Curvature identity on a 4-dimensional Riemannian manifold and its applications

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(joint work with Y. Euh and J. H. Park)

Conference in Geometry and Global Analysis

Celebrating P. Gilkey's 65th Birthday

December 13-17, 2010

Outline

Gauss-Bonnet Theorem

Let $M = (M, g)$ be a compact oriented surface. Then,

$$2\pi\chi(M) = \int_M K dv_g,$$

where $\chi(M)$, K and dv_g are the Euler number, the Gaussian curvature and the volume element of M, respectively.

Here, $K = \frac{\tau}{2}$, where τ is the scalar curvature of M.

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$g(t)$: any one-parameter deformation of the metric g

$$2\pi\chi(M) = \frac{1}{2}\int_M \tau(t) dv_{g(t)}$$

$$0 = \frac{d}{dt}\Big|_{t=0}\int_M \tau(t) dv_{g(t)} = \int_M \Big(-\rho^{ij} + \frac{\tau}{2}g^{ij}\Big) h_{ij} dv_g$$

for any symmetric $(0,2)$-tensor field $h_{ij} = \frac{d}{dt}\big|_{t=0} g(t)_{ij}$.

$$\rho = \frac{\tau}{2}g$$

The equality holds for any surface.

Question

Does the same phenomenon occur for any $2n(n \geqq 2)$-dimensional Riemannian manifold?

- Can we deduce a curvature identity from the generalized Gauss-Bonnet theorem for a $2n$-dimensional compact oriented Riemannian manifold?
- Is this identity valid for any $2n$-dimensional Riemannian manifold?

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$M = (M, g)$: $2n$-dim. compact oriented Riemannian manifold

h_{2k} : Gauss-Bonnet curvature tensor determined by the complete contraction of the Gauss-Kronecker curvature R^k of order $2k$.

$$h_{2k} = \frac{1}{(2k)!} c^{2k} R^k \quad (1 \leqq k \leqq n),$$

where c is the contraction map. Here, R^k is defined by the exterior product of the curvature tensor R with itself k-times in the ring of curvature structures on M.

[1] O. Kowalski, On the Gauss-Kronecker curvature tensors, Math. Ann. **203** (1973), 335–343.

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$$\mathcal{H}_4(g) = \int_M \left\{|R|^2 - 4|\rho|^2 + \tau^2\right\} dv_g \quad (k = 2),$$

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$$\chi(M) = \alpha\mathcal{H}_{2n}(g) \quad (k = n)$$

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For example, if $n = 2$, the generalized Gauss-Bonnet formula is given by

$$\chi(M) = \frac{1}{32\pi^2}\mathcal{H}_4(g).$$

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Let $M = (M, g)$ be a 4-dimensional compact oriented Riemannian manifold. Then,

$$\check{R} - 2\check{\rho} - L\rho + \tau\rho - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)g = 0$$

holds on M.

Here,

$$\check{R} : \check{R}_{ij} = \sum_{a,b,c} R_{abci}R^{abc}{}_j, \quad \check{\rho} : \check{\rho}_{ij} = \sum_a \rho_{ai}\rho^a{}_j, \quad L : (L\rho)_{ij} = 2\sum_{a,b} R_{iabj}\rho^{ab},$$

where R is the curvature tensor of M defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ (∇ : Levi-Civita connection) and ρ is the Ricci tensor of M.

[2] M. Berger, Quelques formules de variation pour une structure riemannienne, Ann. Sci. École Norm. Sup. 4^e Serie 3 (1970), 285–294.

Labbi, 2008

Let $M = (M, g)$ be a $2n$-dimensional compact oriented Riemannian manifold. Then,

$$h_{2n}g - \frac{1}{(2n-1)!}c^{2n-1}R^n = 0.$$

[3] M.-L. Labbi, Variational properties of the Gauss-Bonnet curvatures, Calc. Var. 32 (2008), 175–189.

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Outline

Theorem A

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Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then,

$$\check{R} - 2\check{\rho} - L\rho + \tau\rho - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)g = 0 \tag{1}$$

holds on M.

