p-Laplacian and topology of manifolds

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I. Introductory examples

We are given an m-dimensional Riemannian manifold $(X^m, \langle , \rangle)$. A natural way to detect the geometry and the topology of X is to view X either as the domain or as the target space of some interesting class of maps. Clearly, the Riemannian structure adds information on X and therefore the interesting maps should take them into account.

Let us consider a couple of (classical) examples to give some flavour of ideas and techniques and to introduce (some of) the main ingredients.

Let M, N be compact, with $Sec_N \leq 0$. Let $f : M \to N$ be a smooth map. Then we have

Th. 1 (Eells-Sampson, Hartman)

$$\exists u : M \to N : \int_M |du|^2 = \min \left\{ \int_M |dh|^2 : h \text{ homotopic to } f \right\}.$$

The minimizer u satisfies the (system of) equations

$$-\Delta u := \delta(du) = 0$$

i.e. u is a **harmonic map**. Note: u is smooth by elliptic regularity. In particular, the validity of a Liouville type result

$$\Delta u = 0 \Longrightarrow u = \text{const}$$

gives that f is topologically trivial. For instance, we have the following

Th. 2 (Eells-Sampson) *M cmpt, $Ric_M \geq 0$ and N cmpt, $Sec_N \leq 0$.*

(a) If $Ric_M(p_0) > 0$ for some $p_0 \in M \Longrightarrow$ Liouville for harmonic maps $\Longrightarrow$ every smooth $f : M \to N$ is homotopically trivial.

(b) If $Sec_N < 0$ then either the harmonic map $u : M \to N$ is constant or $u(M) = \Gamma$ a closed geodesic of N.

Proof. Let $u : M \to N$ be harmonic. The **Bochner-Weitzenböck** formula states

$$\frac{1}{2}\Delta |du|^2 = |Ddu|^2 + \sum_i \langle du\left(Ric_M\left(E_i\right)\right), du\left(E_i\right)\rangle$$
$$- \sum_{i,j} Sec_N(du(E_i) \wedge du(E_j)) \left|du(E_i) \wedge du(E_j)\right|^2.$$

Since $Ric_M \geq 0$ and $Sec_N \leq 0$,

$$\Delta |du|^2 \geq 0,$$

equality holding iff $Ddu = 0$. Use Stokes theorem with $X = |du|^2 \nabla |du|^2$:

$$0 = \int_M \mathrm{div}\,(X) \geq \int_M \left|\nabla |du|^2\right|^2 \geq 0 \Rightarrow |du| \equiv \text{const.}$$

and du is parallel. If $Ric_M(p_0) > 0$ then $d_{p_0}u = 0$ and this implies $du = 0$. Similarly if $Sec_N < 0$ and $du \neq 0$, since $du(E_i) \wedge du\left(E_j\right) = 0$ we obtain that $u(M)$ is 1-dimensional. Since $Ddu = 0 \Rightarrow u$ maps geodesics into geodesics$\Rightarrow$ $u(M) \subset \Gamma$ geodesic. If Γ is not closed then u is homotopically trivial. But (M cmpt) it can be shown that u **minimizes energy in its homotopy class**$\Rightarrow$ $u \equiv$ const. Contradiction. It is now easy to obtain $u(M) = \Gamma$. ■

Now, some classical applications.

Application I. We first illustrate a use of X as a target space.

Th. 3 (Preissman) *X cmpt, $Sec < 0$. Then $\mathbb{Z}^2 \not\subset \pi_1(X)$.*

Proof. By contradiction, $\mathbb{Z}^2 \subset \pi_1(X)$. Fix any injective homomorphism $\rho : \pi_1(T^2) \simeq \mathbb{Z}^2 \to \pi_1(X)$ with T^2 the flat torus. Since $Sec_X \leq 0$, by the general theory of aspherical spaces, we can assume that $\exists$smooth nonconst map $u : T^2 \to X$ which induces ρ up to some $\alpha \in \mathrm{Aut}(\pi_1(X))$, say $\alpha \circ \rho = u_\#$. By Eells-Sampson-Hartman, we can take u harmonic. Liouville Theorem$\Rightarrow$ $u(T^2) =$closed geodesic of X. Therefore, $u_\#$ maps the generators of $\pi_1(T^2)$ onto a single loop$\Rightarrow$ $u_\#$ is not injective. Contradiction. ∎

The flat-torus theorem by Lawson-Yau and Gromoll-Wolf can be obtained along the same line.

Application II. Now we illustrate a use of X as a source space.

