About the Lorentzian version of Nash's theorem on isometric embeddings

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Santiago de Compostela, December '10,
in honour to Professor Peter Gilkey

0. Introduction

Theorem

(Nash, Ann. Math.'56.) *Any smooth Riemannian manifold* (M, g) *can be isometrically embedded in [an arbitrarily small open subset* U *of]* $\mathbb{R}^N$, *for some* N.

Notes:

- Smooth: C^∞ —but C^3 is enough
- Value of N: if $m =$dim (M), Nash's bound was $N = (m+1)(3m(m+1)/2 + 4m)$
- Günther's '89 bound: max $\{2m + m(m+1)/2, m+5+m(m+1)/2\}$ (optimal?, it can be lowered in many particular cases)

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We do not worry about these bounds but for the following:

Is a Lorentzian version available?

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Independent simple arguments by Greene (Memoirs AMS'70) and Clarke (Proc. London'70) show:

Theorem

Any smooth manifold M endowed with a pseudo-metric (or, equivalently a possibly degenerate quadratic form) can be isometrically embedded in semi-Euclidean space $\mathbb{R}^N_\nu$ for sufficiently large dimension N and index ν.

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(just by reducing the problem to the Riemannian one).
However, the question is not so trivial if the index ν is not allowed to be arbitrary. That is, we focus in:

Which Lorentzian manifolds can be isometrically embedded in some $\mathbb{L}^N$?

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About this problem:

1. Our viewpoint will be global; locally such embeddings always exist (see for example JC Díaz Ramos & E García Río '04 about consequences for the curvature in Gilkey's book '01, further developed JC Díaz Ramos, E García Río, B. Fiedler, P. Gilkey '05).

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3. The most important class of Lorentzian manifolds (from the viewpoint of hyperbolic equations, General Relativity, etc.) are the *globally hyperbolic spacetimes*
4. Clarke '70 claimed that any globally hyperbolic spacetime can be isometrically embedded in $\mathbb{L}^N$.
5. However, his proof was affected by the so-called *folk problems* of causally-constructed functions.

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Theorem

Any globally hyperbolic s-t. admits a steep temporal function τ

by using techniques which are not affected by the folk problems of smoothability. Moreover, τ will be *Cauchy*, recall:

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Folk problems and structure of globally hyperbolic (M, g):

1. Cornerstone: $\exists$ a Cauchy time function (R. Geroch, JMP'70)
   $\Longrightarrow$ Topological splitting $M \cong_{\mathrm{top}} \mathbb{R} \times S$
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0. Introduction

Contents:

1. Reduction to Nash Riemannian theorem
   - (a) Greene's result for arbitrary metrics
   - (b) Lorentzian preliminary conventions
   - (c) Characterization of embeddability in $\mathbb{L}^N$
   - (d) Consequences for conformal embeddings
2. Background: causal volume functions and "folk problems"
   - (a) Future/past volume functions
   - (b) Relation with the causal ladder of spacetimes
   - (c) Geroch's topological construction
   - (d) Folk problems related to smoothability
3. Steep temporal functions on globally hyperbolic spacetimes

(a) Greene's result for arbitrary metrics

Result for pseudo-metrics (with an associated quadratic form with no restrictions: degenerate, signature-changing...)

1. Reduction to Nash: (a) Greene's result

(a) Greene's result for arbitrary metrics

Result for pseudo-metrics (with an associated quadratic form with no restrictions: degenerate, signature-changing...)

Theorem

(Greene '70). *Assume that any Riemannian metric on $M = M^m$ admits an isometric embedding in $\mathbb{R}^N$. Then, any pseudo-metric g on M admits an isometric embedding ψ_g in $\mathbb{R}^{N+2m+1}_{2m+1}$.*

(For M compact as well as isometric immersions, the dimension and index can be reduced in 1.)

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Proof. By Whitney theorem there exists a closed (proper) embedding $\psi : M \to \mathbb{R}^{2m+1}$ and by the claim below we can assume that [notation: g_0 natural metric in Euclidean space]

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is Riemannian. By assumption there exists an isometric embedding $\psi_R : (M, g_R) \to \mathbb{R}^N$ and the required one is

$$\psi_g(x) = (\psi(x), \psi_R(x)) \in \mathbb{R}^{2m+1}_{2m+1} \times \mathbb{R}^N.$$

□

1. Reduction to Nash: (a) Greene's result

Claim

Let $\phi : M \to \mathbb{R}^{N'}$ be a (smooth) proper embedding, and g be a pseudo-metric on M. Then, there exists a positive function $f : \mathbb{R}^{N'} \to \mathbb{R}$ such that the embedding $\phi_f := (f \circ \phi) \cdot \phi$ satisfies:

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is a Riemannian metric.

