3)$	3-stein	112
$E_{8(8)} = \dfrac{E_8}{D_8}$	E_8		50 1,800	$(k=2)$ $(k=3)$	3-stein	128
$D_6^{R,2} = \dfrac{SO(12)}{SO(10) \times SO(2)}$	B_2		100	$(k=3)$	exceptional (see lemma 4.4) Hermitian Symmetric	20

Proof. Assume that $(T_1M, \bar{g}, \phi, \xi, \eta)$ is H-contact. Then, by the Main theorem, M is 2-stein. In [1], the Tables I(39), II(40) and III(41, 42) were presented, listing k-stein symmetric spaces for all values of k. Part (1)(15), (2)(15) and (3)(15) of the statement of Theorem 3 correspond to the 2-stein examples in Tables II, I, III of [1] respectively.

P. Carpenter, A. Gray and T. J. Willmore, The curvature of Einstein symmetric spaces, *Quart. J. Math. Oxford*, **33** (1982), 45–64.

J. A. Wolf, *Spaces of constant curvature*, McGRAW-Hill Book Company, 1967.

The sufficiency part follows easily, as well, from the same Tables and the Main Theorem.

There are plenty of simply connected, Einstein, symmetric spaces which are not two-point homogeneous and have H-contact unit tangent sphere bundles.We have a negative answer of the Question 1(14).

Proof. Assume that $(T_1M, \bar{g}, \phi, \xi, \eta)$ is H-contact. Then, by the Main theorem, M is 2-stein. In [1], the Tables I(39), II(40) and III(41, 42) were presented, listing k-stein symmetric spaces for all values of k. Part (1)(15), (2)(15) and (3)(15) of the statement of Theorem 3 correspond to the 2-stein examples in Tables II, I, III of [1] respectively.

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Examples

The unit tangent sphere bundle T_1M of 4-dimensional strictly almost Kähler Einstein manifold $M = (M, J, g)$ which is locally neither a real space form nor a complex space form, and not even a locally symmetric space.

P. Nurowski and M. Pruzanowski, A four-dimensional example of Ricci flat metric admitting almost-Kähler non-Kähler structure, *Class. Quantum Grav.*, **16 (3)** (1999), L9–L13.

T. Oguro, K. Sekigawa and A. Yamada, Four-dimensional almost Kähler Einstein and weakly $*$-Einstein manifolds, *Yokohama Math. J.*, **47** (1999), 75–91

Let M be an 4-dimensional Einstein manifold and p be any point of M. Then, we may choose an orthonormal basis $\{e_i\}$ (known as the Singer-Thorpe basis) at each point $p \in M$ such that

$$\begin{cases} R_{1212} = R_{3434} = a, \quad R_{1313} = R_{2424} = b, \quad R_{1414} = R_{2323} = c, \\ R_{1234} = d, \quad R_{1342} = e, \quad R_{1423} = f, \\ R_{ijkl} = 0 \ \text{ whenever just three of the indices } i, j, k, l \text{ are distinct.} \end{cases} \tag{12}$$

I. M. Singer and J. A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, In: Global Analysis, Papers in Honor of K. Kodaira, pp 355–365: Princeton University Press (1969).

It is known that M is 2-Einstein if and only if

$$\pm d = a + \frac{\tau}{12}, \quad \pm e = b + \frac{\tau}{12}, \quad \pm f = c + \frac{\tau}{12}. \tag{13}$$

K. Sekigawa and L. Vanhecke, Volume-preserving geodesic symmetries on four dimensional 2-stein spaces, *Kodai Math. J.*, **9** (1986), 215–224.