Th. 4 (Lohkamp remark) *There is no metric $\langle , \rangle$ on $\mathbb{R}^m$ such that:*

(a) $\langle , \rangle = \mathrm{can}_{\mathbb{R}^m}$ *on* $\mathbb{R}^m \backslash \mathbb{B}_1$ *and*

(b.1) $Ric \geq 0$ *in* $\mathbb{B}_1$, *(b.2)* $Ric\,(x_0) > 0$ *for some* $x_0 \in \mathbb{B}_1$

Proof. By contradiction, suppose $\langle , \rangle$ exists. Cut an m-cube $\mathcal{C}$ centered at 0 s.t. $\mathbb{B}_1 \subset \mathcal{C}$. Glue the faces of $\mathcal{C}$ to obtain an m-torus X with $Ric_X \geq 0$ and $Ric_X\,(x_0) > 0$. Now, in the homotopy class of the identity map **id** : $X \to T^m$ there is a harmonic map $u : X \to T^m$. Liouville theorem$\Rightarrow$ $u \equiv$ **cnst** $\Rightarrow$ **id** homotopically trivial. Contradiction. ■

What is impressive is that there exists $\langle , \rangle$ on $\mathbb{R}^m$ satisfying (a) and $Ric < 0$ on $\mathbb{B}_1$ (**Lohkamp** $Ric < 0$**-balls**)

II. p-harmonic maps and functions

The previous examples involve (2-)harmonic maps. The concept was introduced by Eells-Sampson in the mid '60s and extends the notion of harmonic function.

Let $u : (M, \langle , \rangle_M) \to (N, \langle , \rangle_N)$ be a smooth map. The Hilbert-Schmidt norm of its differential $du \in \Gamma\left(T^*M \otimes u^{-1}(TN)\right)$ is denoted by $|du|$. Let $p > 1$.

Def. 1 *The map u is said to be p-**harmonic** if*

$$\Delta_p u := -\delta\left(|du|^{p-2}\, du\right) = 0,$$

*where δ is the formal adjoint of d with respect to the standard L^2-inner product on vector valued 1-forms. The operator $\Delta_p u$ is called the p-**Laplacian** (or p-**tension field**) of u. In case $u \in C^1$ the above condition has to be interpreted in the sense of distributions, i.e., $(\Delta_p u, \eta) = -\int_M \left\langle |du|^{p-2}\, du, d\eta \right\rangle$.*

Let X be a vector field along u with compact support in $\Omega \subset\subset M$. Define the variation with fixed boundary $u_t(x) = \exp_{u(x)} tX(x)$. Then

$$\left.\frac{d}{dt}\right|_{t=0} \int_\Omega |du_t|^p = -p \int_\Omega \langle \Delta_p u, X \rangle ,$$

proving that p-**harmonic maps are stationary points of the** p-**energy functional** for this kind of variations.

Note that

$$\Delta_p u = |du|^{p-2} \Delta u + du \left(\nabla |du|^{p-2} \right).$$

In the special case $N = \mathbb{R}$ one can also speak of p-**subharmonic function** whenever $\Delta_p u \geq 0$ and of p-**superharmonic function** if $\Delta_p u \leq 0$.

II.a. p-harmonic maps as "canonical" representatives

We are interested in **complete non-compact domains**. It is then natural to prescribe asymptotic (decay) properties to maps, more precisely on the energy of maps. Say that $f : M \to N$ has **finite p-energy** if $|df|^p \in L^1(M)$. According to results by R. Schoen and S.T. Yau, F. Burstall, B. White, S.W. Wei, p-harmonic maps can be considered as canonical representatives of homotopy class of maps with finite p-energy into nonpositively curved targets.

Th. 5 *Let $(M, \langle , \rangle_M)$ be complete and $(N, \langle , \rangle_N)$ be compact with $Sec_N \leq 0$. Fix a smooth map $f : M \to N$ with finite p-energy $|df|^p \in L^1(M)$, $p \geq 2$. Then, in the homotopy class of f, there exists a p-harmonic map $u \in C^{1,\alpha}(M, N)$ with $|du|^p \in L^1(M)$. If $p = 2$ then $u \in C^\infty(M, N)$.*

Some consequences and questions that arise naturally from the existence thm:

(a) **Trivial homotopy type**. Liouville type thms under geometric assumptions on $M \Rightarrow$ a map $f : M \to N$ with finite p-energy must be topologically trivial.

(b) **Comparison of homotopic p-harmonic maps**. How many p-harmonic maps with finite p-energy are there in a given homotopy class ?

In case $p = 2$ (harmonic case) both questions in the complete setting are answered in deep seminal works by Schoen-Yau (the compact case is due to P. Hartman). They proved:

(α) vanishing results for harmonic maps assuming that either $Ric_M \geq 0$ or M is a stable minimal hypersurface in $\mathbb{R}^{m+1}$;

(β) comparison of homotopic harmonic maps and uniqueness of the harmonic representative, assuming $\mathrm{vol}\,(M) < +\infty$.

Schoen-Yau vanishing results alluded to in (α) have been unified and extended by allowing a controlled amount of negative Ricci curvature.