Here,

$$\check{R} : \check{R}_{ij} = \sum_{a,b,c} R_{abci}R^{abc}{}_{j}, \quad \check{\rho} : \check{\rho}_{ij} = \sum_{a} \rho_{ai}\rho^{a}{}_{j}, \quad L : (L\rho)_{ij} = 2\sum_{a,b} R_{iabj}\rho^{ab}.$$

Proof of Theorem A

Let $\{e_i\}$ be an orthonormal basis of T_pM. Then, we rewrite equation (1) as

$$\sum_{a,b,c} R_{abci}R_{abcj} - 2\sum_{a}\rho_{ai}\rho_{aj} - 2\sum_{a,b} R_{iabj}\rho_{ab} + \tau\rho_{ij} - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)\delta_{ij} = 0. \tag{2}$$

We choose a Chern basis of T_pM at any point $p \in M$, namely, an orthonormal basis of T_pM satisfying

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Step 1. Set $i = j$ in (2), then

$$\sum_{a,b,c} {R_{abci}}^2 - 2\sum_a {\rho_{ai}}^2 - 2\sum_{a,b} R_{iabi}\rho_{ab} + \tau\rho_{ii} - \frac{1}{4}|R|^2 + |\rho|^2 - \frac{\tau^2}{4} = 0 \qquad (1 \leqq i \leqq 4).$$

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Step 2. Set $i \neq j$ in (2), then

$$\sum_{a,b,c} R_{abci}R_{abcj} - 2\sum_a \rho_{ai}\rho_{aj} - 2\sum_{a,b} R_{iabj}\rho_{ab} + \tau\rho_{ij} = 0 \quad (1 \leqq i, j \leqq 4)$$

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Since this is a tensor equation, it is valid for any orthonormal basis. □

From Theorem A, we can prove the following classical theorem [4].

Corollary

$M' = (M', g')$: 3-dimensional Riemannian manifold

$\{e'_a\}$: orthonormal basis of $T_{p'}M'$ at any point $p' \in M'$

The following equality

$$R'_{abcd} = \rho'_{ad}\delta_{bc} - \rho'_{ac}\delta_{bd} + \delta_{ad}\rho'_{bc} - \delta_{ac}\rho'_{bd} - \frac{\tau}{2}(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})$$

$(1 \leqq a, b, c, d \leqq 3)$ holds.

[4] H. Weyl, Reine Infinitesimalgeometrie, Math. Z. 2 (1918), 384–411.

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sketch of proof. $M = M' \times \mathbb{R}$

Setting $i = j = 4$ in (2), $\frac{1}{4}\left(|R'|^2 - 4|\rho'|^2 + \tau'^2\right) = 0$.

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Corollary

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then the following curvature condition

$$\check{W} = \frac{1}{4}|W|^2 g$$

holds on M. Here, W is the Weyl curvature tensor of M and $\check{W}$: $\check{W}_{ij} = W_{abci}W^{abc}{}_{j}$.

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sketch of proof. We substitute the following equality

$$R_{ijkl} = W_{ijkl} - \frac{1}{2}(\rho_{il}g_{jk} - \rho_{ik}g_{jl} + g_{il}\rho_{jk} - g_{ik}\rho_{jl}) + \frac{\tau}{6}(g_{il}g_{jk} - g_{ik}g_{jl})$$

into the equality (2) in Theorem A.

Corollary

A 4-dimensional Einstein manifold satisfies the condition

$$\check{R} = \frac{1}{4}|R|^2 g. \tag{3}$$

sketch of proof. $\rho_{ij} = \frac{\tau}{4} g_{ij}$ in (2).

However, the converse of the above is not necessarily valid.

Remark, Corollary can be also proved by making use of a Singer-Thorpe basis of a 4-dimensional Einstein manifold.

[5] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, Global Analysis, (Papers in Honor of K. Kodaira) Princeton University Press, Princeton, 1969, 355–365.

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An n-dimensional Einstein manifold is called *super-Einstein* if the following curvature identity is satisfied

$$\check{R} = \frac{1}{n}|R|^2 g. \tag{4}$$

Remark. The constancy of $|R|^2$ follows from (4) automatically in dimensions $n > 4$ (see [6], Lemma 3.3). For a 4-dimensional super-Einstein manifold, the constancy of $|R|^2$ is not automatically satisfied but it is usually required (see [7]).

