Asymptotics of the heat exchange

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Abstract

Let K be a compact subset in Euclidean space $\mathbb{R}^m$, and let $E_K(t)$ denote the total amount of heat in $\mathbb{R}^m \backslash K$ at time t, if K is kept at fixed temperature 1 for all $t \geq 0$, and if $\mathbb{R}^m \backslash K$ has initial temperature 0. For two disjoint compact subsets K_1 and K_2 we define the heat exchange $H_{K_1,K_2}(t) = E_{K_1}(t) + E_{K_2}(t) - E_{K_1 \cup K_2}(t)$. We obtain the leading asymptotic behaviour of $H_{K_1,K_2}(t)$ as $t \to 0$ under mild regularity conditions on K_1 and K_2.

1 Introduction

Let K be a compact set in $\mathbb{R}^m$, and let $u : \mathbb{R}^m \setminus K \times [0, \infty) \to [0, \infty)$ be the unique weak solution of the heat equation

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in \mathbb{R}^m \backslash K, \quad t > 0, \tag{1}$$

with initial condition

$$u(x; 0) = 0, \quad x \in \mathbb{R}^m \backslash K, \tag{2}$$

and with boundary condition

$$u(x; t) = 1, \quad x \in \partial K, \quad t \geq 0. \tag{3}$$

Let

$$E_K(t) = \int_{\mathbb{R}^m \backslash K} u(x; t) dx. \tag{4}$$

Then $E_K(t)$ represents the total amount of heat in $\mathbb{R}^m \backslash K$ at time t, if K is kept at fixed temperature 1 for all $t \geq 0$, while $\mathbb{R}^m \backslash K$ has initial temperature 0.

In this paper we investigate the reduction of the heat flow from K in the presence of a second disjoint compact subset. Let K_1, K_2, be a pair of disjoint compact subsets in $\mathbb{R}^m$, and define the heat exchange $H_{K_1,K_2} : [0, \infty) \to \mathbb{R}$ by

$$H_{K_1,K_2}(t) = E_{K_1}(t) + E_{K_2}(t) - E_{K_1 \cup K_2}(t). \tag{5}$$

It is well-known [1, 2], that if K_1, and K_2 have C^∞ smooth boundaries ∂K_1 and ∂K_2 respectively, then there exists coefficients $b_n^{(1)}, b_n^{(2)}, n \in \mathbb{N}$ such that for any $N \in \mathbb{N}$

$$E_{K_i}(t) = \sum_{n=1}^{N} b_n^{(i)} t^{n/2} + O(t^{(N+1)/2}),\ t \to 0, i = 1, 2. \tag{6}$$

Since the $b_n^{(i)}$ are locally computable in terms of the curvature and its derivatives we have that

$$E_{K_1 \cup K_2}(t) = \sum_{n=1}^{N} \sum_{i=1}^{2} b_n^{(i)} t^{n/2} + O(t^{(N+1)/2}), \quad t \to 0. \tag{7}$$

It follows that for any $N \in \mathbb{N}$

$$H_{K_1,K_2}(t) = O(t^{(N+1)/2}), \quad t \to 0. \tag{8}$$

The main result of this paper is a refinement and generalization of (8) (Theorem 1). Let $B(x;\epsilon)$ denote the closed ball with centre x and radius ϵ, and denote by $Cap(K)$ the Newtonian capacity of a compact set K in $\mathbb{R}^m$ if $m \geq 3$. Let $|A|$ denote the Lebesgue measure of a Borel set A in $\mathbb{R}^m$. We make the following hypotheses.

(i) If $m = 2$ then

$$|K \cap B(x;\epsilon)| > 0, \quad x \in K, \quad \epsilon > 0. \tag{9}$$

(ii) If $m \geq 3$ then

$$Cap(K \cap B(x;\epsilon)) > 0, \quad x \in K, \epsilon > 0. \tag{10}$$

Theorem 1. *Let K_1, K_2 be disjoint compact subsets in $\mathbb{R}^m$. Suppose that both K_1 and K_2 satisfy* (9) *if $m = 2$ and* (10) *if $m \geq 3$. Then*

$$\lim_{t\to 0} t \log H_{K_1,K_2}(t) = -d^2/4, \tag{11}$$

where

$$d = \min\{|x_1 - x_2| : x_1 \in \partial K_1, x_2 \in \partial K_2\}. \tag{12}$$

Moreover, $t \to H_{K_1,K_2}(t)$ is strictly increasing on $[0,\infty)$.

A classical theorem of F.Spitzer [10] asserts that if K is compact and non-polar then

$$E_K(t) = Cap(K)t + o(t), \qquad t \to \infty \qquad (m \geq 3), \tag{13}$$

$$E_K(t) = \frac{4\pi t}{\log t} + O\left(\frac{t}{(\log t)^2}\right), \qquad t \to \infty \qquad (m = 2). \tag{14}$$

The large time behaviour of $H_{K_1,K_2}(t)$ can be read-off from (13) and (14) respectively:

$$H_{K_1,K_2}(t) = (Cap(K_1) + Cap(K_2) - Cap(K_1 \cup K_2))t + o(t), t \to \infty (m \geq 3), \tag{15}$$

$$H_{K_1,K_2}(t) = \frac{4\pi t}{\log t} + O\left(\frac{t}{(\log t)^2}\right), t \to \infty, (m = 2). \tag{16}$$

It is straight forward to read-off refinements to (15), (16) from the results in [4, 5, 10].