The negative part of the curvature is measured via a spectral assumption. Suppose $Ric_M \geq -a(x)$. Let $\mathcal{L} = -\Delta - a(x)$. By definition

$$\lambda_1(\mathcal{L}) := \inf \left\{ \frac{\int |\nabla \varphi|^2 - a(x)\varphi^2}{\int \varphi^2} : \varphi \in C_c^\infty(M) \setminus \{0\} \right\}.$$

Th. 6 (P.-Rigoli-Setti [2]) *Let M be complete, noncmpt, $Ric_M \geq -a(x)$ with $\lambda_1(\mathcal{L}) \geq 0$. Let N be complete, $Sec_N \leq 0$. Then a harmonic map $u : M \to N$ with finite energy $|du| \in L^2$ must be constant.*

Proof. Starting point: Bochner formula+refined Kato (RHS)

(B) $$|du|\left(\Delta|du|+a(x)|du|\right)=|Ddu|^2-|\nabla|du||^2\geq\frac{1}{m}|\nabla|du||^2.$$

By assumption $\lambda_1(-\Delta-a(x))\geq 0$. According to FischerColbrie-Schoen

(FCS) $$\exists v>0:\Delta v+a(x)v=0.$$

In the spirit of the generalized maximum principle define

$$0\leq w=\frac{|du|}{v}\in L^2\left(M,v^2d\mathrm{vol}\right).$$

Then, the v^2-Laplacian of w satisfies

$$\Delta_{v^2}w:=v^{-2}\operatorname{div}\left(v^2\nabla w\right)\geq 0,$$

i.e. w is Δ_{v^2}-subharmonic.

By an L^2-Liouville Theorem (a la Yau) for Δ_{v^2}-subharmonic functions on the manifold with density $\left(M, v^2 d\mathrm{vol}\right)$, we obtain $w := \frac{|du|}{v} \equiv \mathbf{cnst}$

Whence, combining equations $(B) + (FCS)$ we deduce $\nabla |du| \equiv 0$ and either $|du| \equiv 0$ or $Ric_M \geq 0$.

By Calabi and Yau, $Ric_M \geq 0 \Rightarrow \mathrm{vol}(M) = +\infty$. Since $\mathbf{cnst} \equiv |du| \in L^2$ we must conclude $|du| = 0$. ■

Rmk 1 (P.-Rigoli-Setti [2]) *Actually, a very similar proof gives a more general vanishing result for $|du| \in L^{2\gamma}(M)$ and $\lambda_1(-\Delta - Ha(x)) \geq 0$ where γ and H are related by $(m-1)/m < \gamma \leq H$.*

Rmk 2 (on the spectral assumption) *Let* $\mathcal{L} = -\Delta - a(x)$. *Intuitively,* $\lambda_1(\mathcal{L}) \geq 0$ *relies on the fact that* $a_+(x) = \max\{a(x), 0\}$ *is small in some integral sense.*

For instance, assume an Euclidean L^2 *Sobolev inequality*

$$\|\varphi\|_{L^{\frac{2m}{m-2}}} \leq S \|\nabla\varphi\|_{L^2}, \forall\varphi \in C_c^\infty$$

for some $S > 0$. *Then, by Sobolev and Hölder inequalities,*

$$\int \left(|\nabla\varphi|^2 - a\varphi^2\right) \geq S^{-2} \|\varphi\|^2_{L^{\frac{2m}{m-2}}} - \|a_+\|_{L^{\frac{m}{2}}} \|\varphi\|^2_{L^{\frac{2m}{m-2}}}.$$

Thus

$$\|a_+\|_{L^{\frac{m}{2}}} \leq S^{-2} \Longrightarrow \lambda_1(\mathcal{L}) \geq 0.$$

As for general comparisons alluded to in (β) we have the following classical

Th. 7 (Schoen-Yau) *Let $u, v : M \to N$ be homotopic harmonic maps with $|du|^2 + |dv|^2 \in L^1$. If* $\mathrm{vol}\,(M) < +\infty$ *and $Sec_N < 0$ then, either $u = v$ or $u\,(M)\,, v\,(M) \subset \Gamma$ geodesic of N.*

Proof. Focus on some key points. Lift u, v to π_1-equivariant harmonic maps $u', v' : M' \to N'$ between universal coverings (π_1 acts by isometries). **Then** $(u', v') : M^{'} \to N' \times N'$ **is (equivariant) harmonic.** Define

$$\rho\,(x) = \mathrm{dist}_{N'} \circ (u'(x'), v'(x')) : M \to \mathbb{R}_{\geq 0}$$

where x' is any point in the fiber over x. Since N' is Cartan-Hadamard then $\mathrm{dist}_{N'}$ is convex. **Harmonic maps pull-back convex functions to subharmonic functions.** Therefore $\Delta\rho \geq 0$. Consider $h = \sqrt{1 + \rho^2}$. Then, $\Delta h \geq 0$. Moreover, $|du|^2 + |dv|^2 \in L^1 \Longrightarrow |\nabla h|^2 \in L^1$. Now use a Liouville-type theorem to deduce $h \equiv \mathrm{const}$. This implies $\rho \equiv \mathrm{const}$. Etc... $\blacksquare$

Project: extend Schoen-Yau theory to $p \neq 2$, thus obtaining topological information on higher energy maps.