[6] E. Boeckx and L. Vankecke, *Unit Tangent sphere bundles with constant scalar curvature*, Czechoslovak Math. J. 51 (126) (2001), 523–544.

[7] A. Gray and T. J. Willmore, Mean-value theorems for Riemannian manifolds, Proc. Roy. Soc. Edinburgh Sect. A 92 (1982), 343–364.

We shall call a 4-dimensional Riemannian manifold satisfying the curvature condition (4) (i.e., $\check{R} = \frac{1}{4}|R|^2 g$) *weakly Einstein*.

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By the definition, a weakly Einstein manifold is a generalization of a 4-dimensional Einstein manifold.

The following examples show that a weakly Einstein manifold is not necessarily Einstein.

Example (1)

Let M be a Riemannian product manifold of 2-dim. Riemannian manifolds $M_1(c)$ and $M_2(-c)$ of constant Gaussian curvatures c and $-c$ ($c \neq 0$), respectively. Then M satisfies (3) but it is not Einstein.

Example (2)

Let $\mathfrak{g} = span_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$ *be a 4-dimensional real Lie algebra equipped with the following Lie bracket operation:*

$$[e_1, e_2] = ae_2, \qquad [e_1, e_3] = -ae_3 - be_4, \qquad [e_1, e_4] = be_3 - ae_4,$$
$$[e_2, e_3] = 0, \qquad [e_2, e_4] = 0, \qquad [e_3, e_4] = 0,$$

where $a(\neq 0)$, b *are constant.* $<,>$ *on* $\mathfrak{g}$ *is defined by* $< e_i, e_j >= \delta_{ij}$. *Let* G *be a connected and simply connected solvable Lie group with the Lie algebra* $\mathfrak{g}$ *of* G *and* g *the* G*-invariant Riemannian metric on* G *determined by* $<,>$. *By direct calculations, we have*

$$R_{1212} = a^2, \quad R_{1313} = a^2, \quad R_{1414} = a^2,$$
$$R_{2323} = -a^2, \quad R_{2424} = -a^2, \quad R_{3434} = a^2,$$

and otherwise being zero up to sign. From these, G *is not Einstein since the Ricci curvature components satisfy* $\rho_{11} = -3a^2$ *but* $\rho_{22} = a^2$.
However G *satisfies* (3)*, thus* G *is weakly Einstein.*

Outline

Proposition

Let $M = (M, g)$ be a weakly Einstein manifold and $\{e_i\}$ an orthonormal Ricci eigenbasis of T_pM corresponding to the eigenvalues λ_i $(1 \leqq i \leqq 4)$ at any point $p \in M$. Then, the following equalities

$$R_{1212}{}^2 = R_{3434}{}^2, \quad R_{1313}{}^2 = R_{2424}{}^2, \quad R_{1414}{}^2 = R_{2323}{}^2$$

hold and the following cases $(a) \sim (d)$ never occur:

(a) $\lambda_1 = \lambda_2 = \lambda_3 (\neq 0),\ \lambda_4 = 0,$

(b) $\lambda_1 = \lambda_2 = \lambda_4 (\neq 0),\ \lambda_3 = 0,$

(c) $\lambda_1 = \lambda_3 = \lambda_4 (\neq 0),\ \lambda_2 = 0,$

(d) $\lambda_2 = \lambda_3 = \lambda_4 (\neq 0),\ \lambda_1 = 0.$

Especially, if M is Einstein, then the following holds:

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323}.$$

By Proposition, we see that the following examples are not weakly Einstein manifolds.

Example (3)

$$M = M^2(c_1) \times M^2(c_2), \quad (c_1, c_2 : \mathit{constant},\ c_1^2 \neq c_2^2)$$

Example (4)

$$M = M^3(c) \times \mathbb{R}, \quad (c(\neq 0) : \mathit{constant})$$

Remark. The statement "any 4-dimensional Riemannian manifold satisfies the curvature condition (3)" (in [8] p. 165) seems a printing mistake.

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Let $M = (M, g)$ be a 4-dimensional Riemannian manifold.

We here introduce some special kinds of orthonormal basis of T_pM at any point $p \in M$ and explain their intermediate relationships.

