Geometric modelling of the solutions of some Monge-Ampère equations

in honor of Peter B. Gilkey

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AIM:
Present 3 different geometric models

Theorem Jörgens-Calabi-Pogorelov
J-C-P - geometric version
An improper affine hypersphere defined as strictly convex graph over $\mathbb{R}^n$ *is an elliptic paraboloid.*

Theorem - J-C-P - PDE version
Let $f : \mathbb{R}^n \to \mathbb{R}$ *with* $x := (x^1, \cdots, x^n) \mapsto f(x)$ *be a strictly convex* C^∞*-function defined for all* $x \in \mathbb{R}^n$. *If* f *satisfies the Monge-Ampère equation*

$$\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right) = 1$$

then f *must be a quadratic polynomial.*

n=2: K. Jörgens (1954)
$n = 3, 4, 5$: E. Calabi (1958)
$n \geq 2$: A.V. Pogorelov (1972), PDE-proof
$n \geq 2$: S.Y. Cheng, S.T. Yau (1986), geom. proof.

Geometric background

Tools: We use different relative geometries for modelling proofs.

1. J-C-P Theorem:

. $2 \leq n$, use *Blaschke's geometry.*

2. First extension J-C-P Theorem:

. $n \leq 4$, use the *Calabi metric*.

3. Second extension J-C-P Theorem:

. $2 \leq n$, conformal deformation of

. the *Calabi metric*.

Relative hypersurfaces - a short survey

$x : M^n \to \mathbb{R}^{n+1}$ satisfies:

- x is non-deg. (here: loc. str. convex),
- *normalization:* pair (U, Y):

. $Y : M^n \to (\mathbb{R}^{n+1})$ is transversal

. $U : M^n \to (\mathbb{R}^{n+1})^*$ conormal with

. $< U, dx >= 0$ and $< U, Y >= 1$.

Invariance group: general affine transf.

Gauss equations

$$\overline{\nabla}_v dx(w) = dx(\nabla_v w) + h(v,w)Y,$$

h *relative metric*,
∇ torsion free, Ricci symmetric *induced* connection

$$\overline{\nabla}_v dU(w) = dU(\nabla^*_v w) + \frac{1}{n-1} Ric^*(v,w)(-U),$$

∇^* torsion free, Ricci symmetric *conormal* connection

Weingarten equation

$$dY(v) = -\,dx(Sv).$$

S *relative shape operator*

All coefficients are *affinely invariant*

Facts and Notation.
Relative theory for x loc. str. convex:
- any $U : M^n \to (\mathbb{R}^{n+1})^*$ is immersion,
- h is pos. definite Riem. metric.

- $A(v,w) := \nabla(h)_v w - \nabla^*_v w$ is symmetric,
- assoc. *cubic form* $A^\flat$ is tot. symm.:

. $A^\flat(u,v,w) := h(u, A(v,w))$

- $nT^\flat(v) := trace(w \mapsto A(v,w))$ closed,
- $\|A\|^2 =: n(n-1) \cdot J$ *Pick invariant.*

Blaschke hypersurfaces *Invar. group:*
Unimodular (equiaffine) group (incl. transl).
There exist:
- unique unimod. invar. *affine normal*
- unique unimod. invar. *Blaschke metric.*

Charact. Blaschke geom.: $T^\flat = 0$.

Constant normal: Calabi metric $\mathfrak{H}$:
Describe geom. in terms of a local convex graph with constant transversal field.
Calabi, Pogorelov.

Special hypersurfaces

- **proper affine sphere** center x_o,
  $y = \lambda \cdot (x - x_o)$, $\lambda \neq 0$;
  Blaschke geometry:
  characterization $S = H \cdot id$,
  $H = const \neq 0$.

- **improper affine sphere**
  Blaschke geom.: $y = const$.
  characterization: $S = 0$.

  Calabi geom.: $T^\flat = 0$.

- **hyperquadric**
  within Blaschke geom.
  characterization: $A = 0$.
  loc. str. convex: $J = 0$.

Note: The class of affine spheres is

very large - see PDEs

So far only partial classifications exist under very strong additional assumptions.