Examples

Let M be a 4-dimensional real half-space given by
$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 |\ x_1 > 0, (x_2, x_3, x_4) \in \mathbb{R}^3\}$. We define a Riemannian metric g and almost complex structure J on M respectively by

$$g = (g_{ij}) = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_1 + \frac{x_3^2}{4x_1} & -\frac{x_2 x_3}{4x_1} & \frac{x_3}{2x_1} \\ 0 & -\frac{x_2 x_3}{4x_1} & x_1 + \frac{x_2^2}{4x_1} & -\frac{x_2}{2x_1} \\ 0 & \frac{x_3}{2x_1} & -\frac{x_2}{2x_1} & \frac{1}{x_1} \end{pmatrix}, \tag{14}$$

$$J = (J_j^i) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & \frac{x_3}{2x_1} & -\frac{x_2}{2x_1} & \frac{1}{x_1} \\ 1 & 0 & 0 & 0 \\ \frac{x_2}{2} & -x_1 - \frac{x_3^2}{4x_1} & \frac{x_2 x_3}{4x_1} & -\frac{x_3}{2x_1} \end{pmatrix}, \tag{15}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $J(\frac{\partial}{\partial x_j}) = \sum_i J_j^i(\frac{\partial}{\partial x_i})$.

Then, (J, g) is an almost Hermitian structure on M and the Kähler form Ω is given by

$$\Omega = -x_1 dx_1 \wedge dx_3 - \frac{x_2}{2} dx_2 \wedge dx_3 + dx_2 \wedge dx_4. \tag{16}$$

From (16), $d\Omega = 0$, and hence (M, J, g) is an almost Kähler manifold.

Now, we define vector fields e_1, e_2, e_3, e_4 on M respectively by

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{x_1}} \frac{\partial}{\partial x_1}, \qquad & e_2 &= \frac{1}{\sqrt{x_1}} \frac{\partial}{\partial x_3} + \frac{x_2}{2\sqrt{x_1}} \frac{\partial}{\partial x_4}, \\ e_3 &= \sqrt{x_1} \frac{\partial}{\partial x_4}, \qquad & e_4 &= \frac{1}{\sqrt{x_1}} \frac{\partial}{\partial x_2} - \frac{x_3}{2\sqrt{x_1}} \frac{\partial}{\partial x_4}. \end{aligned} \tag{17}$$

Then, $\{e_i\}_{i=1,2,3,4}$ is a unitary frame field on M with $e_2 = Je_1, \ e_4 = Je_3$.

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From the definition of the frame field $\{e_i\}$, we have

$$
\begin{aligned}
&\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \\
&\nabla_{e_2} e_1 = \frac{1}{2x_1\sqrt{x_1}} e_2, \quad \nabla_{e_2} e_2 = -\frac{1}{2x_1\sqrt{x_1}} e_1, \\
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Further, we obtain

$$\begin{cases} R_{1212} = R_{3434} = -\frac{1}{2x_1^3}, \quad R_{1313} = R_{2424} = \frac{1}{x_1^3}, \\ R_{1414} = R_{2323} = -\frac{1}{2x_1^3}, \\ R_{1234} = -\frac{1}{2x_1^3}, \quad R_{1342} = \frac{1}{x_1^3}, \quad R_{1423} = -\frac{1}{2x_1^3}, \\ \text{all others are zero.} \end{cases} \tag{19}$$

Then M is 2-stein and $\{e_i\}_{i=1,2,3,4}$ is a global Singer-Thorpe basis on M, and further $M = (M, J, g)$ is a strictly almost Kähler manifold of pointwise constant holomorphic sectional curvature $C(p) = \frac{1}{2x_1^3}$ $(p = (x_1, x_2, x_3, x_4) \in M)$ and $\rho = 0$, $\rho^* = \frac{1}{{x_1}^3} g$. $|R|$ is not constant, and hence (M, g) is not locally symmetric space. Thus, from the Szabo's result M can not be locally two-point homogeneous.