A Singer-Thorpe basis $\{e_i\}$ is an orthonormal basis satisfying the following conditions

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323},$$
$$R_{ijjk} = 0 \quad (i \neq k).$$

Theorem (2.1)

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then, M is an Einstein manifold if and only if M admits a Singer-Thorpe basis at each point of M.

[5] I. M. Singer and J. A. Thorpe, The curvature of 4-dimensional Einstein spaces, Global Analysis, (Papers in Honor of K. Kodaira) Princeton University Press, Princeton, 1969, 355–365.

A Chern basis $\{e_i\}$ is an orthonormal basis of T_pM at any point $p \in M$ satisfying $R_{1213} = R_{1214} = R_{1223} = R_{1224} = R_{1314} = R_{1323} = 0$.

A Ricci eigenbasis $\{e_i\}$ is an orthonormal basis of T_pM corresponding to the Ricci eigenvalues λ_i $(1 \leqq i \leqq 4)$ at p, namely, satisfying the following condition $Qe_i = \lambda_i e_i$ $(1 \leqq i \leqq 4)$.

Here, we assume that an orthonormal basis $\{e_i\}$ of T_pM is a Ricci eigenbasis and at the same time a Chren basis. Then, we have

$$R_{2434} = R_{2334} = R_{1434} = R_{1334} = R_{2324} = R_{1424} = 0.$$

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$$R_{ijjk} = 0 \quad (i \neq k), \quad (1 \leqq i, j, k \leqq 4). \tag{5}$$

Conversely, if (5) holds with respect to an orthonormal basis $\{e_i\}$, then the basis $\{e_i\}$ is a Ricci eigenbasis and Chern basis at the same time.

Question. Can we always choose the orthonormal basis which is a Ricci eigenbasis and at the same time a Chern basis?

The following example shows that an orthonormal Ricci eigenbasis is not necessarily a Chern basis.

Example (5)

Let $\mathfrak{g} = span_{\mathbb{R}}\{e_1, e_2, e_3, e_4\}$ be a 4-dimensional real Lie algebra equipped with the following Lie bracket operation:

$$[e_1, e_2] = 2e_2, \qquad [e_1, e_3] = -e_3, \qquad [e_1, e_4] = 2e_3 - e_4,$$
$$[e_2, e_3] = 0, \qquad [e_2, e_4] = 0, \qquad [e_3, e_4] = 0,$$

and $<,>$ the inner product on $\mathfrak{g}$ given by $< e_i, e_j >= \delta_{ij}$.

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and $<,>$ the inner product on $\mathfrak{g}$ given by $< e_i, e_j >= \delta_{ij}$.

Let G be a connected and simply connected solvable Lie group with the Lie algebra $\mathfrak{g}$ of G and g the G-invariant Riemannian metric on G determined by $<,>$. Then, we have

$$\begin{aligned} &R_{1212} = 4, \quad R_{1414} = 4, \quad R_{2323} = -2, \\ &R_{2424} = -2, \quad R_{1314} = -2, \quad R_{2324} = 2, \end{aligned} \tag{6}$$

and otherwise being zero up to sign. Then, the orthonormal basis $\{e_i\}$ is a Ricci eigenbasis satisfying $Qe_i = \lambda_i e_i$ $(1 \leqq i \leqq 4)$, where $\lambda_1 = -8$, $\lambda_2 = 0$, $\lambda_3 = 2$, $\lambda_4 = -2$. However, from (6), the basis $\{e_i\}$ does not satisfy (5). This means that the basis $\{e_i\}$ is a Ricci eigenbasis but not a Chern basis.

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold and $\{e_i\}$ an orthonormal basis of the tangent space T_pM at a point $p \in M$. If the basis $\{e_i\}$ satisfies

$$R_{1212}{}^2 = R_{3434}{}^2, \quad R_{1313}{}^2 = R_{2424}{}^2, \quad R_{1414}{}^2 = R_{2323}{}^2 \tag{7}$$

and

$$R_{ijjk} = 0 \ \ (i \neq k), \quad (1 \leqq i, j, k \leqq 4) \tag{8}$$

then we shall call the orthonormal basis a *generalized Singer-Thorpe basis*.

The result of Singer and Thorpe [5] is well-known as a characterization of a 4-dimensional Einstein manifold. The following theorem is a generalization of their result, which is a characterization of a weakly Einstein manifold.

