The Dirac eta series and twisted eta invariants of $\mathbb{Z}_p$-manifolds and equivariant bordism

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The eta invariant and equivariant bordism of flat manifolds with cyclic holonomy group of odd prime order, *Annals of Global Analysis and Geometry (AGAG)*, **37**, 2010.

Outline

1. Introduction
2. $\mathbb{Z}_p$-manifolds
3. Spectral asymmetry: $\eta(s)$ and the η-invariant
4. Equivariant bordism
5. Appendix: number theoretical tools
6. Epilogue

η-invariant result

Theorem (A)

Let p be an odd prime and $0 \leq \ell \leq p-1$. Consider a spin $\mathbb{Z}_p$-manifold (M, ε) of dimension n. Then

$$\bar{\eta}_\ell \equiv 0 \quad mod\ \mathbb{Z}$$

unless $p = n = 3$. Furthermore,

$$\bar{\eta}_\ell - \bar{\eta}_0 \equiv 0 \quad mod\ \mathbb{Z}$$

Bordism result

Theorem (B)

Let $(M, \varepsilon, \sigma_p)$ and $(M, \varepsilon, \sigma_0)$ denote a $\mathbb{Z}_p$-manifold M equipped with a spin structure ε and with the canonical and the trivial $\mathbb{Z}_p$-structures respectively. Then

$$[(M, \varepsilon, \sigma_p)] - [(M, \varepsilon, \sigma_0)] = 0$$

in the reduced equivariant spin bordism group $\widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$

Settings

General setting

- M = (compact) Riemannian manifold
- $E \to M \quad = \quad$ vector bundle of M
- $D : \Gamma^\infty(E) \to \Gamma^\infty(E) \quad = \quad$ elliptic differential operator

Our interest

- M = compact flat manifold
- D = twisted spin Dirac operator

  [but also Laplacians and Dirac-type operators]

Spectrum

Let M be a compact Riemannian manifold

Definition

The **spectrum** of D on M is the set

$$Spec_D(M) = \{\lambda \in \mathbb{R} \,:\, Df = \lambda f,\ f \in \Gamma^\infty(E)\} = \{(\lambda, d_\lambda)\}$$

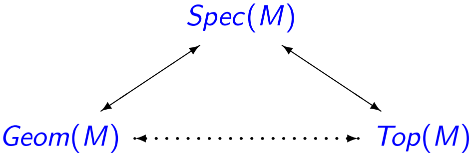
of eigenvalues counted with multiplicities

- $Spec_D(M) \subset \mathbb{R}$ is discrete
- $0 \leq |\lambda_1| \leq \cdots \leq |\lambda_n| \nearrow \infty$
- $d_\lambda = \dim H_\lambda < \infty, \qquad H_\lambda = \lambda$-eigenspace

Spectral geometry

Goal: to study

- $Spec(M)$
- relations between $Spec(M)$ with $Geom(M)$ and $Top(M)$



Some problems

of (our) interest

1. Computation of the spectrum
2. Isospectrality
3. **Spectral asymmetry** (this talk)

Definition

$Spec_D(M)$ is **asymmetric** $\quad \Leftrightarrow \quad \exists\, \lambda \neq 0$ such that $d_\lambda \neq d_{-\lambda}$

Eta series

To study this phenomenon Atiyah-Patodi-Singer '73 introduced

- The **eta series**:

$$\eta_D(s) = \sum_{\lambda \neq 0} \frac{sgn(\lambda)}{|\lambda|^s} = \sum_{\lambda \in \mathcal{A}} \frac{d_\lambda^+ - d_\lambda^-}{|\lambda|^s} \qquad Re(s) > \frac{n}{d}$$

  where $n = \dim M$, $d = \text{ord } D$

- has a meromorphic continuation to $\mathbb{C}$ called the **eta function**, also denoted by $\eta_D(s)$, with (possible) simple poles in $\{s = n - k : k \in \mathbb{N}_0\}$

Eta invariants

- the **eta invariant**:

$$\eta_D = \eta_D(0)$$

  - not trivial that $\eta(0) < \infty$ [APS '76], [Gilkey '81]
  - spectral invariant, globally defined
  - does not depend on the metric

- the **reduced eta invariant**:

$$\bar{\eta}_D = \frac{1}{2}(\eta_D + \dim\ker D)$$

Relation with Index Theorems

- For M closed, the Index Theorem of APS states

$$\mathrm{Ind}(D) = \int_M \alpha_0$$

- For M with boundary $\partial M = N$
(under certain boundary conditions)

$$\underbrace{\mathrm{Ind}(D)}_{top} = \underbrace{\int_M \alpha_0}_{geom} - \underbrace{\bar{\eta}_{D_N}}_{spec}$$

Relation with Index Theorem: spin Dirac operator

M with boundary N

- D = Dirac operator

$$\mathrm{Ind}(D) = \int_M \hat{A}(p) - \tfrac{1}{2}(\eta_{D_N} + h)$$

where $h = \dim\ker D_N$

- D = signature operator, $\dim M = 4k$

$$\mathrm{Sign}(D) = \int_M L(p) - \eta_{D_N}$$

Particular setting and notations

From now on we consider

- p = odd prime in $\mathbb{Z}$
- M = compact flat manifold with holonomy group $F \simeq \mathbb{Z}_p$
- ε = spin structure on M
- ρ_ℓ = character of $\mathbb{Z}_p$, $0 \leq \ell \leq p-1$
- D_ℓ = Dirac operator twisted by ρ_ℓ

Problems considered

Spectral asymmetry

for any (M, ε) compute:

1. the eta series $\eta_\ell(s)$ associated to D_ℓ
2. the reduced eta invariants $\bar{\eta}_\ell$
3. the relative eta invariants $\bar{\eta}_\ell - \bar{\eta}_0$

Bordism groups

in addition, can we say something about
the *reduced equivariant spin bordism group* $\widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$?

Compact flat manifolds

- A **flat manifold** is a Riemannian manifold with $K \equiv 0$
- Any compact flat n-manifold M is isometric to

$$M_\Gamma = \Gamma\backslash\mathbb{R}^n, \qquad \Gamma \simeq \pi_1(M)$$

where Γ is a *Bieberbach group*, i.e.
a **discrete, cocompact, torsion-free** subgroup of

$$\mathrm{I}(\mathbb{R}^n) \simeq \mathrm{O}(n) \rtimes \mathbb{R}^n$$

- $\gamma \in \Gamma \;\Rightarrow\; \gamma = BL_b$, with $B \in \mathrm{O}(n)$, $b \in \mathbb{R}^n$ and

$$BL_b \cdot CL_c = BCL_{C^{-1}b+c}$$

Algebraic properties

- The map

$$r : \mathrm{I}(\mathbb{R}^n) \to \mathrm{O}(n) \qquad BL_b \mapsto B$$

induces the exact sequence

$$0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1$$

- $\Lambda =$ lattice of $\mathbb{R}^n$ (the lattice of pure translations)
- $F \simeq \Lambda\backslash\Gamma \subset \mathrm{O}(n)$ is finite, called the **holonomy group** of Γ
- One says that M is an F-**manifold**
- fact:

$$n_B := \dim (\mathbb{R}^n)^B \geq 1 \qquad \forall\, BL_b \in \Gamma$$

Holonomy representation

- The action by conjugation on Λ by $F \simeq \Lambda\backslash\Gamma$

$$B\, L_\lambda\, B^{-1} = L_{B\lambda}$$

defines the **integral holonomy representation**

$$\rho : F \to GL_n(\mathbb{Z})$$

- This ρ is far from determining a flat manifold uniquely
- There are (already in dim 4) non-homeomorphic orientable flat manifolds $M_\Gamma, M_{\Gamma'}$ with the same integral holonomy representation, i.e.

$$\rho_\Gamma = \rho_{\Gamma'} \qquad \text{but} \qquad M_\Gamma \not\simeq M_{\Gamma'}$$