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Proof. Obvious if M is compact: choose f as a (big enough) constant c.
Otherwise, for each compact $K_r = \phi^{-1}(\overline{B_0(r)}), r = 1, 2, \ldots$ take the constant c_r corresponding to K_r, and choose f radial and monotone with $f|_{K_r \setminus K_{r-1}} \geq c_r$. $\square$

(b) Lorentzian preliminary conventions

- Causal character of a tangent vector $v \in T_pM$ in a Lorentzian manifold $(-, + \ldots, +)$ (analogously for curves, hypersurfaces):
  - v is causal when timelike $g(v, v) < 0$, or lightlike $g(v, v) = 0, \mathbf{v} \neq \mathbf{0}$
  - Otherwise, spacelike: $g(v, v) > 0$, or $v = 0$.
- (M, g) spacetime: time-orientable (and time-oriented, when necessary) connected smooth Lorentzian $n-$manifold. (No restrictive: any Lorentzian manifold isometrically embeddable in $\mathbb{L}^N$ must be time-orientable). One can speak on future and past directed causal vectors.

1. Reduction to Nash: (b) Lorentzian conventions

- The (piecewise) smooth timelike (resp. causal) curves define the chronological $\ll$ (resp. causal $\leq$) relation.
- Future and past of points (analogously subsets)
  - Chronological fut. $I^+(p) = \{q \in M : p \ll q\}$ (future-directed timelike curve from p to q)
  - Causal future $J^+(p) = \{q \in M : p \leq q\}$ (fut.-dir. causal curve from p to q, or $p = q$)
  - Analogously $I^-(p), J^-(p)$.
  - For an open subset $U \subset M$ regarded as spacetime: $I^+(p, U), J^-(p, U)$...
  - $\mathbf{J(p,q) = J^+(p) \cap J^-(q)}$ $(J(p,S) = J^+(p) \cap J^-(S))$

1. Reduction to Nash: (b) Lorentzian conventions

- Time-separation (or Lorentzian distance):
  $d : M \times M \to [0, +\infty]$

$$d(p,q) = \begin{cases} 0, \ \text{if } \mathcal{C}_{p,q} = \emptyset \\ \sup\left\{L(\alpha), \ \alpha \in \mathcal{C}_{p,q}\right\}, \ \text{if } \mathcal{C}_{p,q} \neq \emptyset \end{cases}$$

  $\mathcal{C}_{p,q}$ space of future-directed causal curves from p to q }

- Temporal function: smooth function τ with $\nabla\tau$ timelike and past-directed
  —in particular, it is a time-function: continuous function which increases on any future-directed causal curve

(c) Characterization of embeddability in $\mathbb{L}^N$

Theorem

For a Lorentzian manifold (M, g), it is equivalent:

(i) (M, g) *admits a isometric embedding in* $\mathbb{L}^N$ *for some* $N \in \mathbb{N}$.

(ii) (M, g) *(is a stably causal spacetime which) admits a* steep temporal function τ *($g(\nabla\tau, \nabla\tau) \leq -1$).*

In this case, d is finite.

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Lemma

If $i : M \to \mathbb{L}^N$ is an isometric embedding, then:
(a) the natural time coordinate $t = x^0$ of $\mathbb{L}^N$ induces a steep temporal function on M, and
(b) the time-separation d of (M, g) is finite-valued.

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Proof. (a) $x^0 \circ i$ is clearly temporal, and it is steep because $1 \equiv |\nabla^0 x^0|(i(M)) \leq |\nabla(x^0 \circ i)|$ the latter as $\nabla(x^0 \circ i)_p$ is the projection of $\nabla^0 x^0_{i(p)}$ onto the tangent space $di(T_pM)$ (and its orthogonal $di(T_pM)^\perp$ in $T_{i(p)}\mathbb{L}^N$ is spacelike).

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(b) The finiteness of d is consequence of the finiteness of the time-separation d_0 on $\mathbb{L}^N$ and the inequality
$d(p, q) \leq d_0(i(p), i(q))$ for all $p, q \in M$.
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In this case, d is finite.

Proof. From the lemma , only (ii) $\Rightarrow$ (i) needs to be proved.

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Lemma

If (M, g) admits a temporal function τ then the metric g admits a decomposition

$$g = -\beta d\tau^2 + g_\tau,$$

(g_{τ_0}: Riemannian metric on the slice $S_{\tau_0} = \tau^{-1}(\tau_0)$ varying locally smoothly with τ_0 –globally the topology of S_{τ_0} may change), where $\beta = |\nabla\tau|^{-2}$.
In particular, if τ is steep then $\beta \leq 1$.