Theorem (2.2)

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then, M is a weakly Einstein manifold if and only if M admits a generalized Singer-Thorpe basis at each point of M.

$M = (M, g)$: a compact oriented weakly Einstein manifold.

Then, from Theorem (2.2), we can choose a generalized Singer-Thorpe basis $\{e_i\}$ of T_pM at any point $p \in M$ compatible with the orientation of M.

We set

$$\begin{aligned} &\alpha'_1 = R_{1212}, \qquad &\alpha'_2 = R_{1313}, \qquad &\alpha'_3 = R_{1414}, \\ &\alpha''_1 = R_{3434}, \qquad &\alpha''_2 = R_{2424}, \qquad &\alpha''_3 = R_{2323}, \\ &\beta_1 = R_{1234}, \qquad &\beta_2 = R_{1342}, \qquad &\beta_3 = R_{1423}. \end{aligned} \tag{9}$$

Set $\mathbf{a}' = (\alpha'_1, \alpha'_2, \alpha'_3)$, $\mathbf{a}'' = (\alpha''_1, \alpha''_2, \alpha''_3)$ and $\mathbf{b} = (\beta_1, \beta_2, \beta_3)$ and denote the canonical inner product by $<,>$ on the 3-dimensional Euclidean space $\mathbb{R}^3$.

Set $|\mathbf{x}| = \sqrt{< \mathbf{x}, \mathbf{x} >}$ for any $\mathbf{x} \in \mathbb{R}^3$. Then, $|\mathbf{a}'| = |\mathbf{a}''|$ by virtue of (7).

Now, we denote the Euler number and the first Pontrjagin number of M by $\chi(M)$ and $p_1(M)$, respectively.

Then, from (9), applying the similar arguments as in [9], we have the following equalities:

$$\chi(M) = \frac{1}{4\pi^2}\int_M \{< \mathbf{a}', \mathbf{a}'' > + |\mathbf{b}|^2\} dv_g$$

and

$$p_1(M) = \frac{1}{2\pi^2}\int_M < \mathbf{a}' + \mathbf{a}'', \mathbf{b} > dv_g,$$

where dv_g is the volume element of M. Now, we set

$$\mathbf{a} = \frac{1}{2}(\mathbf{a}' + \mathbf{a}''). \tag{10}$$

[9] N. Hitchin, Compact four-diemensional Einstein manifolds, J. Differential Geometry 9 (1974), 435–441.

By taking account of (10),

$$\chi(M) = \frac{1}{4\pi^2}\int_M \{2|\mathbf{a}|^2 - |\mathbf{a}'|^2 + |\mathbf{b}|^2\} dv_g,$$

$$p_1(M) = \frac{1}{2\pi^2}\int_M 2 < \mathbf{a}, \mathbf{b} > dv_g.$$

Then, we have the following:

$$\begin{aligned}
&2\chi(M) \pm p_1(M)\\
=&\frac{1}{2\pi^2}\int_M \{|\mathbf{a}|^2 + |\mathbf{b}|^2 \pm 2 < \mathbf{a}, \mathbf{b} > + |\mathbf{a}|^2 - |\mathbf{a}'|^2\} dv_g\\
=&\frac{1}{2\pi^2}\int_M \{|\mathbf{a} \pm \mathbf{b}|^2 + |\mathbf{a}|^2 - |\mathbf{a}'|^2\} dv_g\\
\geqq&\frac{1}{2\pi^2}\int_M \{|\mathbf{a}|^2 - |\mathbf{a}'|^2\} dv_g.
\end{aligned}$$

Theorem (2.3)

Let $M = (M, g)$ be a compact weakly Einstein manifold. Then, the following inequality holds on M:

$$2\chi(M) \pm p_1(M) \geqq C, \tag{11}$$

where $C = \frac{1}{2\pi^2} \int_M \{|\mathbf{a}|^2 - |\mathbf{a}'|^2\} dv_g \leqq 0$.

Remark. Since $p_1(M) = 3\sigma(M)$ ($\sigma(M)$ is the Hirzebruch signature of M), from Theorem (2.3), the inequality (11) reduces to the Hitchin inequality [9]

$$2\chi(M) \geqq 3|\sigma(M)|, \tag{12}$$

for the case where M is Einstein. Thus, (11) in Theorem (2.3) is regarded as the generalization of the Hitchin inequality (12).