Theorem J-C-P
- sketch of a geometric proof
Use Blaschke's geometry

1. Charact. improper aff. spheres:
x loc. str. convex h-surf. in $\mathbb{R}^{n+1}$
Assume:
x graph described by str. conv. $f : \Omega \to \mathbb{R}$
s.t. affine normal $Y = (0, \cdots, 0, 1)$
and $\mathbb{R}^n$ tangent plane in $0 \in \mathbb{R}^{n+1}$.
Then:
x loc. strict. conv. improp. aff. h-sphere
$\Longleftrightarrow$ *f satisfies det$(\partial_i \partial_j f) = const > 0$.*

2. Completeness:
x loc. str. convex improp. aff. hyp-sphere
Then:
(i) $Ric(G) \geq 0$
(ii) graph over $\mathbb{R}^n$ (*Eucl. compl.*) implies:
. metric G complete (*affine compl.*)
. (S.Y. Cheng, S.T. Yau)
Proof: use structure equat., derive
estimates up to 3-rd order derivatives.

3. First PDE for the Pick invariant

. on improper affine hyper-spheres:

Calabi, R. Schneider, Cheng-Yau, et al.:

$$\frac{n}{2}\,\Delta J \;\;\geq\;\; \frac{1}{(n-1)}\|\nabla A\|^2 + n(n+1)J\,(J+H)$$

$$\frac{\Delta\,J}{J} \geq\; 2\,(n+1)J \quad \text{(lower estim. if } J \neq 0\text{)}.$$

4. Geodesic distance function: r

$x := \left(x^1, \cdots, x^n\right) \in \mathbb{R}^n;\quad p, p_o \in M$:

graph $\quad M := \left\{\left(x, x^{n+1}\right) \mid x^{n+1} = f(x)\right\}.$

geod. dist. fctn. (Blaschke metric).

. $\qquad r : p \mapsto r(p_o, p) =: r.$

$\bar{B}(p_o, a) = \{p \in M \mid r \leq a\}$ geod. ball.

Test function: Define $F : B(p_o, a) \to \mathbb{R}$

$$F(p) := (a^2 - r^2)^2 \;\cdot J(p),$$

Note:

(i) boundary behaviour: $\quad F|_{\partial B} \downarrow 0,$

(ii) F attains sup. at interior $p^* \in B(p_o, a)$

. $\quad J\,(p^*) > 0, \quad r^2 \in C^2$ around p^*.

5. Second PDE for the Pick invariant:
Δ is G-Laplacian - calculate $\Delta\, F$:

$\frac{\Delta J}{J} \;\le\; 24\,\frac{r^2\|\mathrm{grad}\, r\|^2}{(a^2-r^2)^2} + 4\,\frac{\|\mathrm{grad}\, r\|^2}{a^2-r^2} + 4\,\frac{r\Delta r}{a^2-r^2}.$
On compl. Riem. mfd. with $Ric \ge 0$ the *Laplacian comparison thm* allows
to estimate Δr;   here:

$$r\Delta r \le (n-1).$$

Simplify above PDE   for all $p \in \bar{B}(p_o, a)$:

$\frac{\Delta J}{J}(p) \;\le\; 24\,\frac{r^2}{(a^2-r^2)^2} \;+\; \frac{4n}{(a^2-r^2)}$, upper est.

6. Combine PDEs
Combine (3) and (5), then at $p^* \in \bar{B}(p_0, a)$:

$$(a^2-r^2)^2\; J(p^*) \;\le\; \frac{2(n+6)}{n+1}\cdot a^2$$

at any interior $p \in B(p_0, a)$ with $r = r(p)$:

$J(p) \le \frac{2(n+6)}{n+1}\cdot\frac{a^2}{(a^2-r^2)^2} \;=\; \frac{2(n+6)}{n+1}\cdot\frac{1}{a^2\left(1-\frac{r^2}{a^2}\right)^2}$.

7. Conclusion:
G complete; let $a \to \infty$,
then $J(p) \le J(p^*) \to 0$,
thus (improper) quadric - elliptic paraboloid.

Two extensions of the J-C-P Theorem

(A)

$u : \mathbb{R}^n \to \mathbb{R}$ *strictly convex graph*

. $\qquad (\xi) \mapsto u(\xi).$

If u satisfies the M-A PDE

$$\det\left(\frac{\partial^2 u}{\partial\xi_i \partial\xi_j}\right) = \exp\left\{-\sum c_i \frac{\partial u}{\partial \xi_i} - c_o\right\},$$

where c_o, c_1,...,$c_n \in \mathbb{R}$, then:

u must be a quadratic polynomial.

Precisely: Only in case that $c_i = 0$ for $i = 1, ..., n$ there exists a global solution, namely quad. polynom.