Proof. Decomposition: restrict g.
Value of β: $d\tau(\nabla\tau) = g(\nabla\tau, \nabla\tau) = -\beta\,(d\tau(\nabla\tau))^2$. $\square$

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

Proof of Th. (steep temporal function $\Rightarrow$ embeddability in $\mathbb{L}^N$).
Using the decomposition of previous lemma, the auxiliary Riemannian metric

$$g_R := (4-\beta)d\tau^2 + g_\tau$$

admits a Nash isometric embedding

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The required isometric embedding $i : (M, g) \hookrightarrow \mathbb{L}^{N_0+1}$ is just:

$$i(\tau, x) = (2\tau, i_{\text{nash}}(\tau, x)) .$$

1. Reduction to Nash: (c) embeddability in $\mathbb{L}^N$

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However, even this question is harmless for conformal embeddings.

(d) Consequences for conformal embeddings

(1) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$.

Proof. (1) Let τ be any temporal function. Then τ is temporal for any conformal metric, and steep for $g^* = \sqrt{|\nabla t|}g$ $(|\nabla^* \tau|^* \equiv 1)$.

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(1) A Lorentzian manifold is a stably causal spacetime if and only if it admits a conformal embedding in some $\mathbb{L}^N$.

(2) In this case, there is a representative of its conformal class whose time-separation (Lorentzian distance) function is finite-valued.

Proof. (2) For $g^* = \sqrt{|\nabla t|} g$ as above, (M, g^*) is isometrically embeddable, and, then, its time-separation d^* is finite.

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(2) In this case, there is a representative of its conformal class whose time-separation (Lorentzian distance) function is finite-valued.

(3) A stably causal spacetime is conformal to a spacetime non-isometrically embeddable in $\mathbb{L}^N$ if [*and only if*] it is not globally hyperbolic .

Proof. (3) Such spacetimes are conformal to a spacetime with infinite-valued d and, thus, non-isometrically embeddable. $\square$

1. Reduction to Nash: (d) conformal embeddings

Summing up, we will conclude:

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Remark. As a difference with the Riemannian case, there is a (very neat) obstruction to the existence of isometric and conformal embeddings.

2. Background: (a) volume functions

(a) Future/past volume functions

Definition

Admissible Borel measure on M for Geroch-type construction:

1. $m(M) < \infty$,
2. $m(U) > 0$ if $U \neq \emptyset$ is open
3. $m(\partial I^{\pm}(z)) = 0, \forall z \in M$.

The one associated to a Riemannian metric with finite volume suffices

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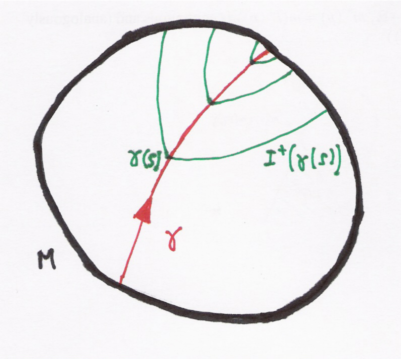
Definition

Associated past/future *volume* functions on M:

- $t^- : M \to \mathbb{R}$, $t^-(p) = m(I^-(p))$
- $t^+ : M \to \mathbb{R}$, $t^+(p) = -m(I^+(p))$

Let $\gamma : (a, b) \to M$ fut.-pointing causal: $s \to t^{\pm}(\gamma(s))$ is non-decreasing.



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Let $M \to \mathbb{R}$ be a (non-necessarily continuous) function on M:

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Remark: in general, $t^{\pm}$ are not generalized time functions (for ex.: when there exist closed causal curves). To understand this well...

(b) Relation with the causal ladder of spacetimes

- To obtain spacetimes both, physically realistic and mathematical interesting, it is useful to impose conditions on the global causality of the spacetime.
- Such conditions are always conformally invariant
- This yields a causal ladder or hierarchy of spacetimes.
- The steps directly related to volume functions are:

$\boxed{\text{Globally hyperbolic}}$

$\Downarrow$

$\boxed{\text{Causally simple}}$

$\Downarrow$

$\boxed{\text{Causally continuous}}$

$\Downarrow$

$\boxed{\text{Stably causal}}$

$\Downarrow$

$\boxed{\text{Strongly causal}}$

$\Downarrow$

$\boxed{\text{Distinguishing}}$

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- **Characterization**: (M, g) is chronological $\iff t^-$ (resp. t^+) is strictly increasing on any future-directed **timelike** curve.