Theorem (2.3)

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Theorem (2.4)

Let $M = (M, g)$ be a compact weakly Einstein manifold. Then, the following inequality holds on M:

$$2\chi(M) \pm p_1(M) \geqq -\frac{1}{16\pi^2}\int_M |R|^2 dv_g, \tag{13}$$

where $\chi(M)$ and $p_1(M)$ are denoted the Euler number and the first Pontrjagin number of M, respectively.

sketch of proof. In Theorem (2.3) the constant C satisfies the inequality $C \geqq -\frac{1}{16\pi^2}\int_M |R|^2 dv_g$.

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The following example illustrates Theorems (2.3) and (2.4).

Example (6)

Let M_1 and M_2 be a unit 2-sphere and a compact oriented surface of genus m ($m \geqq 2$) with constant Gaussian curvature -1, respectively, and further, M be the Riemannian product of M_1 and M_2, $M = M_1 \times M_2$. Then, M is a compact, oriented weakly Einstein manifold which is a special case of Example 1. Then, by taking account of the Künneth formula, the Gauss-Bonnet formula and the formulas in [9], we have

$$\begin{aligned} &\chi(M) = 4(1-m),\ p_1(M) = 0 \ (\textit{thus, } \sigma(M) = 0), \\ &Vol(M) = 16(m-1)\pi^2, \quad C = 8(1-m). \end{aligned} \tag{14}$$

Therefore, from (14), *the equality signs of the inequalities* (11) *and* (13) *in Theorems* (2.3) *and* (2.4) *hold for M respectively, but M does not satisfy the Hitchin inequality* (12).

Outline

Let M be a 4-dimensional oriented smooth manifold and $\mathfrak{M} = \mathfrak{M}(M)$ be the space of all Riemannian metrics on M. We denote the squared L^2-norm functionals of the curvature tensor, the Ricci tensor and the scalar curvature by $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively.

It is well-known that the Euler number $\chi(M)$ of a 4-dimensional compact, oriented Riemannian manifold $M = (M, g)$ is given by the following generalized Gauss-Bonnet formula:

$$32\pi^2\chi(M) = \mathcal{A}(g) - 4\mathcal{B}(g) + \mathcal{C}(g). \tag{15}$$

A metric $g \in \mathfrak{M}$ is $\mathcal{A}$-critical if and only if g satisfies $\mathcal{A}(g) < +\infty$ and the following equation

$$R_{abci}R^{abc}{}_{j} + 2\nabla^{a}\nabla_{a}\rho_{ij} - \nabla_{j}\nabla_{i}\tau - 2\rho_{ia}\rho_{j}{}^{a} + 2\rho^{ab}R_{iabj} - \frac{1}{4}|R|^{2}g_{ij} = 0. \quad (16)$$

A metric $g \in \mathfrak{M}$ is called $\mathcal{B}$-critical if and only if g satisfies $\mathcal{B}(g) < +\infty$ and the following equation

$$2\rho^{ab}R_{iabj} + \frac{1}{2}(\triangle\tau)g_{ij} + \nabla^{a}\nabla_{a}\rho_{ij} - \nabla_{j}\nabla_{i}\tau - \frac{1}{2}|\rho|^{2}g_{ij} = 0. \quad (17)$$

A metric $g \in \mathfrak{M}$ is $\mathcal{C}$-critical if and only if g satisfies $\mathcal{C}(g) < +\infty$ and the following equation

$$\tau\rho_{ij} - \nabla_{j}\nabla_{i}\tau + (\triangle\tau)g_{ij} - \frac{1}{4}\tau^{2}g_{ij} = 0. \quad (18)$$

Remark. In general, although critical Riemannian metrics were first defined on a compact case, it is easy to generalize the definition when M is not compact by considering variations of Riemannian metrics with compact support.

Remark. If $g \in \mathfrak{M}$ is $\mathcal{A}$-critical, $\mathcal{B}$-critical or $\mathcal{C}$-critical then $\triangle \tau = 0$.

Especially, if M is compact oriented and g is a critical metric of one of the functionals $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$, the scalar curvature τ is constant since τ is a harmonic function on M.