Geometric properties

Bieberbach theorems

- $T_\Lambda \to M_\Gamma, \qquad M_\Gamma = T_\Lambda/F = (\mathbb{R}^n/\Lambda)/(\Gamma/\Lambda)$
- diffeomorphic $\Leftrightarrow$ homeomorphic $\Leftrightarrow$ homotopically equivalent

$$M_\Gamma \simeq M_{\Gamma'} \quad \Leftrightarrow \quad \Gamma \simeq \Gamma' \quad \Leftrightarrow \quad \pi_n(M_\Gamma) = \pi_n(M'_\Gamma)$$

since $\pi_n(M_\Gamma) = 0$ for $n \geq 2$

- In each dimension, there is a finite number of affine equivalent classes of compact flat manifolds

Geometric properties

- Every finite group can be realized as the holonomy group of a compact flat manifold [Auslander-Kuranishi '57]
- Every compact flat manifold bounds, i.e.,
  if M^n is a compact flat manifold,
  then there is a N^{n+1} such that $\partial N = M$ [Hamrick-Royster '82]

The trivial example: 2-torus

Let

$$\Lambda = \mathbb{Z}^2 = \mathbb{Z}Le_1 \oplus \mathbb{Z}Le_2$$

the canonical lattice then

$$\mathbb{T}^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2 = \langle L_{e_1}, L_{e_2} \rangle \backslash \mathbb{R}^2 = \Lambda \backslash \mathbb{R}^2$$

is a 2-torus

The first non-trivial example

A $\mathbb{Z}_2$-manifold in dimension 2

The Klein bottle:

$$\mathbb{K}^2 = \langle \left[\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right] L_{\frac{e_2}{2}}, L_{e_1}, L_{e_2} \rangle \backslash \mathbb{R}^2$$

where

$$\Lambda = \mathbb{Z}^2 = \mathbb{Z} L e_1 \oplus \mathbb{Z} L e_2, \qquad F \simeq \langle \left[\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right] \rangle \simeq \mathbb{Z}_2$$

Note that

$$\mathbb{K}^2 \simeq \mathbb{Z}_2 \backslash \mathbb{T}^2$$

$\mathbb{Z}_p$-manifolds

We will now describe the $\mathbb{Z}_p$-manifolds M_Γ

- M_Γ satisfies

$$0 \to \Lambda \simeq \mathbb{Z}^n \to \Gamma \to \mathbb{Z}_p \to 1$$

- M_Γ can be thought to be constructed by

integral representations of $\mathbb{Z}_p = \mathbb{Z}[\mathbb{Z}_p]$-modules

- $\mathbb{Z}_p$-modules were classified by Reiner [Proc AMS '57]
- $\mathbb{Z}_p$-manifolds were classified by Charlap [Annals Math '65]

Reiner $\mathbb{Z}_p$-modules

Any $\mathbb{Z}_p$-module is of the form

$$\Lambda(a,b,c,\mathfrak{a}) := \mathfrak{a} \oplus (a-1)\,\mathcal{O} \oplus b\,\mathbb{Z}[\mathbb{Z}_p] \oplus c\,Id$$

where

- $a, b, c \in \mathbb{N}_0$, $a+b>0$
- ξ = primitive p^{th}-root of unity
- $\mathcal{O} = \mathbb{Z}[\xi]$ = ring of algebraic integers in $\mathbb{Q}(\xi)$
- $\mathfrak{a}$ = ideal in $\mathcal{O}$
- $\mathbb{Z}[\mathbb{Z}_p]$ = group ring over $\mathbb{Z}$
- Id = trivial $\mathbb{Z}_p$-module

$\mathbb{Z}_p$-actions

- The actions on the modules are given by **multiplication by** ξ
- In matrix form, the action of ξ on $\mathcal{O}$ and $\mathbb{Z}[\mathbb{Z}_p]$ are given by

$$C_p = \begin{pmatrix} 0 & & & & -1 \\ 1 & 0 & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 0 & -1 \\ & & & 1 & -1 \end{pmatrix} \in GL_{p-1}(\mathbb{Z}), \qquad J_p = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \in GL_p(\mathbb{Z})$$

- The action on $\mathfrak{a}$ is given by $C_{p,a} \in GL_{p-1}(\mathbb{Z})$ with $C_{p,a} \sim C_p$
- $n_{C_p} = n_{C_{p,a}} = 0$, $n_{J_p} = 1$

Properties of $\mathbb{Z}_p$-manifolds

Proposition

Let $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ be a $\mathbb{Z}_p$-manifold with $\Gamma = \langle\gamma, \Lambda\rangle$, $\gamma = BL_b$. Then

- $(BL_b)^p = L_{b_p}$ *where* $b_p = \sum_{j=0}^{p-1} B^j b \in L_\Lambda \smallsetminus (\sum_{j=0}^{p-1} B^j)\Lambda$
- *As a $\mathbb{Z}_p$-module,* $\Lambda \simeq \Lambda(a, b, c, \mathfrak{a})$, *with* $c \geq 1$ *and*

$$n = a(p-1) + bp + c$$

- *a, b, c are uniquely determined by the $\simeq$ class of Γ*
- *Γ is conjugate in $I(\mathbb{R}^n)$ to a Bieberbach group $\tilde\Gamma = \langle\tilde\gamma, \Lambda\rangle$ with $\tilde\gamma = BL_{\tilde b}$ where* $B\tilde b = \tilde b$ *and* $\tilde b \in \frac{1}{p}\Lambda \smallsetminus \Lambda$

Properties of $\mathbb{Z}_p$-manifolds

Proposition (continued)

- $n_B = 1 \Leftrightarrow (b, c) = (0, 1)$ *and in this case* $\gamma = BL_b$ *can be chosen so that* $b = \frac{1}{p} e_n$
- *One has*

$$H_1(M_\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^{b+c} \oplus \mathbb{Z}_p^a$$
$$H^1(M_\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^{b+c}$$

  and hence $n_B = b + c = \beta_1$
- M_Γ *is orientable*

The models

For our purposes, **it will suffice to work with the "models"**

$$M_{p,a}^{b,c}(\mathfrak{a}) = \langle BL_{\frac{e_n}{p}}, \Lambda_{p,a}^{b,c}(\mathfrak{a}) \rangle \backslash \mathbb{R}^n$$

where

$$\Lambda_{p,a}^{b,c}(\mathfrak{a}) = X_{\mathfrak{a}} L_{\mathbb{Z}^n} X_{\mathfrak{a}}^{-1} = X_{\mathfrak{a}} \mathbb{Z}^{n-c} \overset{\perp}{\oplus} \mathbb{Z}^c$$

for some $X_{\mathfrak{a}} \in GL_n(\mathbb{R})$

The models

and

$$B = \operatorname{diag}(\underbrace{B_p, \ldots, B_p}_{a+b}, \underbrace{1, \ldots, 1}_{b+c}) \in SO(n)$$

with

$$B_p = \begin{pmatrix} B(\frac{2\pi}{p}) & & & \\ & B(\frac{2\cdot 2\pi}{p}) & & \\ & & \ddots & \\ & & & B(\frac{2q\pi}{p}) \end{pmatrix} \qquad q = [\tfrac{p-1}{2}]$$

$$B(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad t \in \mathbb{R}$$

Exceptional $\mathbb{Z}_p$-manifolds

- In Charlap's classification there is a distinction between *exceptional* and *non-exceptional* $\mathbb{Z}_p$-manifolds
- A $\mathbb{Z}_p$-manifold is called **exceptional** if

$$\Lambda \simeq \Lambda(a, 0, 1, \mathfrak{a})$$

- We will use exceptional $\mathbb{Z}_p$-manifolds $M_{p,a}^{0,1}(\mathfrak{a})$, hence odd dimension

$$n = a(p-1) + 1$$

Example: the "tricosm"

- It is the only 3-dimensional $\mathbb{Z}_3$-manifold
- It is exceptional: $M_{3,1} = M_{3,1}^{0,1}(\mathcal{O})$, with $\mathcal{O} = \mathbb{Z}[\frac{2\pi i}{3}]$
- As a $\mathbb{Z}_3$-module, $\Lambda \simeq \mathbb{Z}[e^{\frac{2\pi i}{3}}] \oplus \mathbb{Z}$
- with $\mathbb{Z}_3$-(integral) action given by $C = \begin{pmatrix} 0 & -1 & \\ 1 & -1 & \\ & & 1 \end{pmatrix}$
- Thus

$$M_{3,1} = \langle BL_{\frac{e_3}{3}}, L_{f_1}, L_{f_2}, L_{e_3} \rangle \backslash \mathbb{R}^3$$

with

$$B = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & \\ \sqrt{3}/2 & -1/2 & \\ & & 1 \end{pmatrix} \in \mathrm{SO}(3)$$

where f_1, f_2, e_3 is a $\mathbb{Z}$-basis of $\Lambda_{3,1} = X\mathbb{Z}^2 \oplus \mathbb{Z}$ and $X \in \mathrm{GL}_3(\mathbb{R})$ is such that $X\,C\,X^{-1} = B$