(B)

$f : \mathbb{R}^n \supset \Omega \to \mathbb{R}$ *strictly convex graph*

. $\qquad (x) \mapsto f(x), \qquad \mathfrak{H}^\alpha$ *complete.*

If f satisfies the M-A PDE

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \exp\left\{\sum d_i \frac{\partial f}{\partial x_i} + d_o\right\},$$

where d_o, d_1,...,$d_n \in \mathbb{R}$, then:

f must be a quadratic polynomial.

Different proofs:

- **(A)**

Restrict to $n \leq 4$; use Calabi's metric $\mathfrak{H}$,
u on $\mathbb{R}^n$ globally defined;
for $n \geq 5$ the proof is very lengthy and technical, see the monograph
A.M. Li, R. Xu, U. Simon, F. Jia:
Affine Bernstein Problems and Monge-Ampère Equations,
World Sci. 2010.

- **(B)**

$n \geq 2$; conformal deformation:
$\mathfrak{H}^{(\alpha)} := \rho^{(\alpha)}\, \mathfrak{H}$ complete.

Tools: Legendre transformation

$\Omega \subset \mathbb{R}^n$ domain, $x := (x^1, ..., x^n) \in \Omega$,
$f : \Omega \to \mathbb{R}$ loc. strongly convex C^∞.

Legendre transformation of f:
new coord.: $\xi_i := \frac{\partial f}{\partial x^i}$, $\xi := (\xi_i)_{i=1,...,n}$
Leg. transf. fct. $u(\xi) := \sum x^i \frac{\partial f}{\partial x^i} - f(x)$

Ω^* *Legendre transform domain* of f,
$\Omega^* := \{ \xi(x) \mid x \in \Omega \}$, $u : \Omega^* \to \mathbb{R}$.
$x^i = \frac{\partial u}{\partial \xi_i}$, $f(x) = \sum \xi_i \frac{\partial u}{\partial \xi_i} - u(\xi)$.

Relation $x \in \Omega \leftrightarrow \xi \in \Omega^*$ bijective

$$\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} = \frac{\partial x^i}{\partial \xi_j}, \qquad \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \xi_i}{\partial x^j}.$$

Inverse matrices:

$$\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j}\right) \qquad \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)$$

Advantage: Appropriate use of coord's.

Tools: Graph & Calabi metric.

$x \in \Omega \subset \mathbb{R}^n$ domain,
$f : \Omega \to \mathbb{R}$ loc. strongly convex C^∞,
convex graph hypersurface
$M := \{(x, f(x)) \mid x^{n+1} = f(x), \quad x \in \Omega\}$
relative normalization:
$Y := (0, 0, ..., 1), \qquad U = (-f_1, ..., -f_n, 1).$

Legendre coord.: Gauß basis $\quad \{\partial_i\}$
loc. notation: $f_{ij} := \partial_i \partial_j f, \qquad f_{ij} \cdot f^{jk} = \delta^i_k.$

. $\qquad u_{ij} := \partial_i \partial_j u, \qquad u_{ij} \cdot u^{jk} = \delta^i_k.$

Calabi metric

$$\sum u_{ij} d\xi_i d\xi_j = \mathfrak{H} = \sum f_{ij} dx^i dx^j$$

f loc. str. convex $\Leftrightarrow$ (f_{ij}) pos. def.
$\Leftrightarrow$ $\mathfrak{H}$ Riem.

Invariants ($\mathfrak{H}$-geometry) in terms of f:

Christoffel symbols $\Gamma_{ij}^{k} = \frac{1}{2}\sum f^{kl} f_{ijl}$

cubic form $A_{ijk} = -\frac{1}{2} f_{ijk}$

Weingarten tensor
(impr. aff. sph.): $B_{ij} = 0$

Tchebychev vector $T := \frac{1}{n}\sum f^{ij} A_{ij}^{k} \partial_k$

Pick invariant

$$J = \frac{1}{4n(n-1)} \sum f^{il} f^{jm} f^{kr} f_{ijk} f_{lmr}$$

Invariants ($\mathfrak{H}$-geom.) in terms of u:
analogously

Auxiliary functions: ρ **and** Φ

Use Calabi metric $\mathfrak{H}$:

$$\rho := \left[\det\left(\frac{\partial^2 u}{\partial\xi_i\,\partial\xi_j}\right)\right]^{\frac{1}{n+2}} = \left[\det\left(\frac{\partial^2 f}{\partial x_i\,\partial x_j}\right)\right]^{-\frac{1}{n+2}}$$