Neither of the above proofs work in this general contest due to the (nonlinear) structure of the p-Laplace operator Δ_p. A list of difficulties:

(A) **Vanishing thm.** No refined Kato inequalities, low regularity of maps, Bochner formula presents new terms, no possibility of combine solutions of PDEs

(B) **Comparison theory.** $u, v : M \to N$ p-harmonic$\not\Rightarrow$ $(u, v) : M \to N \times N$ p-harmonic. Moreover:

Th. 8 (Veronelli [8]) *There exist Riemannian manifolds M, N, a convex function $H : N \to \mathbb{R}$ and a p-harmonic map $u : M \to N$, for some $p > 2$, such that $H \circ u : M \to \mathbb{R}$ is not p-subharmonic.*

II.b. New vanishing for finite-energy p-harmonic maps

Th. 9 (P.-Veronelli [6]) *Let* $u : M \to N$ *be a* C^1 p*-harmonic map,* $p \geq 2$*, with* $|du| \in L^q$*. Assume* N *complete with* $Sec_N \leq 0$ *and* M *complete,* $Ric_M \geq -a(x)$ *with* $\lambda_1(-\Delta - Ha(x)) \geq 0$ *for some* $H > q^2/4(q-1)$*. Then* $u \equiv$ const.

Cor. 1 (P.-Veronelli [6]) *Assume* N *cmpt with* $Sec_N \leq 0$ *and* M *complete with* $Ric_M \geq -a(x)$ *and* $\lambda_1(-\Delta - Ha(x)) \geq 0$ *for some* $H > p^2/4(p-1)$*. Then every* $f : M \to N$ *with* $|df|^p \in L^1$ *is homotopic to a constant.*

Proof (idea). Again we start with a Bochner-type inequality

$$|du|\,\Delta\,|du| + a\,(x)\,|du|^2 \geq -\,\langle du, d\Delta u\rangle\,,$$

where, since u is p-harmonic,

$$\Delta u = -\,(p-2)\,du\,(\nabla \log |du|)\,.$$

However:

(a) The RHS is not so nice as in the case $p = 2$ (no sign, no refined Kato). We need manipulations in integral form and a direct use of the spectral assumption with suitably chosen test-functions.

(b) u is not smooth. We use of a version of the approximation procedure by Duzaar-Fuchs. Idea: C^1-approximate u on $M_+ = \{|du| > 0\}$ by smooth u_k (**not** p-harmonic). Prove an L^p-Caccioppoli type inequality for $|du_k|$. The Caccioppoli contains an extra term that vanishes as $k \to +\infty$. Take limits to get a Caccioppoli for $|du|$. Duzaar-Fuchs teach us how to extend this inequality from M_+ to M. ■

II.c. New comparisons for finite-energy p-harmonic maps

We need to record some facts from **potential theory**. Let $1 < p < +\infty$.

Def. 2 *M is p-**parabolic** if* $\Delta_p u \geq 0$, $\sup_M u < +\infty \Rightarrow u \equiv$ const.

There are a number of equivalent definitions of parabolicity. The first one is classical and involves the concept of **capacity**.

Th. 10 *M is p-parabolic* $\Longleftrightarrow \forall K \subset\subset M$,

$$\mathrm{cap}_p\,(K) = \inf \int_M |\nabla \varphi|^p = 0,$$

the infimum being taken with respect to all $\varphi \in C_c^\infty$ such that $\varphi \geq 1$ on K.

Interpretation: every $K \subset\subset M$ has a small mass from the viewpoint of p-harmonic functions.

The next result is known as the **Kelvin-Nevanlinna-Royden criterion** (KNR for short). It is due to T. Lyons and D. Sullivan ($p = 2$) and V. Gol'dshtein and M. Troyanov ($p > 1$).

Th. 11 *M is p-parabolic* $\Longleftrightarrow \forall X \in L^{\frac{p}{p-1}}$ *vector field s.t.* $(\operatorname{div} X)_- \in L^1$,

$$\int_M \operatorname{div} X = 0.$$

Interpretation: from the viewpoint of X, the "boundary" of M is negligible (or X has zero "boundary values"). Therefore, a global version of Stokes theorem holds. In a sense, the celebrated **Gaffney(-Karp) version** of Stokes theorem is in the same spirit: take $p = +\infty$ and $X \in L^1$. Here ∞-**parabolicity = geodesic completeness** (thanks to Troyanov for this remark).