Spin group and maximal torus

- The **spin group** $\mathrm{Spin}(n)$ is the universal covering of $\mathrm{SO}(n)$

$$\pi : \mathrm{Spin}(n) \xrightarrow{2} \mathrm{SO}(n) \qquad n \geq 3$$

- A maximal torus of $\mathrm{Spin}(n)$ is given by

$$T = \{x(t_1, \ldots, t_m) : t_1, \ldots, t_m \in \mathbb{R}, m = [\tfrac{n}{2}]\}$$

$$x(t_1, \ldots, t_m) := \prod_{j=1}^{m} (\cos t_j + \sin t_j \, e_{2j-1} e_{2j})$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$

Spin representations

The **spin representation** of Spin(n) is the restriction (L_n, S_n) of any irreducible representation of $Cliff(\mathbb{C}^n)$

- $\dim_{\mathbb{C}} \mathsf{S}_n = 2^{[n/2]}$
- (L_n, S_n) is irreducible if n is odd
- (L_n, S_n) is reducible if n is even, $\mathsf{S}_n = \mathsf{S}_n^+ \oplus \mathsf{S}_n^-$
- $L_n^{\pm} := L_{n|\mathsf{S}_n^{\pm}}$ are the **half-spin representations**

Characters of spin representations

Characters of L_n, $L_n^{\pm}$ are known on the maximal torus

Lemma (Miatello-P, TAMS '06)

$$\chi_{L_n}(x(t_1,\ldots,t_m)) = 2^m \prod_{j=1}^{m} \cos t_j$$

$$\chi_{L_n^{\pm}}(x(t_1,\ldots,t_m)) = 2^{m-1}\Big(\prod_{j=1}^{m} \cos t_j \pm i^m \prod_{j=1}^{m} \sin t_j\Big)$$

where $m = [n/2]$

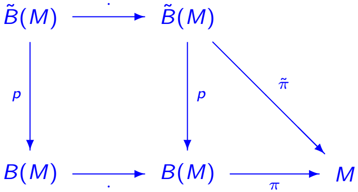
Spin structures

Let

- M = orientable Riemannian manifold
- $B(M)$ = SO(n)-principal bundle of oriented frames on M

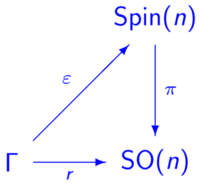
A **spin structure** on M is

- an equivariant double covering $p : \tilde{B}(M) \to B(M)$
- $\tilde{B}(M)$ is a Spin(n)-principal bundle of M, i.e.



Spin structures on compact flat manifolds

- The spin structures on M_Γ are in a 1–1 correspondence with group homomorphisms ε commuting the diagram



Spin structures on compact flat manifolds

- Let M_Γ be a $\mathbb{Z}_p$-manifold, $\Gamma = \langle \gamma, \Lambda = \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n \rangle$. Then ε is determined by

$$\varepsilon(\gamma) \qquad \text{and} \qquad \delta_j := \varepsilon(L_{f_j}) \in \{\pm 1\} \qquad 1 \leq j \leq n$$

- $\exists$ necessary and sufficient conditions on $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ for defining a spin structure on M_Γ when $F \simeq \mathbb{Z}_2^k$ or $F \simeq \mathbb{Z}_n$

  [Miatello-P, Math Z. '04]

Spin structures on flat manifolds

- Not every flat manifold is spin [Vasquez '70]
- Flat tori are spin [Friedrich '84]
- $\mathbb{Z}_2^k$-manifolds are not spin (in general) but

  $\mathbb{Z}_2$-manifolds are always spin [Miatello-P '04]

Spin structures on $\mathbb{Z}_p$-manifolds

Existence

- every F-manifold with $|F|$ odd is spin (Vasquez, JDG '70)
- thus every $\mathbb{Z}_p$-manifold is spin

Number

- if M is spin, the spin structures are classified by $H^1(M, \mathbb{Z}_2)$
- If M is a $\mathbb{Z}_p$-manifold, since $H^1(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{b+c}$,

$$\#\{\text{spin structures of } M\} = 2^{b+c} = 2^{\beta_1}$$

Spin structures on the models $M_{p,a}^{b,c}(\mathfrak{a})$

Proposition

A $\mathbb{Z}_p$-manifold M admits exactly 2^{β_1} spin structures, only one of which is of trivial type.

If $M = M_{p,a}^{b,c}(\mathfrak{a})$, its 2^{b+c} spin structures are explicitly given by

$$\varepsilon_{|\Lambda} = \big(\underbrace{1,\ldots,1}_{a(p-1)}, \underbrace{\delta_1,\ldots,\delta_1}_{p}, \ldots, \underbrace{\delta_b,\ldots,\delta_b}_{p}, \delta_{b+1},\ldots,\delta_{b+c-1}, (-1)^{h+1}\big)$$

$$\varepsilon(\gamma) = (-1)^{(a+b)[\frac{q+1}{2}]+h+1}\, x_{a+b}\big(\tfrac{\pi}{p}, \tfrac{2\pi}{p}, \ldots, \tfrac{q\pi}{p}\big)$$

with $h = 1, 2$.

Spin structures on the models $M_{p,a}^{b,c}(\mathfrak{a})$

Notations:

- $\varepsilon_{|\Lambda} = \big(\varepsilon(L_{f_1}), \ldots, \varepsilon(L_{f_n})\big) \in \{\pm 1\}^n$
- $x_a(t_1, t_2, \ldots, t_q) := x(\underbrace{t_1, t_2, \ldots, t_q}_{1}, \ldots, \underbrace{t_1, t_2, \ldots, t_q}_{a}) \quad a \in \mathbb{N}$

Spin structures on exceptional $\mathbb{Z}_p$-manifolds

Remark

If M is an exceptional $\mathbb{Z}_p$-manifold, i.e. $M \simeq M_{p,a}^{0,1}(\mathfrak{a})$, then M has only 2 spin structures $\varepsilon_1, \varepsilon_2$ given by

$$\varepsilon_{h|\Lambda} = \left(1, \ldots, 1, (-1)^{h+1}\right)$$
$$\varepsilon_h(\gamma) = (-1)^{a[\frac{q+1}{2}]+h+1} \, x_a\left(\tfrac{\pi}{p}, \tfrac{2\pi}{p}, \ldots, \tfrac{q\pi}{p}\right)$$

with $h = 1, 2$. In particular, ε_1 is of trivial type

Twisted Dirac operators on flat manifolds

- Let (M_Γ, ε) = compact flat spin n-manifold
  $\rho : \Gamma \to U(V)$ = unitary representation such that $\rho_{|\Lambda} = 1$
- The **spin Dirac operator twisted by** ρ is

$$D_\rho = \sum_{i=1}^{n} L_n(e_i) \frac{\partial}{\partial x_i}$$

where $\{e_1, \ldots, e_n\}$ is an o.n.b. of $\mathbb{R}^n$

Twisted Dirac operators on flat manifolds

D_ρ acts on smooth sections of the spinor bundle

$$D_\rho : \Gamma^\infty(\mathcal{S}_\rho(M_\Gamma, \varepsilon)) \to \Gamma^\infty(\mathcal{S}_\rho(M_\Gamma, \varepsilon))$$

where

$$\mathcal{S}_\rho(M_\Gamma, \varepsilon) = \Gamma\backslash(\mathbb{R}^n \times (\mathsf{S}_n \otimes V)) \to \Gamma\backslash\mathbb{R}^n$$

$$\gamma \cdot (x, \omega \otimes v) = \big(\gamma x, L\big(\varepsilon(\gamma)\big)(\omega) \otimes \rho(\gamma)v\big)$$