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- (M, g) is distinguishing if $p \neq q \Rightarrow I^{\pm}(p) \neq I^{\pm}(q)$

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Causally continuous

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Strongly causal

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Distinguishing

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Causal

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Chronological

- (M, g) is distinguishing if $p \neq q \Rightarrow I^{\pm}(p) \neq I^{\pm}(q)$
- **Characterization**: (M, g) distinguishing $\iff t^{-}, t^{+}$ are strictly increasing on any future-directed causal curv, i.e. $t^{\pm}$ generalized time functions (non-necessarily continuous).

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2. Background: (b) causal ladder

- A spacetime is stably causal if it remains causal under C^0 perturbations of the metric (or if it remains causal when its cones are opened slightly)

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- The existence of a time/temporal function seems a big gap with previous conditions...

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Theorem

For any (M, g), they are equivalent:

1. *To be stably causal*
2. *To admit a time function* t *(continuous strictly increasing on any future-directed causal curve)*
3. *To admit a temporal function* $\mathcal{T}$ *(smooth with timelike gradient everywhere)*

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- (1) $\Rightarrow$ (2) (Hawking) Even though volume functions are only generalized time functions, a true (continuous) time function can be obtained by integrating the t^- of close metrics $g_\lambda, \lambda \in [0,1]$ with wider lightcones: $t(p) = \int_0^1 t_\lambda^-(p) d\lambda.$

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- (3) $\Rightarrow$ (1) Not difficult

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  Equivalently, **if the volume functions $t^{\pm}$ are time functions**

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- (M, g) is causally continuous when (M, g) is distinguishing and $I^{\pm}(p)$ vary continuously with p
  Equivalently, **if the volume functions $t^{\pm}$ are time functions**
- (M, g) is causally simple if it is causal and $J^{\pm}(p)$ is the closure of $I^{\pm}(p)$
  Beware: d may reach the value ∞ and, so, not all these spacetimes are isometrically embeddable in Ł^N

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- (M, g) is globally hyperbolic if it is causal and it does not contain naked singularities: $J(p, q) := J^+(p) \cap J^-(p)$ compact for all p, q.

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- Natural strengthening of the causal requirements...
- ... but implies spectacular properties for the spacetime!

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(Characterization of global hyperbolicity). For a spacetime (M, g), the following conditions are equivalent *(Geroch'70):*

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(i) (M, g) *is globally hyperbolic.*
(ii) (M, g) *admits a Cauchy hypersurface, that is, a subset* S *which is crossed exactly once by any inextendible timelike curve*
(iii) (M, g) *admits a Cauchy time function, i.e., an onto time function* $t : M \to \mathbb{R}$ *such that all its levels* $S_{t_0} = t^{-1}(t_0)$*,* $t_0 \in \mathbb{R}$*, are (acausal) Cauchy hypersurfaces.*

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plus "sharp smooth versions" (Bernal & MS '03,'05)
(iv) (M, g) admits a spacelike Cauchy hypersurface (a smooth hypersurface which is spacelike and Cauchy).
(v) (M, g) admits a Cauchy temporal function, i.e., an onto temporal function $t : M \to \mathbb{R}$ such that all its levels S_{t_0}, $t_0 \in \mathbb{R}$, are Cauchy hypersurfaces ($\Rightarrow$ orthogonal splitting)

(c) Geroch's topological construction (JMP'70)
(Existence of a Cauchy time function in any glob hyp spacetime in terms of volume functions)

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Cauchy hypersurface: subset $S \subset M$ which is intersected exactly once by any inext. timelike curve ("the whole 'space' at an instant of time")

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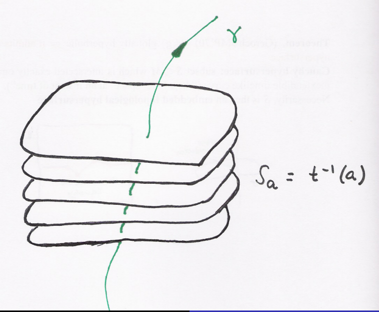
Remarks:

- Necessarily, S is then an *embedded topological hypersurface*
- Cauchy hyp. yield an alternative definition of glob. hyp. as: ((M, g) is globally hyperbolic $\Leftrightarrow$ it admits a Cauchy hypersurface)
- We will focus on one of the implications by Geroch:

2. Background: (c) Geroch's construction

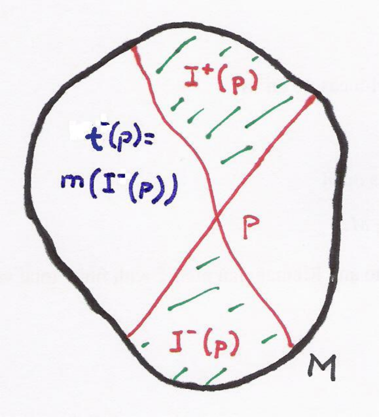
Theorem

If M is glob. hyp., there exists a ("Cauchy time function") $t : M \to \mathbb{R}$ continuous and onto such that:
(1) t is strictly increasing on any future-directed causal curve (and then a time function).
(2) $S_a := t^{-1}(a)$ Cauchy hyp. $\forall a \in \mathbb{R}$.
(As a consequence, M is homeomorphic to $\mathbb{R} \times S$).



2. Background: (c) Geroch's construction

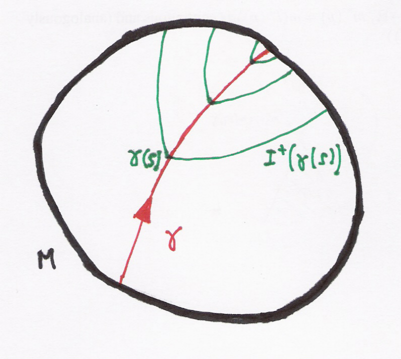
Idea of the proof. Consider the volume functions $t^{\pm}$



2. Background: (c) Geroch's construction

If $\gamma : (a, b) \to M$ is causal, fut.-directed and inextensible:

1. $s \to t^+(\gamma(s))$ (resp. $t^-(\gamma(s))$) is strictly increasing [t^-, t^+ were time functions]
2. $\lim_{s\to b} t^+(\gamma(s)) = 0 = \lim_{s\to a} t^-(\gamma(s))$
3. $\lim_{s\to a}(-t^+(\gamma(s))), \lim_{s\to b} t^-(\gamma(s)) > 0$

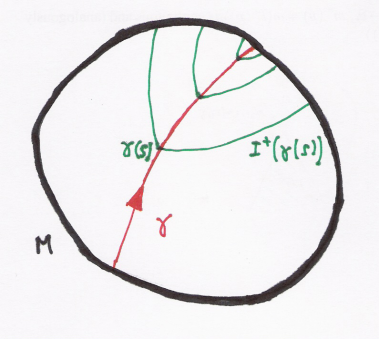


2. Background: (c) Geroch's construction

Required "Cauchy time" function:

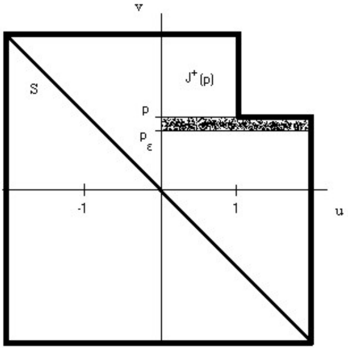
$$t(z) = \log\left(-t^-(z)/t^+(z)\right)$$

$$\left.\begin{array}{l} \lim_{s\to b} t(\gamma(s)) = \infty \\ \lim_{s\to a} t(\gamma(s)) = -\infty \end{array}\right\} \Longrightarrow t = \text{const. is Cauchy}$$



(d) Folk problems related to smoothability

Remark. From the constructive proof, $t, t^{\pm}$ is not always smooth:



$M \subset \mathbb{L}^2$, (null coord. u, v) Diagonal S Cauchy hyp, t^+

2. Background: (d) folk problems

Given Geroch's result, *folk questions* for glob. hyp. spacetimes:

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(1) Find a **(smooth) spacelike** Cauchy hyp. (Sachs & Wu, Bulletin AMS '77)
■ Even more (Brunetti/Ruzzi): can any smooth spacelike compact submanifold with boundary be extended to a spacelike Cauchy hyp.?

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(1) Find a **(smooth) spacelike** Cauchy hyp. (Sachs & Wu, Bulletin AMS '77)
■ Even more (Brunetti/Ruzzi): can any smooth spacelike compact submanifold with boundary be extended to a spacelike Cauchy hyp.?