Clou: Use Legendre coord. twofold:
ρ can be considered in both coord. syst's.
Calculate (in appropr. coord. system):

$$\Delta f = n + \frac{n+2}{2\rho}\,\mathfrak{H}(\mathrm{grad}\,\rho, \mathrm{grad}\,f), \quad x-\mathrm{coord}$$

$$\Delta u = n - \frac{n+2}{2\rho}\,\mathfrak{H}(\mathrm{grad}\,\rho, \mathrm{grad}\,u), \quad \xi-\mathrm{coord}$$

$$\Phi := \frac{4n^2}{(n+2)^2}\,\|T\|^2 = \frac{\|grad\,\rho\|^2}{\rho^2}, \quad x \text{ or } \xi$$

Geometry: Calabi geometry:
$\|T\|^2 = 0 \;\Leftrightarrow\;$ improper affine sphere
Analysis: If $\Phi = 0$ then $\rho = const \neq 0$, thus we arrive at original M-A-equation

$$\det\left(\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}\right) = const$$

J-C-P extended: Proofs

Steps **(1)** and **(2)**: use Calabi metric $\mathfrak{H}$

1. Step. Write M-A-PDE in terms of ρ:

$$\Delta\,\rho = \frac{n+4}{2}\,\frac{\|\mathrm{grad}\,\rho\|^2}{\rho}.$$

2. Step. Let $f \in C^\infty$ str. conv. satisf. M-A-PDE. Calc. $\Delta\Phi$ for $\Phi := \frac{\|grad\ \rho\|^2}{\rho}$.

$$\Delta\Phi \;\geq\; \frac{n}{(n-1)}\,\frac{\|grad\,\Phi\|^2}{\Phi} \;+$$

$$+\,\frac{n^2-3n-10}{2(n-1)}\,\mathfrak{H}(grad\,\Phi, grad\,\ln\rho) + \frac{(n+2)^2}{(n-1)}\,\Phi^2.$$

Lemma: Let $f \in C^\infty$ str. conv. describe a graph hyps. and satisfy M-A-PDE. Then:

$$\Delta\,J \geq J^2 - 10(n+2)^8\,\Phi^2.$$

Ext. (A) - proof: Use Calabi metric $\mathfrak{H}$, $2 \leq n \leq 4$.

3. Step. W. l. o. g. assume:
$u(0) = 0,\ grad\, u(0) = 0,\ u(\xi) \geq 0,\ \xi \in \mathbb{R}^n$.
For $C > 0$ define level set

$$S_u(0, C) := \{\xi \mid u(\xi) < C\}$$

and *test function* $F : S_u(0, C) \to \mathbb{R}$ by:

$$F(\xi) := \exp\left\{-\frac{m}{C - u(\xi)}\right\} \Phi(\xi) \geq 0.$$

Note:
(i) boundary behaviour: $F|_{\partial B} \downarrow 0$,
(ii) F attains sup. at interior $p^* \in B(p_o, a)$.

Calculate ΔF at p^*; this gives:

$$\frac{\Delta\Phi}{\Phi} - \frac{\sum(\Phi_{,i})^2}{\Phi^2} - \gamma' \sum(u_{,i})^2 - \gamma\, \Delta u \leq 0$$

where $\gamma := \frac{m}{(C-u)^2}$ and $\gamma' := 2\, \frac{m}{(C-u)^3}$.

4. Step. Combine PDEs from **2** and **3**:

$$B \cdot \Phi \;+\; \left(\tfrac{1}{2(n-1)}\,\gamma^2 - \gamma\,'\right) \sum (u_{,i})^2 \;\leq\; n\gamma.$$

Note:

(i) For $n \leq 4$:

$$B := \tfrac{(n+2)^2(2-(n-3)^2)}{2(n-1)} \geq 0.$$

(ii) Choose $m = 8(n-1)C$, then

$$\left(\tfrac{1}{2(n-1)}\,\gamma^2 - \gamma\,'\right) = \tfrac{16(n-1)C}{(C-u)^3} \cdot \tfrac{(C+u)}{(C-u)} > 0.$$

(iii) $\exp\left\{-\frac{m}{C-u}\right\} \frac{m^2}{(C-u)^2}$ univ. up. bound.

Finally with $b = b(n)$ at p^*:

$$\exp\left\{-\tfrac{8(n-1)C}{C-u}\right\} \Phi \leq \tfrac{b}{C}$$

F attains its sup at p^*,

thus inequal. yields on $S_u(0, C)$.

Euclid. compl:

$C \to \infty$ then $\Phi(p^*) = 0$, thus $\Phi(p) \equiv 0$,

therefore $\rho = const.$

We arrive at the J-C-P MA-equation.