Spectrum of D_ρ on compact flat manifolds

- The spectrum of D_ρ on (M_Γ, ε) is

$$Spec_{D_\rho}(M_\Gamma, \varepsilon) = \{\lambda = \pm 2\pi\mu : \mu = ||v||, v \in \Lambda_\varepsilon^*\}$$

with multiplicities

$$d_{\rho,\mu}^{\pm}(\Gamma, \varepsilon)$$

where

$$\Lambda_\varepsilon^* = \{u \in \Lambda^* : \varepsilon(L_\lambda) = e^{2\pi i \lambda \cdot u} \quad \forall\, \lambda \in \Lambda\}$$

Multiplicities

Theorem (Miatello-P, TAMS '06)

(i) *for $\mu > 0$:*

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = \frac{1}{|F|} \sum_{\gamma = BL_b \in \Lambda\backslash\Gamma} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\, \chi_{L^{\pm\sigma(u,x_\gamma)}_{n-1}}(x_\gamma)$$

with $(\Lambda^*_{\varepsilon,\mu})^B = \{v \in \Lambda^*_\varepsilon \,:\, Bv = v, ||v|| = \mu\}$

(ii) *for $\mu = 0$:*

$$d_{\rho,0}(\Gamma,\varepsilon) = \begin{cases} \frac{1}{|F|} \sum_{\gamma\in\Lambda\backslash\Gamma} \chi_\rho(\gamma)\, \chi_{L_n}(\varepsilon(\gamma)) & \varepsilon_{|\Lambda} = 1 \\ 0 & \varepsilon_{|\Lambda} \neq 1 \end{cases}$$

Notations

From now on we consider

- $p = 2q + 1$ an odd prime
- $M = \mathbb{Z}_p$-manifold of dim n
- ε_h = spin structure on M, $1 \leq h \leq 2^{b+c}$
- For $0 \leq \ell \leq p-1$, the characters

$$\rho_\ell : \mathbb{Z}_p \to \mathbb{C}^* \qquad k \mapsto e^{\frac{2\pi i k\ell}{p}}$$

- D_ℓ = Dirac operator twisted by ρ_ℓ
- $d^{\pm}_{\ell,\mu,h} := d^{\pm}_{\rho_\ell,\mu}(M, \varepsilon_h)$

The eta series for $\mathbb{Z}_p$-manifolds

- We will compute $\eta_{D_\ell}(s)$ for any $\mathbb{Z}_p$-manifold, $0 \le \ell \le p-1$
- Recall that

$$\eta_{\ell,h}(s) = \sum_{\pm 2\pi\mu \in \mathcal{A}} \frac{d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}}{(2\pi\mu)^s}$$

- Although the expressions for $d^\pm_{\ell,\mu,h}$ are not explicit, the differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$ can be computed

An important reduction for flat manifolds

- By a result in [Miatello-P, TAMS '06],

$$n_B > 1 \quad \forall\, BL_b \in \Gamma \quad \Rightarrow \quad Spec_D(M) \text{ is symmetric}$$

thus

$$d^{+}_{\ell,\mu,h} = d^{-}_{\ell,\mu,h} \quad \Rightarrow \quad \eta_D(s) \equiv 0$$

- For $\mathbb{Z}_p$-manifolds, since $n_B = 1 \Leftrightarrow (b,c) = (0,1)$ then

$$\eta(s) \equiv 0 \quad \text{for } \textbf{non-exceptional } \mathbb{Z}_p\text{-manifolds}$$

An important reduction

- Thus, it **suffices** to compute

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}, \qquad \eta_{\ell,h}(s), \qquad \eta_{\ell,h}$$

for the **exceptional** $\mathbb{Z}_p$-manifolds only

- In particular,

we can assume that $M = M^{0,1}_{p,a}(\mathfrak{a})$

(i.e. $b = \frac{1}{p} e_n$)

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Key lemma

For an exceptional $\mathbb{Z}_p$-manifold (M, ε_h) we have

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = \kappa_{p,a} \sum_{k=1}^{p-1} (-1)^{k(h+1)} \left(\tfrac{k}{p}\right)^a e^{\frac{2\pi i k\ell}{p}} \sin\left(\tfrac{2\pi\mu k}{p}\right)$$

where

$$\kappa_{p,a} = (-1)^{(\frac{p^2-1}{8})a+1}\, i^{m+1}\, 2\, p^{\frac{a}{2}-1}$$

and $(\frac{\cdot}{p})$ is the Legendre symbol

Sketch of proof I

- Apply the general multiplicity formula to this case

$$d^{\pm}_{\ell,\mu,h} = \frac{1}{p} \sum_{k=0}^{p-1} e^{\frac{2\pi i \ell k}{p}} \sum_{u \in (\Lambda^*_{\varepsilon_h,\mu})^{B^k}} e^{-2\pi i u \cdot b_k} \chi_{L_{n-1}^{\pm\sigma(u,x_{\gamma^k})}}(\varepsilon_h(\gamma^k))$$

- note that $(\Lambda^*_{\varepsilon_h})^{B^k} = (\mathbb{Z} + \frac{1}{h})e_n$ and hence

$$(\Lambda^*_{\varepsilon_h,\mu})^{B^k} = \{\pm\mu e_n\}$$

Sketch of proof II

- Thus, we get

$$d^{\pm}_{\ell,\mu,h} = \tfrac{1}{p}\left(2^{m-1}|\Lambda^*_{\varepsilon_h,\mu}| + \sum_{k=1}^{p-1} e^{\frac{2\pi i k\ell}{p}} S^{\pm}_{\mu,h}(k)\right)$$

where

$$S^{\pm}_{\mu,h}(k) := e^{\frac{-2\pi i \mu k}{p}} \chi_{L^{\pm}_{n-1}}(\varepsilon_h(\gamma^k)) + e^{\frac{2\pi i \mu k}{p}} \chi_{L^{\mp}_{n-1}}(\varepsilon_h(\gamma^k))$$

(only 2-terms sums)

Sketch of proof III

- Note that

$$\varepsilon_h(\gamma^k) = (-1)^{s_{h,k}}\, x_a\Big(\frac{k\pi}{p}, \frac{2k\pi}{p}, \dots, \frac{qk\pi}{p}\Big)$$

for $1 \le k \le p$, where

$$s_{h,k} := k([\tfrac{q+1}{2}]a + h + 1)$$

- Compute

$$\chi_{L^{\pm}_{n-1}}(\varepsilon_h(\gamma^k)) = (-1)^{s_{h,k}}\, 2^{m-1}\Big\{\Big(\prod_{j=1}^{q}\cos(\tfrac{jk\pi}{p})\Big)^a \pm i^m \Big(\prod_{j=1}^{q}\sin(\tfrac{jk\pi}{p})\Big)^a\Big\}$$

- compute the blue trigonometric products

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Proposition

Let (M, ε_h) be an exceptional $\mathbb{Z}_p$-manifold and put $r = [\frac{n}{4}]$. Then

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} \in p\mathbb{Z}$$

for $0 \le \ell \le p-1$, $h = 1, 2$. More precisely,

(i) *If a* **is even** *then*

$$d^+_{0,\mu,h} - d^-_{0,\mu,h} = 0$$

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = \begin{cases} \pm(-1)^r p^{\frac{a}{2}} & p \mid h(\ell \mp \mu) \\ 0 & \textit{otherwise} \end{cases}$$

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Proposition (continued)

(ii) *If* a **is odd** *then*

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = (-1)^{q+r} \left(\left(\tfrac{2(\ell-\mu)}{p}\right) - \left(\tfrac{2(\ell+\mu)}{p}\right) \right) p^{\frac{a-1}{2}}$$

In particular,

$$d^+_{0,\mu,h} - d^-_{0,\mu,h} = \begin{cases} 0 & p \equiv 1\,(4) \\ (-1)^r\, 2\left(\tfrac{2\mu}{p}\right) p^{\frac{a-1}{2}} & p \equiv 3\,(4) \end{cases}$$

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Sketch of proof

- Rewrite $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$ in terms of "character Gauß sums"

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = \begin{cases} -i^{m+1}\, 2\, p^{\frac{a}{2}-1} F_h^{\chi_0}(\ell, c_\mu) & a \text{ even} \\ -i^{m+1}\, 2\, p^{\frac{a}{2}-1} (-1)^{\left(\frac{p^2-1}{8}\right)} F_h^{\chi_p}(\ell, c_\mu) & a \text{ odd} \end{cases}$$

where
χ_0 = trivial character mod p
χ_p = quadratic character mod p

- Compute the blue Gauß sums

The eta series $\eta_{\ell,h}(s)$

$\eta_{\ell,h}(s)$ can be computed in terms of **Hurwitz zeta functions**

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

where

$$\alpha \in (0,1] \qquad Re\,(s) > 1$$

Note that

$$\zeta(s,1) = \zeta(s)$$

The eta series $\eta_{\ell,h}(s)$

Theorem

Let (M, ε_h) *be an* **exceptional** $\mathbb{Z}_p$*-manifold. Put* $r = [\frac{n}{4}]$, $t = [\frac{p}{4}]$.