(2) Find a Cauchy **temporal** function (i.e. additionally t smooth with past-pointing timelike gradient and Cauchy hyp, as levels)
$\rightsquigarrow$ such a function would yield the structural orthogonal splitting
$(M, g) \equiv (\mathbb{R} \times S, g = -\beta dt^2 + g_t)$,
■ Even more (Bär/Ginoux/Pfäffle): given a spacelike Cauchy hyp. S, find a Cauchy temporal function with one of the levels equal to S

(3) Prove that functions such as

$$f(x) = \int_{H^+(\tau(x),\sigma(x))} \mu$$

where μ is some (admissible) measure. Here, τ is a Cauchy temporal function which splits the spacetime, σ a sort of spacelike radial coordinate and

$$H^+(t,s) = J^+(\tau^{-1}(0)) \cap J^-(\tau^{-1}(t) \cap \sigma^{-1}([0,s])).$$

■ A proof of the smoothness of such functions would complete Clarke's proof on embeddability and would have interest in its own right.

2. Background: (d) folk problems

Difficulties to solve them with the strictly involved tools:

1. Try to approximate Geroch's time function by smooth ones (Seifert '77):
   BUT even a smooth one may have degenerate Cauchy hypersurfaces.
2. Try to use a different admissible measure for the job (Dieckmann '88):
   BUT for the related problem of smoothability of time functions, no admissible measure can make $t^{\pm}$ be a time function (this happened iff the spacetime was causally continuous).

2. Background: (d) folk problems

- The procedure developed in AN Bernal, MS. '03, '05 '06 yields a Cauchy temporal function (and a temporal function in the stably causal case, as well as solve the other refined problems)

- The procedure developed in AN Bernal, MS. '03, '05 '06 yields a Cauchy temporal function (and a temporal function in the stably causal case, as well as solve the other refined problems)
- Next, we will construct a steep Cauchy temporal function by a modification of this procedure (Müller & MS, '11)
  This re-proves and simplifies widely (even though only in the globally hyperbolic case) the proof of the existence of a Cauchy temporal function.

(a) Technical tools

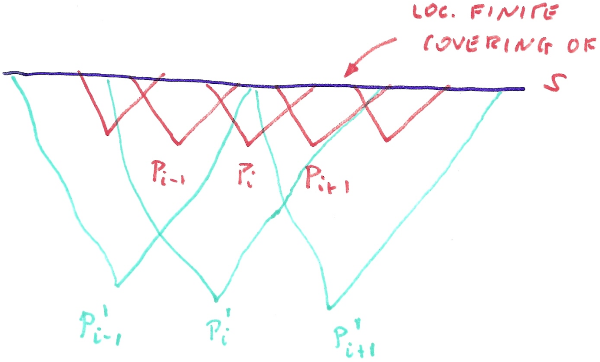
(1) We assume the existence of a Cauchy time function t as in Geroch's, each $S_a = t^{-1}(a)$ Cauchy.

(2) Function

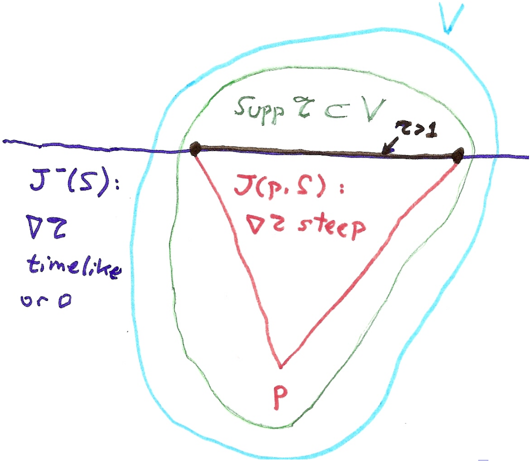
$$j_p : M \to \mathbb{R}, \quad j_p(q) = \exp(-1/d(p,q)^2).$$

Restricted to a convex neighborhood of p, this is a smoothed version of the Lorentzian distance to p (smooth even at 0).

(3) For any Cauchy S, a fat cone covering:
sequence of pairs of points $p'_i \ll p_i, i \in \mathbb{N}$ such that both, $\mathcal{C}' = \{I^+(p'_i) : i \in \mathbb{N}\}$ and $\mathcal{C} = \{I^+(p_i) : i \in \mathbb{N}\}$ yield a locally finite covering of S.



(4) For any Cauchy $S = S_a$, $p \in J^-(S)$ and $V \supset J(p, S)$, a smoooth function τ steep temporal on $J(p, S)$ with support in V ("τ steep on the forward cone $J(p, S)$").