Ext. (B) - proof $n \geq 2$.
Use metrics $\mathfrak{H}$ and conformal $\mathfrak{H}^\alpha := \rho^\alpha \cdot \mathfrak{H}$.

3. Step. Geod. dist. fct. r for $\mathfrak{H}^\alpha$−metric: $\bar{B}^\alpha(p_o, a)$ geod. ball.
Aim: Estimate for $\frac{1}{\rho^\alpha} \cdot \Phi$ on $\bar{B}^\alpha(p_o, a)$.
We need **two** test functions on $\bar{B}^\alpha$.

$$\mathcal{A} := max\{(a^2 - r^2)^2 \cdot \tfrac{1}{\rho^\alpha} \cdot \Phi\},$$

$$\mathcal{B} := max\{(a^2 - r^2)^2 \cdot \tfrac{1}{\rho^\alpha} \cdot J\}.$$

Boundary behaviour as above.
Reason for 2 test-funct's:
To estimate $\frac{1}{\rho^\alpha} \cdot \Phi$, we have to apply the *Laplacian Comparison Theorem* again. For this we need

$$Ric^\alpha \geq -D \left(\tfrac{1}{\rho^\alpha} \cdot \Phi + \tfrac{1}{\rho^\alpha} \cdot J \right)$$

where $D = D(n, \alpha) \in \mathbb{R}$. A very long calculation gives:

Lemma: ; Let $f \in C^\infty$ str. conv. describe a graph hypers. with $\mathfrak{H}^\alpha$-metric and satisfy the M-A-PDE. Assume $\alpha \neq n+2$. Then there exist a constant $C = C(n,\alpha)$ s.t. on $\bar{B}^\alpha$:

$$\mathcal{A} \leq C\, a^2, \quad \text{and} \quad \mathcal{B} \leq C\, a^2.$$

4. Step. $\mathcal{A} \leq C\, a^2$ implies on B^α:

$$(a^2 - r^2)^2 \cdot \tfrac{1}{\rho^\alpha} \cdot \Phi \leq C\, a^2,$$

$$\tfrac{1}{\rho^\alpha} \cdot \Phi \leq C\, \tfrac{a^2}{(a^2-r^2)^2}\,.$$

For $a \longrightarrow \infty$ we get $\Phi \equiv 0$.
Apply the original J-C-P Theorem.

Comparison of all proofs

JCP: Use Blaschke geometry,
given PDE char. imp. aff. sphere.
Aim: aff. sphere is quadric,
thus show: Pick inv. J vanishes.
Known:
(i) Euclid. complete then Blaschke compl.
(ii) PD-inequ. $\Delta J \geq 2(n+1)J^2$.
Derive: second PDE on geod. ball,
using $F(p) := (a^2 - r^2)^2 \cdot J(p)$, at p^*.
Note: boundary behaviour of F.
Combine inequ.: gives estim.,

$$J(p) \leq J(p^*) \leq const(n) \cdot \frac{1}{a^2\left(1-\frac{r^2}{a^2}\right)^2} \to 0$$

(use Blaschke completeness).

Ext. (A): Given PDE on $\mathbb{R}^n$, Calabi geom.
Aim: x is improper aff. sphere, $T = 0$
Rewrite: Given PDE in terms of $\Phi \sim \|T\|^2$
Derive: second PDE on level set
using $F(p) := exp\left(-\frac{const}{C-u(\xi)}\right) \cdot \Phi$.
Note: boundary behaviour of F.
Combine two PDE inequal.: At p^*:
$exp\left(-\frac{const}{C-u(\xi)}\right) \cdot \Phi \leq \frac{b(n)}{C} \to 0$
(use Euclid. completeness).

Ext. (B): Given PDE on $\Omega \subset \mathbb{R}^n$,
$\mathfrak{H}^\alpha$-completeness. Study **two** test funct. on geod. ball,

$$\mathcal{A} := max\{(a^2 - r^2)^2 \cdot \frac{1}{\rho^\alpha} \cdot \Phi\},$$

$$\mathcal{B} := max\{(a^2 - r^2)^2 \cdot \frac{1}{\rho^\alpha} \cdot J\}.$$

Apply Laplace Comp. Thm.; finally:

$$\frac{1}{\rho^\alpha} \cdot \Phi \leq C \, \frac{a^2}{(a^2-r^2)^2} \, .$$

$a \to \infty$ gives $\Phi \equiv 0$.