(i) *If* a **is even** *then* $\eta_{0,1}(s) = \eta_{0,2}(s) = 0$ *and for* $\ell \neq 0$

$$\eta_{\ell,1}(s) = \frac{(-1)^r}{(2\pi p)^s}\, p^{\frac{a}{2}} \left(\zeta(s, \tfrac{\ell}{p}) - \zeta(s, \tfrac{p-\ell}{p})\right)$$

$$\eta_{\ell,2}(s) = \begin{cases} \frac{(-1)^r}{(2\pi p)^s}\, p^{\frac{a}{2}} \left(\zeta(s, \frac{1}{2} + \frac{\ell}{p}) - \zeta(s, \frac{1}{2} - \frac{\ell}{p})\right) & 1 \leq \ell \leq q \\ \frac{(-1)^r}{(2\pi p)^s}\, p^{\frac{a}{2}} \left(\zeta(s, \frac{1}{2} - \frac{p-\ell}{p}) - \zeta(s, \frac{1}{2} + \frac{p-\ell}{p})\right) & q < \ell < p \end{cases}$$

The eta series $\eta_{\ell,h}(s)$

Theorem (continued)

(ii) *If* a **is odd** *then*

$$\eta_{\ell,1}(s) = \frac{(-1)^{t+r}}{(2\pi p)^s}\, p^{\frac{a-1}{2}} \sum_{j=1}^{p-1} \Big(\big(\tfrac{\ell-j}{p}\big) - \big(\tfrac{\ell+j}{p}\big) \Big)\, \zeta(s, \tfrac{j}{p})$$

$$\eta_{\ell,2}(s) = \frac{(-1)^{q+r}}{(\pi p)^s}\, p^{\frac{a-1}{2}} \sum_{j=0}^{p-1} \Big(\big(\tfrac{2\ell-(2j+1)}{p}\big) - \big(\tfrac{2\ell+(2j+1)}{p}\big) \Big)\, \zeta(s, \tfrac{2j+1}{2p})$$

In particular, $\eta_{0,h}(s) = 0$ *for* $p \equiv 1\,(4)$

Computation of eta invariants

We will now compute, for $0 \le \ell \le p-1$,

- the eta invariants

$$\eta_\ell = \eta_\ell(0)$$

- the reduced eta invariants

$$\bar{\eta}_\ell = \frac{\eta_\ell + \dim\ker D_\ell}{2} \mod \mathbb{Z}$$

- the relative eta invariants

$$\bar{\eta}_\ell - \bar{\eta}_0$$

Eta invariants $\eta_{\ell,h}$

Theorem

Let (M, ε_h) be an exceptional $\mathbb{Z}_p$-manifold. Put $r = [\frac{n}{4}]$, $t = [\frac{p}{4}]$.

(i) *If* a **is even** *then*

$$\eta_{0,h} = 0$$

and for $\ell \neq 0$

$$\eta_{\ell,1} = (-1)^r \, p^{\frac{a}{2}-1} \, (p - 2\ell)$$
$$\eta_{\ell,2} = (-1)^r \, p^{\frac{a}{2}-1} \, 2\big([\tfrac{2\ell}{p}]p - \ell\big)$$

Eta invariants $\eta_{\ell,h}$

Theorem (continued)

(ii) *If* a **is odd** *then*

$$\eta_{\ell,1} = \begin{cases} (-1)^{t+r+1} p^{\frac{a-1}{2}} \, S_1^-(\ell,p) & p \equiv 1\,(4) \\ (-1)^{t+r} p^{\frac{a-1}{2}} \big(S_1^+(\ell,p) + \frac{2}{p} \sum_{j=1}^{p-1} \big(\frac{j}{p}\big) j \big) & p \equiv 3\,(4) \end{cases}$$

$$\eta_{\ell,2} = \begin{cases} (-1)^{q+r+1} p^{\frac{a-1}{2}} \big(S_2^-(\ell,p) - \big(\frac{2}{p}\big) S_1^-(\ell,p) \big) & p \equiv 1\,(4) \\ (-1)^{q+r} p^{\frac{a-1}{2}} \big\{ S_2^+(\ell,p) + \big(\frac{2}{p}\big) \, S_1^+(\ell,p) + \\ \qquad + \big(1 - \big(\frac{2}{p}\big)\big) \frac{2}{p} \sum_{j=1}^{p-1} \big(\frac{j}{p}\big) j \big\} & p \equiv 3\,(4) \end{cases}$$

Eta invariants $\eta_{\ell,h}$

where

Notation

$$S_1^{\pm}(\ell,p) := \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right)$$

$$S_2^{\pm}(\ell,p) := \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right] p-2\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right] p-1} \left(\tfrac{j}{p}\right)$$

Eta invariants $\eta_{\ell,h}$

Sketch of proof

- Evaluate $\eta_{\ell,h}(s)$ in $s = 0$, using that $\zeta(0,\alpha) = \frac{1}{2} - \alpha$
- a even trivial, a odd:

$$\eta_{\ell,1}(0) = (-1)^{t+r}\, p^{\frac{a-1}{2}} \sum_{j=1}^{p-1} \left(\left(\tfrac{\ell-j}{p}\right) - \left(\tfrac{\ell+j}{p}\right) \right) \left(\tfrac{1}{2} - \tfrac{j}{p} \right)$$

$$\eta_{\ell,2}(0) = (-1)^{q+r}\, p^{\frac{a-1}{2}} \sum_{j=0}^{p-1} \left(\left(\tfrac{2\ell-(2j+1)}{p}\right) - \left(\tfrac{2\ell+(2j+1)}{p}\right) \right) \left(\tfrac{p-1}{2p} - \tfrac{j}{p} \right)$$

- Study the violet sums!

Eta invariants $\eta_{\ell,h}$: integrality, parity

Corollary

(i) *If* $(p, a) \neq (3, 1)$ *then*

$$\eta_{\ell,h} \in p\mathbb{Z}$$

Furthermore, $\eta_{0,h}$ *is even,* $\eta_{\ell,1}$ *is odd and* $\eta_{\ell,2}$ *is even* $(\ell \neq 0)$

(ii) *If* $(p, a) = (3, 1)$ *then*

$$\eta_{\ell,1} = \begin{cases} -2/3 & \ell = 0 \\ 1/3 & \ell = 1, 2 \end{cases} \qquad \eta_{\ell,2} = 4/3 \quad \ell = 0, 1, 2$$

That is,

$$\eta_{\ell,h} \equiv \tfrac{1}{3} \ \ mod \ \ \mathbb{Z}$$

dim ker D_ℓ

It is known that

$$\begin{aligned} \dim\ker D &= \text{multiplicity of the 0-eigenvalue } (d_0) \\ &= \#\text{ independent harmonic spinors } (h_0) \end{aligned}$$

So, we will compute

$$d_{\ell,0,h} := d_{\ell,0}(\varepsilon_h) = \dim\ker D_{\ell,h}$$

dim ker D_ℓ

Proposition

Let (M, ε_h) be **any** *$\mathbb{Z}_p$-manifold, $1 \le h \le 2^{b+c}$.*

Then $d_{\ell,0,\varepsilon_h} = 0$ for $h \neq 1$ and

$$d_{\ell,0,\varepsilon_1} = \frac{2^{\frac{b+c-1}{2}}}{p}\Big(2^{(a+b)q} + (-1)^{(\frac{p^2-1}{8})(a+b)}\,(p\delta_{\ell,0} - 1)\Big) \in \mathbb{Z}$$