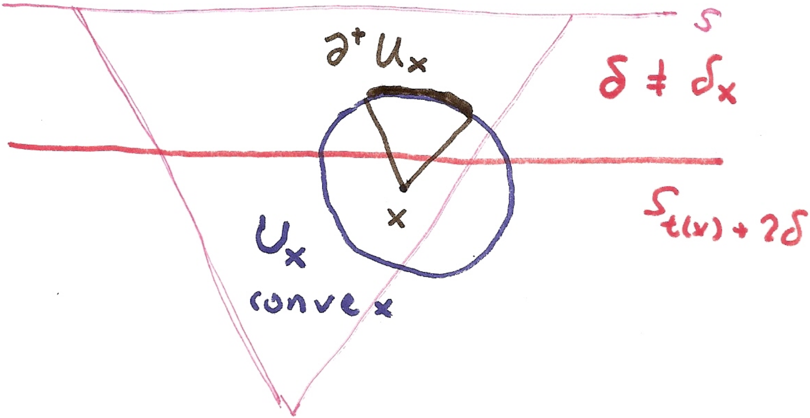


3. Steep temporal functions on glob. hyp. s.-t.

Proposition

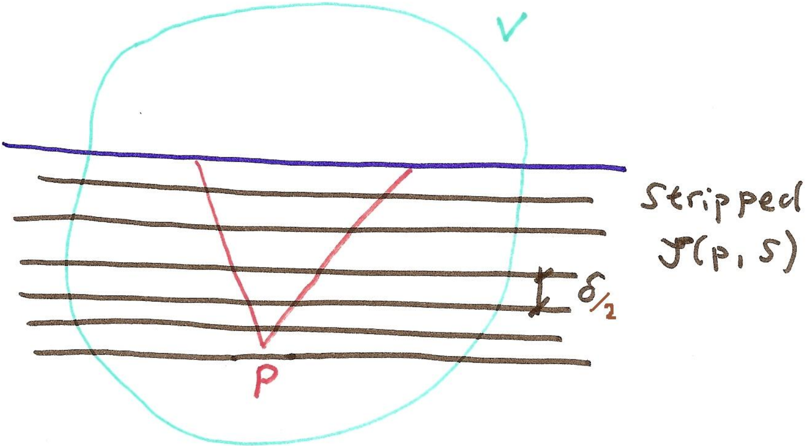
Let S be a Cauchy hypersurface, $p \in J^-(S)$. For all neighborhood V of $J(p,S)$ there exists a smooth function $\tau \geq 0$ such that:
(i) $\mathrm{supp}\,\tau \subset V$
(ii) $\tau > 1$ *on* $S \cap J^+(p)$.
(iii) $\nabla\tau$ *is timelike and past-directed in* $\mathrm{Int}(\mathrm{Supp}\ (\tau) \cap J^-(S))$.
(iv) $g(\nabla\tau, \nabla\tau) < -1$ *on* $J(p,S)$.

Sketch of proof. Choose K compact, $J(p,S) \subset \mathrm{Int}(K)$, $K \subset V$ and $\delta > 0$ s.t.: $\forall x \in K$, $\exists U_x \subset V$ convex with $\partial^+ U_x \subset J^+(S_{t(x)+2\delta})$ (where $\partial^+ U_x := \partial U_x \cap J^+(x)$).



3. Steep temporal functions on glob. hyp. s.-t.

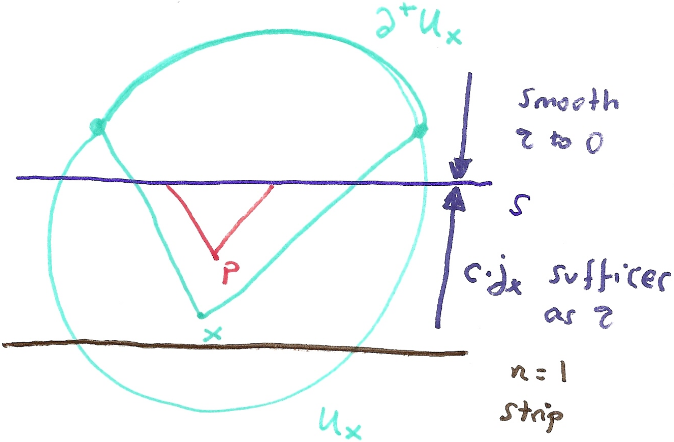
Slice $J(p,S)$ in n strips, $a_0 < t(p) < a_1 < \cdots < a_n = a$ with $a_{i+1} - a_i < \delta/2$



3. Steep temporal functions on glob. hyp. s.-t.

If $n = 1$ strip suffices (otherwise, careful induction!):

- $\tau = cj_x$ on $J^-(S)$ for c large and some close $x \ll p$
- τ is smoothed to 0 on $V \cap J^+(S)$.