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$$\begin{cases} R_{1212} = R_{3434} = -\frac{1}{2x_1^3}, \quad R_{1313} = R_{2424} = \frac{1}{x_1^3}, \\ R_{1414} = R_{2323} = -\frac{1}{2x_1^3}, \\ R_{1234} = -\frac{1}{2x_1^3}, \quad R_{1342} = \frac{1}{x_1^3}, \quad R_{1423} = -\frac{1}{2x_1^3}, \\ \text{all others are zero.} \end{cases} \tag{19}$$

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Next, we shall define the opposite almost complex structure J' on M by

$$J'e_1 = e_2, J'e_2 = -e_1, J'e_3 = -e_4, J'e_4 = e_3. \tag{20}$$

Then, from (18) and (20), $M' = (M, J', g)$ is a Ricci-flat 2-stein Kähler manifold. Thus, from the main Theorem, we see that the corresponding unit tangent sphere bundles of $M = (M, J, g)$ and $M' = (M, J', g)$ are both H-contact with respect to the common standard contact metric structure.

Further, M' is not a space of constant holomorphic sectional curvature. Indeed, using (19) and (20), we see that the holomorphic sectional curvature varies with the variable x_1. These examples can be seen as a counterpart to a result by Calvaruso and Perrone ([1], Theorem 7.3).

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Theorem

Let (M, J, g) be a four-dimensional Kähler manifold. Suppose that M satisfies one of the following properties:

(a) *M has either nonnegative or nonpositive sectional curvature, or*

(b) *M is not Ricci-flat.*

Then, T_1M is H-contact if and only if M has constant holomorphic sectional curvature.

G. Calvaruso and D. Perrone, H-contact unit tangent sphere bundles, *Rocky Mountain J. Math.*, **37** (2007), 1435–1458.

Question 2 Let $M = (M, g)$ be an $n(\geq 3)$-dimensional Riemannian manifold. If the unit tangent sphere bundle T_1M equipped with the standard contact metric structure is H-contact, then is the base Riemannian manifold M Einstein ?

Theorem

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then the unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\bar{g}, \phi, \xi, \eta)$ is H-contact if and only if the base manifold M is 2-stein.

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Further progress on our Main theorem

Given an arbitrary g-natural metric G on the tangent bundle TM of a Riemannian manifold (M, g), there exist smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, where $i = 1, 2, 3$, such that

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) =& (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\ & + (\beta_1 + \beta_3)(r^2) g_x(X, u), g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) =& G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g_x(X, Y) \\ & + \beta_2(r^2) g_x(X, u), g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) =& \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{aligned}$$

for every u, X, $Y \in T_x(M)$, where $r^2 = g_x(u, u)$.

Kaluza-Klein metrics, as commonly defined on principal bundles, are obtained for

$$\alpha_2(t) = \beta_2(t) = \beta_1(t) + \beta_3(t) = 0. \tag{21}$$

Theorem

If (M, g) is an Einstein manifold and $\tilde{G}$ is a Riemannian g-natural metric on T_1M of Kaluza-Klein type, then $(T_1M, \tilde{\eta}, \tilde{G})$ is H-contact if and only if (M, g) is 2-stein.

G. Calvaruso and D. Perrone, Homogeneous and *H*-contact unit tangent sphere bundles, J. Aust. Math. Soc. **88** (2010), 323–377.

Theorem

Let $M = (M, g)$ be an $n(\geq 2)$-dimensional Riemannian manifold whose unit tangent sphere bundle T_1M equipped with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ is H-contact. If $\dim M \neq 4$, then the scalar curvature τ of M, the square norm $|\rho|^2$ of the Ricci tensor and the square norm $|R|^2$ of the curvature tensor are all constant. If $\dim M = 4$, then τ and $|\rho|^2$ are constant, however, $|R|^2$ is not necessarily constant.

Theorem

If T_1M is an η-Einstein manifold ($\bar{\rho} = \alpha\bar{g} + \beta\eta \otimes \eta$), then α, β, τ, $|\rho|^2$, $|R|^2$, and $\bar{\tau}$ are all constant.

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