In particular, if $b + c > 1$ then $d_{\ell,0,1}$ is even for any $0 \le \ell \le p-1$ while if $b + c = 1$ then $d_{0,0,1}$ is even and $d_{\ell,0,1}$ is odd for $\ell \neq 0$.

dim ker D_ℓ

sketch of proof:

- We have

$$d_{\ell,0,\varepsilon_1} = \frac{1}{p} \sum_{k=0}^{p-1} e^{\frac{2\pi i k\ell}{p}} \chi_{L_n}(\varepsilon_1(\gamma^k))$$

and

$$\varepsilon_1(\gamma^k) = (-1)^{k[\frac{q+1}{2}](a+b)} x_{a+b}\left(\frac{k\pi}{p}, \frac{2k\pi}{p}, \dots, \frac{qk\pi}{p}\right)$$

- Thus

$$d_{\ell,0,1} = \frac{2^m}{p} \sum_{k=0}^{p-1} (-1)^{k[\frac{q+1}{2}](a+b)} \left(\prod_{j=1}^{q} \cos\left(\frac{jk\pi}{p}\right)\right)^{a+b} e^{\frac{2\pi i k\ell}{p}}$$

The reduced eta invariant of $\mathbb{Z}_p$-manifolds

Recall that $\bar{\eta}_{\ell,h} = \frac{1}{2}(\eta_{\ell,h} + d_{\ell,0,h}) \quad \mod \mathbb{Z}$

Studying the parities of $\eta_{\ell,h}$ and $d_{0,\ell,h}$ we get our **main result**

Theorem (A)

Let p be an odd prime and $0 \le \ell \le p-1$. Let M be a $\mathbb{Z}_p$-manifold with spin structure ε_h, $1 \le h \le 2^{b+c}$. Then

$$\bar{\eta}_{\ell,h} = \begin{cases} \frac{2}{3} \mod \mathbb{Z} & p = n = 3 \\ 0 \mod \mathbb{Z} & \text{otherwise} \end{cases} \qquad \forall\, \ell, h$$

Moreover, the relative eta invariants are

$$\bar{\eta}_{\ell,h} - \bar{\eta}_{0,h} = 0$$

The exception: the tricosm

- There is only one $\mathbb{Z}_p$-manifold with non-trivial reduced eta invariant
- The tricosm: the only 3-dimensional $\mathbb{Z}_3$-manifold $M = M_{3,1}$

Case $\ell = 0$

- In the untwisted case $\ell = 0$ we have a better insight
- and there is a closer relation with number theory

We can put

- $\eta(s)$ is in terms of the L-**function**

$$L(s, \chi_p) = \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n^s} = \frac{1}{p^s} \sum_{j=0}^{p} \left(\tfrac{j}{p}\right) \zeta(s, \tfrac{j}{p})$$

- η is in terms of **class numbers** h_{-p} of **imaginary quadratic fields** $\mathbb{Q}(\sqrt{-p}) = \mathbb{Q}(i\sqrt{p})$

Case $\ell = 0$, eta series

Theorem ([Miatello-P, PAMQ '08])

Let (M, ε_h) be a $\mathbb{Z}_p$-manifold of dimension n.

If M is exceptional and $n \equiv p \equiv 3\,(4)$, $a \equiv 1\,(4)$ then

$$\eta_{0,1}(s) = \frac{-2}{(2\pi p)^s}\, p^{\frac{a-1}{2}}\, L(s, \chi_p)$$
$$\eta_{0,2}(s) = \frac{2}{(2\pi p)^s}\, p^{\frac{a-1}{2}}\, \left(1 - \left(\tfrac{2}{p}\right) 2^s\right) L(s, \chi_p)$$

In particular,

$$\eta_{0,2}(s) = \left(\left(\tfrac{2}{p}\right) 2^s - 1\right) \eta_{0,1}(s)$$

Otherwise we have $\eta_{0,1}(s) = \eta_{0,2}(s) \equiv 0$

Case $\ell = 0$, eta invariants

Theorem ([Miatello-P, PAMQ '08])

In the non-trivial case before we have $\eta \in p\mathbb{Z}$ except for $n = p = 3$ (the tricosm). More precisely:

(i) *If $p = 3$ then $\eta_{0,1} = -2 \cdot 3^{\frac{a-3}{2}}$ and $\eta_{\varepsilon_2} = 4 \cdot 3^{\frac{a-3}{2}}$*

(ii) *If $p \geq 7$ then*

$$\eta_{0,1} = -2\, p^{\frac{a-1}{2}}\, h_{-p}$$

$$\eta_{0,2} = \left(\left(\tfrac{2}{p}\right) - 1\right) \eta_{\varepsilon_1} = \begin{cases} 0 & p \equiv 7\ (8) \\ 4\, p^{\frac{a-1}{2}}\, h_{-p} & p \equiv 3\ (8) \end{cases}$$

where h_{-p} = the class number of $\mathbb{Q}(\sqrt{-p})$

Case $\ell = 0$, trigonometric expressions

Proposition ([Miatello-P, PAMQ '08])

The eta invariants of an exceptional $\mathbb{Z}_p$-manifold (M, ε_h) can be expressed in the following ways

$$\eta_{0,1} = -p^{\frac{a-2}{2}} \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) \cot\left(\tfrac{\pi k}{p}\right) = -p^{\frac{a-2}{2}} \sum_{k=1}^{p-1} \cot\left(\tfrac{\pi k^2}{p}\right)$$

$$\eta_{0,2} = p^{\frac{a-1}{2}} \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) \csc\left(\tfrac{\pi k}{p}\right)$$

Spin bordism

Let (M_i, ε_i), $i = 1, 2$, be compact oriented spin manifolds of dimension n.

- (M_1, ε_1) and (M_2, ε_2) are **spin bordant** if there exists a compact spin manifold (N, ε) such that

$$\partial N = M_1 \cup -M_2$$

$$\varepsilon_{|M_i} = \varepsilon_i, \quad i = 1, 2$$

- The **spin bordism group** is

$$MSpin_n = \{[(M^n, \varepsilon)]\}$$

$\mathbb{Z}_p$-structures

Let M be a closed compact oriented manifold

- An **equivariant $\mathbb{Z}_p$-structure** σ on M is a principal $\mathbb{Z}_p$-bundle

$$\mathbb{Z}_p \to P \to M$$

- For example, the **trivial $\mathbb{Z}_p$-structure** σ_0

$$\mathbb{Z}_p \to M \times \mathbb{Z}_p \to M$$

Equivariant spin bordism

Let $(M_i, \varepsilon_i, \sigma_i)$, $i = 1, 2$, closed compact oriented spin manifolds with equivariant $\mathbb{Z}_p$-structures.

- $(M_1, \varepsilon_1, \sigma_1)$ is $\mathbb{Z}_p$-**equivariant spin bordant** to $(M_2, \varepsilon_2, \sigma_2)$ if there is a compact oriented spin manifold (N, ε, σ) such that

$$\partial N = M_1 \cup -M_2$$

$$\varepsilon_{|M_i} = \varepsilon_i, \quad \sigma_{|M_i} = \sigma_i, \qquad i = 1, 2$$

- The **equivariant spin bordism group** is

$$MSpin_n(B\mathbb{Z}_p) = \{[(M^n, \varepsilon, \sigma)]\}$$

Reduced equivariant spin bordism

- The forgetful map

$$\mathcal{F} : MSpin_n(B\mathbb{Z}_p) \to MSpin_n$$

given by

$$[(M, \varepsilon, \sigma)] \mapsto [(M, \varepsilon)]$$

- The **reduced equivariant bordism group** is

$$\begin{aligned} \tilde{M}Spin_n(B\mathbb{Z}_p) &= \ker \mathcal{F} \\ &= \{[(M, \varepsilon, \sigma)] : [(M, \varepsilon)] = 0 \text{ in } MSpin_n\} \end{aligned}$$

Reduced equivariant spin bordism

- We have the natural isomorphism

$$MSpin_n(B\mathbb{Z}_p) \simeq \tilde{M}Spin_n(B\mathbb{Z}_p) \oplus MSpin_n$$

- Let $\pi : MSpin_n(B\mathbb{Z}_p) \to \tilde{M}Spin_n(B\mathbb{Z}_p)$ be the projection

$$\pi(M, \varepsilon, \sigma) = [(M, \varepsilon, \sigma)] - [(M, \varepsilon, \sigma_0)]$$

Reduced equivariant spin bordism

Theorem (Gilkey '88)

Let M^n be an oriented spin manifold with an equivariant $\mathbb{Z}_p$-structure on it, p odd prime. Let $\mathcal{M} := (M, \varepsilon, \sigma)$. Let $1 \leq \ell \leq p-1$ and let τ be a representation of the spin group. Then

(i) $(\bar{\eta}^\tau_\ell - \bar{\eta}^\tau_0)(\mathcal{M})$ takes values in $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

(ii) If $\pi(\mathcal{M}) = 0$ in $\tilde{M}Spin_n(B\mathbb{Z}_p)$, then $(\bar{\eta}^\tau_\ell - \bar{\eta}^\tau_0)(\mathcal{M}) = 0$ in $\mathbb{R}/\mathbb{Z}$.