Proof (of $\Rightarrow$). Let $\Omega_j \subset\subset M$ be s.t. $\Omega_j \nearrow M$. Since

$$\mathrm{cap}_p(\Omega_1) = 0,$$

we can choose $0 \le \varphi_j \in C_c^\infty(\Omega_j)$ s.t.

$$\varphi_j = 1 \text{ on } \Omega_1, \text{ and } \left\| \nabla \varphi_j \right\|_{L^p} \to 0.$$

Apply Stokes theorem

$$0 = \int_M \mathrm{div}\left(X \varphi_j \right) = \int_M \varphi_j \, \mathrm{div}\, X + \int_M \left\langle X, \nabla \varphi_j \right\rangle .$$

To conclude, note that

$$\left| \int_M \left\langle X, \nabla \varphi_j \right\rangle \right| \le \|X\|_{L^{\frac{p}{p-1}}} \|\nabla \varphi\|_{L^p} \to 0$$

and

$$\int_M \varphi_j \, \mathrm{div}\, X \to \int_M \mathrm{div}\, X.$$

■

Geometric conditions insuring p-parabolicity rely on volume growth properties.

Th. 12 *Let* $(M, \langle , \rangle)$ *be complete. Consider the following growth conditions:*

(i) $\mathrm{vol}\,(B_R)^{\frac{1}{p-1}} = O\left(R^{1+\frac{1}{p-1}} \log R \log^{(2)} R \cdots \log^{(k)} R\right)$, *as* $R \to +\infty$.

(ii) $\displaystyle\int^{+\infty} \frac{R^{\frac{1}{p-1}}}{\mathrm{vol}(B_R)^{\frac{1}{p-1}}} dR = +\infty.$

(iii) $\displaystyle\int^{+\infty} \frac{dR}{\mathrm{area}(\partial B_R)^{\frac{1}{p-1}}} = +\infty.$

Then, $\quad (i) \underset{\not\Longleftarrow}{\Rightarrow} (ii) \underset{\not\Longleftarrow}{\Rightarrow} (iii) \underset{\not\Longleftarrow}{\Rightarrow} M$ *is* p*-parabolic.*

Ex. 1 *(recall Schoen-Yau Th.)* $\mathrm{vol}\,(M) < +\infty \Rightarrow$ p*-parabolicity,* $\forall p > 1$.

Here is our new global comparison for vector-valued maps.

Th. 13 (Holopainen-P.-Veronelli [1]) *Let* $u, v : M \to \mathbb{R}^n$ *satisfy*

$$\Delta_p u = \Delta_p v$$

and $|du| + |dv| \in L^p$*, for some* $p > 2$*. If* M *is* p*-parabolic then* $u - v \equiv$ const.

Proof (idea). Set $u(x_0) = v(x_0) = 0 \in \mathbb{R}^m$ and $\forall A > 0$, let

$$X_A := \left[dh_A|_{(u-v)} \circ \left(|du|^{p-2} du - |dv|^{p-2} dv \right) \right]^{\sharp},$$

where $h_A(y) := \sqrt{A + |y|^2}$. Apply the KNR criterion to deduce

$$\int_M \operatorname{div} X_A = 0.$$

Take the limit as $A \to +\infty$ and conclude

$$0 = \int_M |du - dv|^p .$$

■

Note that $\mathbb{R}^n$ is contractible, hence u, v are homotopic. Therefore if u, v are p-harmonic, the previous result follows from the next

Th. 14 (P.-Rigoli-Setti [4]) *Let $u : M \to N$ be a C^∞ p-harmonic map with $|du| \in L^p$, $p > 2$. Assume that M is p-parabolic and $Sec_N \leq 0$. If u is homotopic to a constant then $u \equiv$* const.

Very recently, the complete analogue of Schoen-Yau comparison has been finally obtained.

Th. 15 (Veronelli [9]) *Let $u, v : M \to N$ be C^1, homotopic, p-harmonic maps with $|du|^p + |dv|^p \in L^1$. If M is p-parabolic and $Sec_N < 0$ then, either $u = v$ or $u(M), v(M) \subset \Gamma$ geodesic of N.*

III. Sobolev inequalities and p-Laplacian

Say that $(M^m, \langle , \rangle)$ enjoys an $L^{p^*,p}$-**Sobolev inequality**, $1/p - 1/p^* = 1/m$, if

$$(\mathrm{SI}_p) \qquad \|\varphi\|_{L^{p^*}} \leq S_p \|\nabla\varphi\|_{L^p},$$

$\forall \varphi \in C_c^\infty$ and for some constant $S_p > 0$.

Rmk 3 *If M is complete with* vol$(B_R) \leq CR^m$ *then, by density arguments, (SI$_p$) extends to $\varphi \in L^{p^*}$ satisfying $|\nabla\varphi| \in L^p$.*

In $\mathbb{R}^m$ inequality (SI$_p$) holds and the explicit value of the optimal Sobolev constant K_p is known. In general, $K_p \leq S_p$ and **the validity of (SI$_p$)** (especially when combined with curvature conditions) **introduces a number of constrains on the geometry and the topology of** M. Let us consider some examples.