Theorem (2.5)

Let $M = (M, g)$ be a 4-dimensional compact oriented Riemannain manifold. If $g \in \mathfrak{M}$ is $\mathcal{C}$-critical, then the scalar curvature τ vanishes everywhere on M or M is Einstein.

[10] A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, 1987.

Lemma

If $g \in \mathfrak{M}$ satisfies $\mathcal{A}(g) < +\infty$ ($\mathcal{B}(g) < +\infty$, respectively), then $\mathcal{B}(g) < +\infty$ and $\mathcal{C}(g) < +\infty$ ($\mathcal{C}(g) < +\infty$, respectively).

Since the Euler number is the topological invariant for the case where M is compact and oriented, from (15), if g is both $\mathcal{B}$-critical and $\mathcal{C}$-critical then g is also $\mathcal{A}$-critical. This is also valid for the case where M is not compact.

From (16)∼(18) and taking account of (1) and the above Lemma,

Theorem (2.6)

(1) If $g \in \mathfrak{M}$ with $\mathcal{A}(g) < +\infty$ is $\mathcal{B}$-critical and $\mathcal{C}$-critical, then g is $\mathcal{A}$-critical.

(2) If $g \in \mathfrak{M}$ is $\mathcal{A}$-critical and $\mathcal{B}$-critical, then g is $\mathcal{C}$-critical.

(3) If $g \in \mathfrak{M}$ is $\mathcal{A}$-critical and $\mathcal{C}$-critical, then g is $\mathcal{B}$-critical.

[11] M. Berger, P. Gauduchon and E. Mazet, Le specte d'une varieté Riemannianenne, Lecture Notes in Math., 194, Springer-Verlag, Berlin-New York,

Denote the traceless Ricci tensor G of M by

$$G = \rho - \frac{\tau}{4} g. \tag{19}$$

Lemma

Let $M = (M, g)$ be a 4-dimensional Riemannian manifold with parallel Ricci tensor. If $G^{ab} R_{iabj} G^{ij} = 0$ everywhere on M, then M is Einstein or locally a product of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c and $-c$ $(c \neq 0)$.

Theorem (2.7)

Let g be a $\mathcal{B}$-critical metric on a 4-dimensional smooth manifold M. If the scalar curvature τ of g is non-positive constant and the square norm of the Ricci tensor ρ of g is constant and the inequality $G^{ab}R_{iabj}G^{ij} \leqq 0$ holds on M, then (M, g) is Einstein or locally a product of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c and $-c$ $(c \neq 0)$.

Theorem (2.8)

Let g be a $\mathcal{B}$-critical metric on a 4-dimensional compact oriented manifold M. If the scalar curvature τ of g is non-positive and the inequality $G^{ab}R_{iabj}G^{ij} \leqq 0$ holds on M then (M, g) is Einstein or locally a product of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c and $-c$ $(c \neq 0)$.

Theorem (2.9)

Let g be an $\mathcal{A}$-critical metric on a 4-dimensional smooth manifold M. If the scalar curvature τ of g and the square norm of the Ricci tensor ρ of g are constant and the inequality $G^{ab}R_{iabj}G^{ij} \leqq 0$ holds on M, then (M, g) is Einstein or locally a product of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c and $-c$ $(c \neq 0)$.

Theorem (2.10)

Let g be an $\mathcal{A}$-critical metric on a 4-dimensional compact oriented manifold M. If the inequality $G^{ab}R_{iabj}G^{ij} \leqq 0$ holds on M then (M, g) is Einstein or locally a product of 2-dimensional Riemannian manifolds of constant Gaussian curvatures c and $-c$ $(c \neq 0)$.

Remark. Ogiue and Tachibana proved that if the inequality $u_{ab}R^{iabj}u_{ij} < 0$ (at any point where $u \neq 0$) holds for any traceless symmetric (0,2)-tensor field $u = (u_{ij})$ on an n-dimensional compact oriented Riemannian manifold (M, g) then M is a real homology n-sphere.

[12] K. Ogiue and S. Tachibana, Les variétés riemanniennes dont l'opérateur de courbure restreint est positif sont des sphères d'homologie réelle, C. R. Acad. Sci. Paris Sér. 289 (1979), A29–A30.