(iii) If the twisted relative eta invariants $(\bar{\eta}^\tau_\ell - \bar{\eta}^\tau_0)(\mathcal{M})$ vanish for all τ and ℓ, then $\pi(\mathcal{M})$ vanishes in $\tilde{M}Spin_n(B\mathbb{Z}_p)$.

Reduced equivariant spin bordism of $\mathbb{Z}_p$-manifolds

Theorem (A) + (iii) of Theorem Gilkey'88 implies the **second main result**, Theorem (B):

Theorem (Gilkey)

Let $(M, \varepsilon, \sigma_p)$ and $(M, \varepsilon, \sigma_0)$ denote a $\mathbb{Z}_p$-manifold M equipped with a spin structure ε and with the natural and the trivial $\mathbb{Z}_p$-structures

$$\sigma_p : \mathbb{Z}_p \to T_\Lambda \to M, \qquad \sigma_0 : \mathbb{Z}_p \to M \times \mathbb{Z}_p \to M$$

Then

$$[(M, \varepsilon, \sigma_p)] - [(M, \varepsilon, \sigma_0)] = 0 \text{ in } \widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$$

Legendre symbol

Definition

For p an odd prime, the **Legendre symbol** of k mod p is

$$\left(\frac{k}{p}\right) := \begin{cases} 1 & \text{if } x^2 \equiv k\,(p) \text{ has a solution} \\ -1 & \text{if } x^2 \equiv k\,(p) \text{ does not have a solution} \end{cases}$$

if $(k,p) = 1$ and $(\frac{k}{p}) = 0$ otherwise

We have

$$(\tfrac{2}{p}) = (-1)^{\frac{p^2-1}{8}} \qquad (\tfrac{-1}{p}) = (-1)^{\frac{p-1}{2}}$$

Trigonometric products

Lemma

Let $p = 2q + 1$ *be an odd prime,* $k \in \mathbb{N}$ *with* $(k, p) = 1$*. Then*

$$(i) \qquad \prod_{j=1}^{q} \sin\left(\tfrac{jk\pi}{p}\right) = (-1)^{(k-1)(\frac{p^2-1}{8})} \left(\frac{k}{p}\right) 2^{-q} \sqrt{p}$$

$$(ii) \qquad \prod_{j=1}^{q} \cos\left(\tfrac{jk\pi}{p}\right) = (-1)^{(k-1)(\frac{p^2-1}{8})} 2^{-q}$$

Sketch of proof

(i) use

- identities of $\Gamma(z)$

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

$$(2\pi)^{\frac{d-1}{2}}\,\Gamma(z) = d^{z-\frac{1}{2}}\,\Gamma(\tfrac{z}{d})\Gamma(\tfrac{z+1}{d})\cdots\Gamma(\tfrac{z+(d-1)}{d})$$

- Gauß Lemma

$$(-1)^{\sum_{j=1}^{(p-1)/2}[\frac{jk}{p}]} = (-1)^{(k-1)(\frac{p^2-1}{8})}\left(\frac{k}{p}\right)$$

(ii) follows from (i)

Classical character Gauß sums

Definition

For $\ell \in \mathbb{N}_0$ the **character Gauß sum** is

$$G(\ell, p) := G(\chi_p, \ell) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) e^{\frac{2\pi i \ell k}{p}}$$

We have

$$G(\ell, p) = \begin{cases} \left(\frac{\ell}{p}\right) \sqrt{p} & p \equiv 1\ (4) \\ i\left(\frac{\ell}{p}\right) \sqrt{p} & p \equiv 3\ (4) \end{cases}$$

Modified character Gauß sums

Definition

For $p \in \mathbb{P}$, $\ell \in \mathbb{N}_0$, $c \in \mathbb{N}$, $1 \le h \le 2$, χ a character mod p we define

$$G_h^{\chi}(\ell) := \sum_{k=1}^{p-1} (-1)^{k(h+1)} \, \chi(k) \, e^{\frac{\pi i k \,(2\ell+\delta_{h,2})}{p}}$$

$$F_h^{\chi}(\ell, c) := \sum_{k=1}^{p-1} (-1)^{k(h+1)} \, \chi(k) \, e^{\frac{2\pi i \ell k}{p}} \, \sin\left(\frac{\pi k\,(2c+\delta_{h,2})}{p}\right)$$

We want to compute $G_h^{\chi}(\ell)$ and $F_h^{\chi}(\ell, c)$ for

- $\chi = \chi_0 =$ **trivial character** mod p
- $\chi = \chi_p =$ **quadratic character** mod p given by $\left(\frac{\cdot}{p}\right)$

The sums $G_h^{\chi}(\ell)$

$$G_1^{\chi_0}(\ell) = \sum_{k=1}^{p-1} e^{\frac{2\ell\pi i k}{p}}$$

$$G_2^{\chi_0}(\ell) = \sum_{k=1}^{p-1} (-1)^k \, e^{\frac{(2\ell+1)\pi i k}{p}}$$

$$G_1^{\chi_p}(\ell) = \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi i k}{p}}$$

$$G_2^{\chi_p}(\ell) = \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) e^{\frac{(2\ell+1)\pi i k}{p}}$$

The sums $G_h^{\chi_0}(\ell)$

Proposition

We have,

$$G_1^{\chi_0}(\ell) \equiv G_2^{\chi_0}(\ell) \equiv p-1 \quad mod\ p$$

More precisely,

$$G_1^{\chi_0}(\ell) = \begin{cases} p-1 & p \mid \ell \\ -1 & p \nmid \ell \end{cases}$$

$$G_2^{\chi_0}(\ell) = \begin{cases} p-1 & p \mid 2\ell+1 \\ -1 & p \nmid 2\ell+1 \end{cases}$$

The sums $G_h^{\chi_p}(\ell)$

Proposition

We have

$$G_1^{\chi_p}(\ell) = \delta(p) \left(\frac{\ell}{p}\right) \sqrt{p}$$
$$G_2^{\chi_p}(\ell) = \delta(p) \left(\frac{2}{p}\right) \left(\frac{2\ell+1}{p}\right) \sqrt{p}$$

where

$$\delta(p) := \begin{cases} 1 & p \equiv 1 \ (4) \\ i & p \equiv 3 \ (4) \end{cases}$$

In particular, $G_1^{\chi_p}(\ell) = 0$ *if* $p \mid \ell$ *and* $G_2^{\chi_p}(\ell) = 0$ *if* $p \mid 2\ell + 1$

The sums $F_h^{\chi}(\ell, c)$

$$F_1^{\chi_0}(\ell, c) = \sum_{k=1}^{p-1} e^{\frac{2\ell\pi ik}{p}} \sin\left(\tfrac{2c\pi k}{p}\right)$$

$$F_2^{\chi_0}(\ell, c) = \sum_{k=1}^{p-1} (-1)^k \, e^{\frac{2\ell\pi ik}{p}} \sin\left(\tfrac{(2c+1)\pi k}{p}\right)$$

$$F_1^{\chi_p}(\ell, c) = \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi ik}{p}} \sin\left(\tfrac{2c\pi k}{p}\right)$$

$$F_2^{\chi_p}(\ell, c) = \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi ik}{p}} \sin\left(\tfrac{(2c+1)\pi k}{p}\right)$$