III.a. Rigidity under Sobolev inequalities

Th. 16 (Carron, Akutagawa) *Assume the validity of (SI$_p$). Then* $\exists\gamma > 0$ *such that* $\mathrm{vol}\,(B_R) \geq \gamma \mathrm{vol}\,(\mathbb{B}_R)$, *where* $\mathbb{B}_R \subset \mathbb{R}^m$.

Th. 17 (Anderson, Li) *Assume the validity of (SI$_p$) and* $\mathrm{vol}(B'_R) \lesssim \mathrm{vol}\,(\mathbb{B}_R)$ *where* $B'_R \subset M'$, M'*=the universal covering of* M *(e.g.* $Ric_M \geq 0$*). Then* $|\pi_1(M)| < +\infty$.

Ex. 2 *Discussed with Veronelli and Valtorta: in the above Th.,* $\pi_1(M)$ **can achieve all possible cardinalities.** *Let* $M = \mathbb{R}^3 \# \mathbb{L}^3$, *with* $\mathbb{L}^3 = \mathbb{S}^3/\mathbb{Z}_k$ *a lens space. Then, by Seifert-Van Kampen,* $\pi_1(M) \simeq \mathbb{Z}_k$ *and* M' *is a* k*-fold covering of* M. *Note that* M *satisfies the assumptions of Anderson-Li. Indeed, for some* $K \subset\subset M$, *(a)* $Ric_M = 0$ *on* $M \backslash K$ *and (b) the Sobolev inequality (SI$_p$) holds on* $M \backslash K$. *Since* M' *is a finite covering,* $Ric_{M'} = 0$ *off a compact set. Thus, volume comparison*$\Rightarrow$ $\mathrm{vol}(B'_R) \lesssim \mathrm{vol}\,(\mathbb{B}_R)$. *On the other hand (b)*$\Rightarrow$*(SI$_p$) on all of* M *by Carron (*$p = 2$*) and P.-Setti-Troyanov (*$p > 1$*).*

Th. 18 (Ledoux, Xia) *Let $Ric_M \geq 0$ and assume the validity of (SI$_p$). If S_p is sufficiently close to K_p then M is diffeomorphic to $\mathbb{R}^m$. If $S_p = K_p$ then M is isometric to $\mathbb{R}^m$.*

Proof (P.-Veronelli [7]). Crucial point: use the curvature condition to improve Carron volume estimate. Recall that, in $\mathbb{R}^m$, the **equality** in (SI$_p$) is realized by the (radial) Bliss-Aubin-Talenti functions

$$\varphi_\lambda(|x|) = \frac{\beta(m,p)\,\lambda^{\frac{m-p}{p^2}}}{\left(\lambda + |x|^{\frac{p}{p-1}}\right)^{\frac{m}{p}-1}}.$$

which satisfy

$$\int_{\mathbb{R}^m} \varphi_\lambda^{p^*} = 1,$$

and obey the nonlinear Yamabe equation

$$ {}^{\mathbb{R}^m}\Delta_p \varphi_\lambda = -K_p^{-p} \varphi_\lambda^{p^*-1}. $$

Define $\widehat{\varphi}_\lambda : M \to \mathbb{R}$ as $\widehat{\varphi}_\lambda(x) := \varphi_\lambda(r(x))$ and consider the vector field

$$ X_\lambda := \widehat{\varphi}_\lambda \left|\nabla \widehat{\varphi}_\lambda\right|^{p-2} \nabla \widehat{\varphi}_\lambda. $$

Then, by volume comparison, $X_\lambda \in L^1(M)$. Also, by Laplacian comparison,

$$ \Delta_p \widehat{\varphi} \geq -K_p^{-p} \widehat{\varphi}_\lambda^{p^*-1}. $$

Therefore,

$$ \operatorname{div} X_\lambda \geq \widehat{\varphi}_\lambda \Delta_p \widehat{\varphi}_\lambda \geq -K_p^{-p} \widehat{\varphi}_\lambda^{p^*} \in L^1(M) \, . $$

Using the **Karp version of Stokes theorem** we deduce

$$ 0 = \int_M \operatorname{div} X_\lambda \geq \int_M \left|\nabla \widehat{\varphi}_\lambda\right|^p - K_p^{-p} \int_M \widehat{\varphi}_\lambda^{p^*}, $$

that is

(*) $$\frac{\int_M |\nabla \widehat{\varphi}_\lambda|^p}{\int_M \widehat{\varphi}_\lambda^{p^*}} \leq K_p^{-p}.$$

On the other hand, by $\int_M \widehat{\varphi}_\lambda^{p^*} \leq 1$ and by definition of Sobolev constant S_p,

(**) $$\frac{\int_M |\nabla \widehat{\varphi}_\lambda|^p}{\int_M \widehat{\varphi}_\lambda^{p^*}} \geq \frac{\int_M |\nabla \widehat{\varphi}_\lambda|^p}{\left(\int_M \widehat{\varphi}_\lambda^{p^*}\right)^{\frac{p}{p^*}}} \geq S_p^{-p}.$$