Theorem (2.9) is related to the following result.

Theorem (2.11)

A complete locally homogeneous $\mathcal{A}$-critical metric with finite volume on a four-manifold is locally isometric to either an Einstein symmetric space or the product of the standard two-sphere with constant curvature c and the two-dimensional hyperbolic plane with curvature $-c$.

[13] Y. Kang, Locally homogeneous critical metrics on four-dimensional manifolds, J. Korean Math. Soc. 44 (2007), 109–127.

The weakly Einstein manifold in Example 1 is zero-scalar-curved conformally flat. In general, a 4-dimensional zero-scalar-curved conformally flat manifold is weakly Einstein. The following example is a special case of Example 1 whose metric is $\mathcal{A}$-critical.

Example (7)

Let S^2 and $\mathbb{H}^2$ be a unit 2-sphere and a upper hyperbolic 2-plane with constant Gaussian curvature -1, respectively. It is well-known that the group $SL(2,\mathbb{R})$ acts on $\mathbb{H}^2$ transitively as the automorphism group of the canonical Kähler structure $\mathbb{H}^2$. Let $M = (M, g)$ be the Riemannian product of S^2 and $\Gamma\backslash\mathbb{H}^2$, where Γ is the Fuchsian group of the first kind.

For example, if Γ is a modular group and $\Gamma = SL(2, \mathbb{Z})$, where $\mathbb{Z}$ is the set of all integers, then $\Gamma\backslash\mathbb{H}^2$ is a non-compact Riemann surface which is contractible, and further, $Vol(\Gamma\backslash\mathbb{H}^2) = \frac{\pi}{3}$. Therefore, M is non-compact and $Vol(M, g) = \frac{4}{3}\pi^2$, $\chi(M) = 2$, and hence $\mathcal{A}(g) = \frac{8}{3}\pi^2\chi(M)$. On the other hand, let $\Gamma'\backslash\mathbb{H}^2$ be a compact Riemann surface with genus $m(\geqq 2)$ and (M', g') be the Riemannian product of S^2 and $\Gamma'\backslash\mathbb{H}^2$. Then, $\mathcal{A}(g) = -32\pi^2\chi(M)$ holds for all $m(\geqq 2)$ (Example 6.)

Remark. From Example 7, we see that the following statement is wrong:

4.82 Proposition (ii) conformally flat metrics with zero scalar curvature give an absolute minimum for $\mathcal{A}$; this minimum is $-8\pi^2\chi(M)$ ([10] p. 136).

[10] A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer-Verlag, 1987.

Question

- Does there exist a (not conformally flat) half-conformally flat non-Einstein, weakly Einstein manifold with zero-scalar-curvature?

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- Does there exist a (not conformally flat) half-conformally flat non-Einstein, weakly Einstein manifold with zero-scalar-curvature?

Question

- Classify simply connected homogeneous weakly Einstein manifolds.

Outline

Question

Does the same phenomenon occur for any $2n (n \geqq 2)$-dimensional Riemannian manifold?

- Can we deduce a curvature identity from the generalized Gauss-Bonnet theorem for a $2n$-dimensional compact oriented Riemannian manifold?
- Is this identity valid for any $2n$-dimensional Riemannian manifold?

Theorem B

Theorem

$M = (M, g)$: an n-dimensional Riemannian manifold

$S^n = (S^n, g') \subset \mathbb{E}^{n+1}$: an round unit n-sphere

Then, for each fixed point $p \in M$

- *$\exists$ a Riemannian metric $\tilde{g}$ on S^n*
- *$\exists$ neighborhoods U of p*
- *$\exists$ V of the north pole of S^n*

such that $(V, \tilde{g}) \underset{\text{isometry}}{\approx} (U, g)$.

From Theorem B,

Corollary

A curvature identity which holds on any compact Riemannian manifolds is also valid without compactness as it is.

Corollary ([3])

Let $M = (M, g)$ be any $2n$-dimensional Riemannian manifold. Then,

$$h_{2n}g - \frac{1}{(2n-1)!}c^{2n-1}R^n = 0.$$

[3] M.-L. Labbi, Variational properties of the Gauss-Bonnet curvatures, Calc. Var. 32 (2008), 175–189.

Thank you for your attention!