The sums $F_h^{\chi_0}(\ell, c)$

Proposition

We have

1. *If* $p \mid \ell$ *then* $F_h^{\chi_0}(\ell, c) = 0$ *for* $h = 1, 2$
2. *If* $p \nmid \ell$ *then*

$$F_1^{\chi_0}(\ell, c) = \begin{cases} \pm i\, \frac{p}{2} & \text{if } p \mid \ell \mp c \\ 0 & \text{otherwise} \end{cases}$$

$$F_2^{\chi_0}(\ell, c) = \begin{cases} \pm i\, \frac{p}{2} & \text{if } p \mid 2(\ell \mp c) \mp 1 \\ 0 & \text{otherwise} \end{cases}$$

The sums $F_h^{\chi_p}(\ell, c)$

Proposition

We have

$$F_1^{\chi_p}(\ell,c) = i\,\delta(p)\,\Big(\big(\tfrac{\ell-c}{p}\big) - \big(\tfrac{\ell+c}{p}\big)\Big)\,\frac{\sqrt{p}}{2}$$

$$F_2^{\chi_p}(\ell,c) = i\,\delta(p)\,\big(\tfrac{2}{p}\big)\Big(\big(\tfrac{2(\ell-c)-1}{p}\big) - \big(\tfrac{2(\ell+c)+1}{p}\big)\Big)\,\frac{\sqrt{p}}{2}$$

In particular, if $p \mid \ell$ *then*

$$F_1^{\chi_p}(\ell,c) = \begin{cases} 0 & p \equiv 1\,(4) \\ \big(\tfrac{c}{p}\big)\sqrt{p} & p \equiv 3\,(4) \end{cases}$$

$$F_2^{\chi_p}(\ell,c) = \begin{cases} 0 & p \equiv 1\,(4) \\ \big(\tfrac{2}{p}\big)\big(\tfrac{2c+1}{p}\big)\sqrt{p} & p \equiv 3\,(4) \end{cases}$$

Sums involving Legendre symbols

For $0 \leq \ell \leq p-1$, we want to compute the sums

Definition

$$S_1(\ell,p) := \sum_{j=1}^{p-1} \left(\left(\tfrac{\ell-j}{p}\right) - \left(\tfrac{\ell+j}{p}\right) \right) j$$

$$S_2(\ell,p) := \sum_{j=0}^{p-1} \left(\left(\tfrac{2\ell-(2j+1)}{p}\right) - \left(\tfrac{2\ell+(2j+1)}{p}\right) \right) j$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\tfrac{k\ell \pm j}{p}\right) = -\left(\tfrac{k\ell}{p}\right) \qquad k \in \mathbb{Z}$$

$$\sum_{j=0}^{p-1} \left(\tfrac{2\ell \pm (2j+1)}{p}\right) = 0$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\tfrac{\ell+j}{p}\right) j = p \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j$$

$$\sum_{j=1}^{p-1} \left(\tfrac{\ell-j}{p}\right) j = \left(\tfrac{-1}{p}\right) \left(p \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j \right)$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\tfrac{2\ell+j}{p}\right) j = p \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right] p-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j$$

$$\sum_{j=1}^{p-1} \left(\tfrac{2\ell-j}{p}\right) j = \left(\tfrac{-1}{p}\right) \left(p \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right] p-2\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j \right)$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=0}^{p-1} \left(\tfrac{2\ell \pm (2j+1)}{p}\right) j = \sum_{j=1}^{p-1} \left(\tfrac{2\ell \pm j}{p}\right) j - \left(\tfrac{2}{p}\right) \sum_{j=1}^{p-1} \left(\tfrac{\ell \pm j}{p}\right) j$$

Sums involving Legendre symbols

Proposition

$$S_1(\ell,p) = \begin{cases} p\, S_1^-(\ell,p) & p \equiv 1\,(4) \\ -p\, S_1^+(\ell,p) - 2\sum_{j=1}^{p-1} (\frac{j}{p}) j & p \equiv 3\,(4) \end{cases}$$

$$S_2(\ell,p) = \begin{cases} p\Big(S_2^-(\ell,p) - (\frac{2}{p}) S_1^-(\ell,p)\Big) & p \equiv 1\,(4) \\ -p\Big(S_2^+(\ell,p) - (\frac{2}{p}) S_1^+(\ell,p)\Big) + & \\ \qquad +2\big((\frac{2}{p}) - 1\big)\sum_{j=1}^{p-1} (\frac{j}{p}) j & p \equiv 3\,(4) \end{cases}$$

Sums involving Legendre symbols

where we have used the notations

$$S_1^{\pm}(\ell,p) := \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right)$$

$$S_2^{\pm}(\ell,p) := \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right] p-2\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right] p-1} \left(\tfrac{j}{p}\right)$$

Note that

$$S_1^{\pm}(0,p) = S_1^{\pm}(0,p) = 0$$

since $\sum_{1\leq j\leq p-1} \left(\frac{j}{p}\right) = 0$

Sums involving Legendre symbols

Dirichlet's class number formula

We recall

$$\frac{1}{p}\sum_{j=0}^{p-1}\left(\frac{j}{p}\right)j = -2\frac{h_{-p}}{\omega_{-p}} = \begin{cases} -h_{-p} & p \geq 5, \\ -2/3 & p = 3, \end{cases}$$

where

- h_{-p} = class number of $\mathbb{Q}(\sqrt{-p}) \subset \mathbb{Q}(\xi_p)$,
- ω_{-p} = the number of p^{th}-roots of unity of $\mathbb{Q}(\sqrt{-p})$.

In fact, and $h_{-3} = 1$, $\omega_{-3} = 6$ and $\omega_{-p} = 2$ for $p \geq 5$.

Sums involving Legendre symbols

Corollary

For $p \geq 5$,

$$S_1(0,p) = \begin{cases} 0 & p \equiv 1\,(4) \\ -2h_{-p} & p \equiv 3\,(4) \end{cases}$$

$$S_2(\ell,p) = \begin{cases} 0 & p \equiv 1\,(4) \\ 2\big(\big(\tfrac{2}{p}\big) - 1\big)\,h_{-p} & p \equiv 3\,(4) \end{cases}$$

Summary of results

We have

1. considered the "models" $M_{p,a}^{b,c}(\mathfrak{a})$ of $\mathbb{Z}_p$-manifolds
2. given an explicit description of the **spin strucures** of $M_{p,a}^{b,c}(\mathfrak{a})$
3. explicitly computed, for twisted Dirac operators D_ℓ acting on an arbitrary $\mathbb{Z}_p$-manifold $(M_\Gamma, \varepsilon_h)$, the following
   - the **eta series** $\eta_{\ell,h}(s)$
   - the **eta invariants** $\eta_{\ell,h}$
   - the **number of independent harmonic spinors** $d_{\ell,0,h}$
   - the **reduced eta invariants** $\bar{\eta}_{\ell,h} = 0$ (except for $M_{3,1}$)
   - the **relative eta invariants** $\bar{\eta}_{\ell,h} - \bar{\eta}_{0,h} = 0$
4. applied this to study the **reduced $\mathbb{Z}_p$-equivariant spin bordism group** $\widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$ of $\mathbb{Z}_p$-manifolds

Note on methodology

★ There are indirect methods to compute η-invariants (representation techniques, computing $Ind(D)_{geo} - Ind(D)_{top}$)

★ However, we have performed the **direct approach**, that is, we have explicitly computed

1. the spectrum $\pm 2\pi\mu$, $d^{\pm}_{\ell,\mu,h}$
2. the eta series $\eta_\ell(s) = \frac{1}{(2\pi)^s} \sum_{\mu\neq 0} \frac{d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}}{|\mu|^s}$
3. the different eta invariants

$$\eta_\ell, \qquad \bar\eta_\ell = \tfrac{1}{2}(\eta_\ell + \dim\ker D_\ell) \mod \mathbb{Z}, \qquad \bar\eta_\ell - \bar\eta_0$$

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Thanks!