It follows from (*)+(**) that

$$(K_p/S_p)^m \leq \int_M \widehat{\varphi}_\lambda^{p^*} \leq 1 = \int_{\mathbb{R}^m} \varphi_\lambda^{p^*}.$$

With the aid of integration by parts in polar-coordinates, and Bishop-Gromov:

$$\mathrm{vol}\,(B_R) \geq (K_p/S_p)^m\, \mathrm{vol}\,(\mathbb{B}_R)\,,\ \forall R > 0.$$

Now, if $K_p = S_p$, by volume comparison $\mathrm{vol}\,(B_R) = \mathrm{vol}\,(\mathbb{B}_R)$ and we conclude $M = \mathbb{R}^m$ using the equality case in Bishop-Gromov. ■

Rmk 4 (P.-Veronelli [7]) *A similar proof works if we replace* $Ric_M \geq 0$ *with the asymptotic condition* $Ric_M \geq -G(r(x))$ *where* $r(x) = d(x, o)$, $o \in M$ *is a reference origin, and* $G \geq 0$ *satisfies*

$$\int_0^{+\infty} tG(t)\,dt = b_0 < +\infty.$$

The corresponding rigidity (diffeomorphic rigidity) holds under the curvature requirement $Sec_M \geq -G(r(x))$ *when* b_0 *is sufficiently close to* 0.

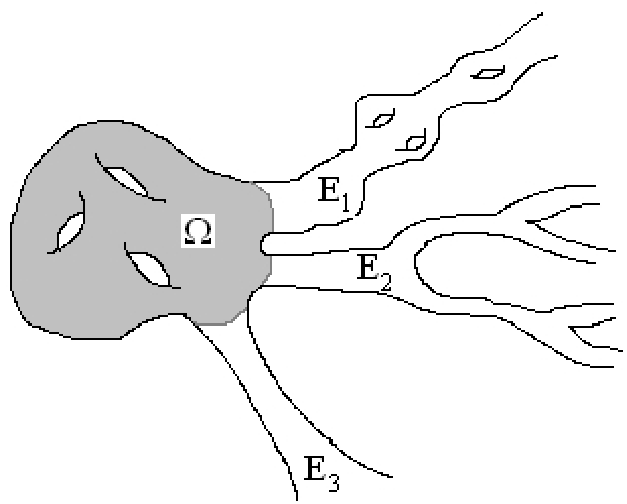
Conj. 1 (Ledoux) *Sharp volume estimate for the optimal Euclidean Sobolev constant holds without any curvature restriction. Namely:*

$$M \text{ complete, } (\mathrm{SI}_p) \text{ holds with } S_p = K_p \Rightarrow \mathrm{vol}(B_R) \geq \mathrm{vol}(\mathbb{B}_R).$$

III.b. Sobolev inequalities and topology at infinity

In the presence of the Sobolev inequality (SI_p) we are able to bulid a link between the analysis of p-harmonic functions and the topology at infinity of the underlying **complete, non-compact** manifold M.

Def. 3 *An* **end** E *of* M *with respect to* $\Omega \subset\subset M$ *is any of the unbounded connected components of* $M \backslash \Omega$*. Say that* M *is* **connected at infinity** *if, for evey smooth* $\Omega \subset\subset M$, $M \backslash \Omega$ *has exactly one end.*



Ex. 3 M =*universal covering of a cmpt manifold* N *with* $\pi_1(N) = \mathbb{Z}^{k \geq 2}$*. Then* M *is connected at infinity. Indeed, by Švarc-Milnor theory,* M *is quasi-isometric to the Cayley graph* $\mathcal{G}$ *of* $\pi_1(N)$*. The number of ends is a quasi-isometry invariant+*$\mathcal{G}$ *connected at infinity*$\Rightarrow$ M *connected at infinity.*

Ex. 4 $M = N \times \mathbb{R}$ *with* N *cmpt is disconnected at infinity.*

Ex. 5 $M = N \times \mathbb{R}^k$*, with* $k \geq 2$*, is connected at infinity.*

Ex. 6 (Cheeger-Gromoll) *Assume* $Ric_M \geq 0$ *and* $Ric_M(x) > 0$ *for some* $x \in M$*. Then* M *is connected at infinity. Indeed, if* $M \backslash \Omega$ *has two unbounded components* E_1, E_2 *then* M *contains a line. Since* $Ric_M \geq 0$ *we have isometric splitting* $M = N \times \mathbb{R}$*. This violates the assumption* $Ric_M(x) > 0$ *somewhere.*

In the presence of a general $L^{q,p}$-Sobolev inequality the curvature assumption in Cheeger-Gromoll Ex. 6 can be considerably relaxed.