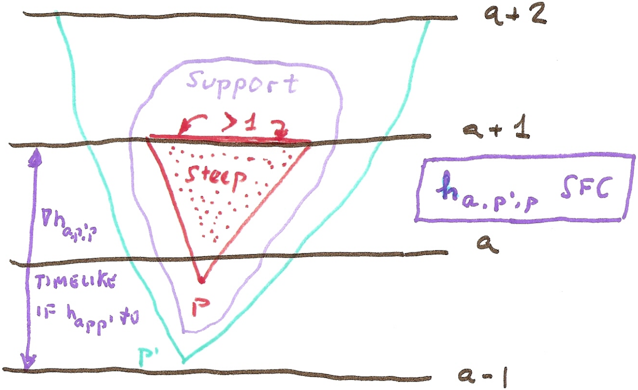


(b) Steps of the proof

- **Step 1**: for any $a \in \mathbb{R}$ and $p' \ll p$, $p, p' \in J(S_{a-1}, S_a)$, construct a *steep forward cone function* (SFC) $h^+_{a,p',p} : M \to [0, \infty)$ which satisfies:

1. $supp(h^+_{a,p',p}) \subset J^+(p', S_{a+2})$,
2. $h^+_{a,p',p} > 1$ on $S_{a+1} \cap J^+(p)$,
3. If $x \in J^-(S_{a+1})$ and $h^+_{a,p',p}(x) \neq 0$ then $\nabla h^+_{a,p',p}(x)$ is timelike and past-directed, and
4. $g(\nabla h^+_{a,p',p}, \nabla h^+_{a,p',p}) < -1$ on $J(p, S_{a+1})$.

3. Steep temporal functions on glob. hyp. s.-t.

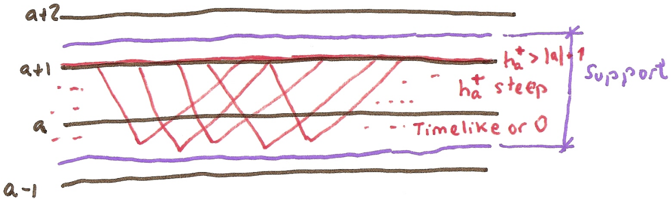


(this is straightforward from the above constructed steep funtions on $J(p, S)$).

3. Steep temporal functions on glob. hyp. s.-t.

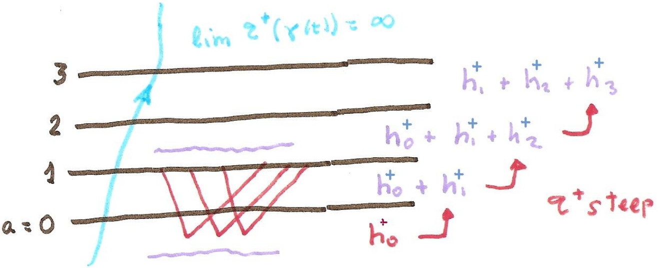
- **Step 2**: by using a fat cone covering $\{p_i' \ll p_i | i \in \mathbb{N}\}$ for $S = S_a$, adjust a locally finite sum of SFC functions to obtain some $h_a^+ > 0$ which satisfies:

1. $supp(h_a^+) \subset J(S_{a-1}, S_{a+2})$,
2. $h_a^+ > |a| + 1$ S_{a+1}, [this will ensure that the finally obtained temporal function is Cauchy]
3. If $x \in J^-(S_{a+1})$ and $h_a^+(x) \neq 0$ then $\nabla h_a^+(x)$ is timelike and past-directed, and
4. $g(\nabla h_a^+, \nabla h_a^+) < -1$ on $J(S_a, S_{a+1})$.



3. Steep temporal functions on glob. hyp. s.-t.

- **Step 3:**
  1. given $h_a^+ \geq 0$ ensure that h_{a+1}^+ can be chosen such that
  $$g(\nabla(h_a^+ + h_{a+1}^+), \nabla(h_a^+ + h_{a+1}^+)) < -1 \qquad \text{on } J(S_{a+1}, S_{a+2}) \tag{1}$$
  So, $h_a^+ + h_{a+1}^+$ is steep on all $J(S_a, S_{a+2})$.
  2. Inductively, construct a Cauchy steep function $\mathcal{T}^+ \geq 0$ on $J^+(S_0)$ with $\mathcal{T}^+(S_a) \geq a$ for $a = 1, 2, \ldots$.
  3. By reversing the time orientation and working on $J^-(S_0)$, obtain the Cauchy steep function $\mathcal{T} = \mathcal{T}^+ - \mathcal{T}^-$ on all M.



3. Steep temporal functions on glob. hyp. s.-t.

By construction this function not only is smooth, temporal and steep, but also satisfies the abstract properties in Geroch's proof which ensure that the levels are Cauchy.