Th. 19 (P.-Setti-Troyanov [5]) *Let $(M^m, \langle , \rangle)$ be a complete manifold satisfying the Sobolev inequality*

$$\|\varphi\|_{L^q} \leq S \|\nabla\varphi\|_{L^p},$$

for some $S > 0$ and $1/p - 1/q \leq 1/m$. Assume that $Ric \geq -a(x)$ where $a(x) \geq 0$ is small in the spectral sense

$$\lambda_1 (-\Delta - Ha(x)) \geq 0,$$

for some $H > p^2/4(p-1)$. Then, M is connected at infinity.

The proof inspires to harmonic function theory developed by P. Li, L.-F. Tam and collaborators. In case $p = 2$, versions of this result are due to P. Li and J. Wang, H.-D. Cao, Y. Shen, S. Zhu. See also [3]. It is done in three steps.

(a) Sobolev inequality (SI_p) $\Rightarrow$ every end E has infinite volume and is “large” in the sense of potential theory, i.e., E is p-**hyperbolic=not** p-**parabolic**.

(b) If M has two p-hyperbolic ends, construct a non-constant p-harmonic function $u \in C^1(M)$ satisfying $|\nabla u| \in L^p$.

(c) Curvature assumption + corresponding vanishing result $\Rightarrow$ $u \equiv$ **const.** (this has been already discussed)

(a) volume and potential theory of ends. Basic idea: since

$$\|\varphi\|_{L^q} \leq S \|\nabla\varphi\|_{L^p},$$

if we fix $K \subset\subset M$ and choose $\varphi = 1$ on K, then

$$\|\nabla\varphi\|_{L^p} \geq S^{-1}\mathrm{vol}(K)^{1/q}.$$

This means

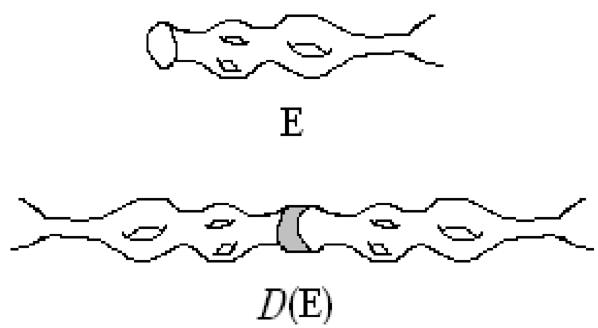
$$\mathrm{cap}_p(K) \geq S^{-1}\mathrm{vol}(K)^{1/q} > 0$$

and the manifold is p-hyperbolic. Also, by Carron-Akutagawa volume estimates,

$$\mathrm{vol}(B_R) \geq CR^m \to +\infty.$$

All these considerations can be localized on each end.

Def. 4 *Say that the end E of M is p-parabolic if its Riemannian double $\mathcal{D}(E)$ is p-parbolic as a manifold without boundary.*



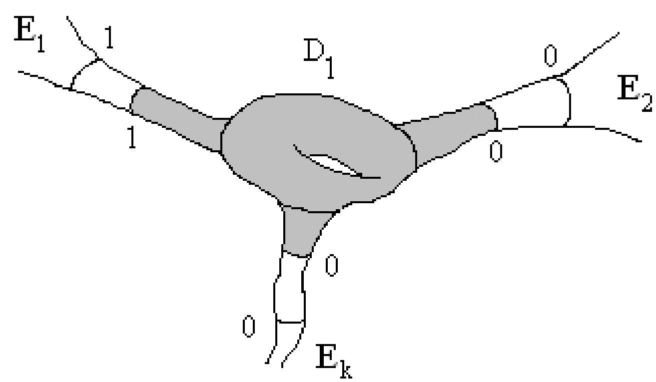
The key point to localize the above arguments on E is the next

Th. 20 (Carron, P.-Setti-Troyanov [5]) *The $L^{q,p}$ Sobolev inequality holds off a compact set if and only if it holds (with a different constant) on all of M*

(b) construction of the p-harmonic function

Let E_1, E_2,...,E_k be the ends of M, $k \geq 2$. By (a) they are p-hyperbolic. Take an exhaustion $D_j \nearrow M$. For every j solve the Dirichlet problem

$$\begin{cases} \Delta_p u_j = 0 & \text{on } D_j \\ u_j = 0 & \text{on } E_1 \cap \partial D_j \\ u_j = 1 & \text{on } (M \backslash E_1) \cap \partial D_j. \end{cases}$$



By the maximum principle $u_j \nearrow$ and, therefore, we can define

$$u(x) = \lim_j u_j(x).$$

Then:

1) u is p-harmonic by the Harnack principle.

2) Using the fact that there are at least two p-hyperbolic ends it can be shown that u is nonconstant.

3) Using capacitary arguments it follows $\left\|\nabla u_j\right\|_{L^p} \leq C$, $\forall j$. This implies $|\nabla u| \in L^p$.

This completes the proof of the Theorem.

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