The main motivation for proving Theorem 1 came from a conjecture [3] on the asymptotic behaviour of the heat trace coefficients in the expansion of the heat trace for a region in $\mathbb{R}^m$. Let Ω be an open, bounded and connected set in Euclidean space $\mathbb{R}^m$ and let Δ_Ω denote the Dirichlet Laplacian for Ω. It is well-known that for any $N \in \mathbb{N}$

$$\operatorname{trace}(e^{t\Delta_\Omega}) = (4\pi t)^{-m/2} \sum_{j=0}^{N} a_j(\Omega) t^{j/2} + O(t^{(N+1-m)/2}), \quad t \to 0, \tag{17}$$

where the $a_j(\Omega), j = 0, 1, 2, \dots$ are locally computable geometric invariants of Ω [6, 9]. Let $\ell(\Omega)$ denote the length of the shortest closed periodic geodesic in Ω. It was conjectured that the behaviour of $a_j(\Omega), j \to \infty$ is related to $\ell(\Omega)$ by formulae (4), (19), (21) in [3]. To see that this cannot hold in general we give two (counter) examples.

Example 2. Let $m = 2$, and let

$$\Omega_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \varepsilon < |x| < 1\}, \tag{18}$$

$$\Omega_\varepsilon^\rho = \{(x_1, x_2) \in \mathbb{R}^2 : |x| < 1, |(x_1 - \rho, x_2)| > \varepsilon\}, \tag{19}$$

where $0 < \varepsilon < 1$ and $0 < \rho < 1 - \varepsilon$. Since the coefficients $a_j(\Omega_\varepsilon), a_j(\Omega_\varepsilon^\rho)$ are locally computable invariants we have that

$$a_j(\Omega_\varepsilon) = a_j(\Omega_\varepsilon^\rho), \quad j = 0, 1, 2, \cdots . \tag{20}$$

On the other hand

$$\ell(\Omega_\varepsilon) = 2(1 - \varepsilon), \quad \ell(\Omega_\varepsilon^\rho) = 2(1 - \varepsilon - \rho). \tag{21}$$

The second example goes back to Example 7.1 in [7].

Example 3. Let $m = 2$, and let

$$\widetilde{P}_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \quad |x| < 1, \ |x_2| < 1 - \varepsilon\}, \tag{22}$$

$$\widetilde{Q}_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 : \quad |x| < 1, \ x_1 < 1 - \varepsilon, \ x_2 < 1 - \varepsilon\}, \tag{23}$$

where $0 < \varepsilon < \frac{1}{5}$. We smooth out the corners of $\partial \widetilde{P}_\varepsilon$ at $x_2 = \pm(1 - \varepsilon)$ and of $\partial \widetilde{Q}_\varepsilon$ at $x_1 = 1 - \varepsilon$, $x_2 = 1 - \varepsilon$ isometrically to obtain convex open sets $P_\varepsilon, Q_\varepsilon$ with C^∞ boundary. We have that

$$a_j(P_\varepsilon) = a_j(Q_\varepsilon), \quad j = 0, 1, 2, \cdots , \tag{24}$$

and

$$\ell(P_\varepsilon) = 2(1 - \varepsilon), \ \ell(Q_\varepsilon) = 2(2 - \varepsilon). \tag{25}$$

Since the Weyl series (17) are identical for the pair $\Omega_\varepsilon, \Omega_\varepsilon^\rho$, we have that the difference of the heat traces cancels up to any order $O(t^N)$:

$$\text{trace}(e^{t\Delta_{\Omega_\varepsilon^\rho}}) - \text{trace}(e^{t\Delta_{\Omega_\varepsilon}}) = O(t^N), \quad t \to 0. \tag{26}$$

Theorem 5 in Section 3 shows that the left hand side of (26) is strictly positive for all t sufficiently small and that

$$\lim_{t\to\infty} t \log(\text{trace}(e^{t\Delta_{\Omega_\varepsilon^\rho}}) - \text{trace}(e^{t\Delta_{\Omega_\varepsilon}})) = -\frac{1}{4}\ell(\Omega_\varepsilon^\rho)^2. \tag{27}$$

While the Weyl series for Ω_ε^ρ does not determine $\ell(\Omega_\varepsilon^\rho)$ comparison of the heat traces for Ω_ε^ρ and Ω_ε does determine the shorter of the two closed periodic geodesics. This is very similar to the result given in Theorem 1.

Similarly one can show that for all t sufficiently small

$$\text{trace}(e^{t\Delta_{Q_\varepsilon}}) - \text{trace}(e^{t\Delta_{P_\varepsilon}}) > 0, \tag{28}$$

and that

$$\lim_{t\to 0} t \log(\text{trace}(e^{t\Delta_{Q_\varepsilon}}) - \text{trace}(e^{t\Delta_{P_\varepsilon}})) = -\frac{1}{4}\ell(P_\varepsilon)^2. \tag{29}$$

Instead of proving (27) for the pair $\Omega_\varepsilon^\rho, \Omega_\varepsilon$ and (29) for the pair $Q_\varepsilon, P_\varepsilon$ respectively, we prove in Section 3 a comparison result (Theorem 4) for the heat contents of regions with isometric obstacles. There we also state the analogous result and generalization of (27) (Theorem 5) for the heat traces of regions with isometric obstacles. The proof of Theorem 1 will be given in Section 2.

2 Proof of Theorem 1

Proof. Let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a Brownian motion associated to the parabolic operator $-\Delta + \frac{\partial}{\partial t}$. For a closed set K in $\mathbb{R}^m$ we define the first hitting time of K by

$$T_K = \inf\{s \geq 0 : B(s) \in K\}. \tag{30}$$

The solution of (1-3) is given by

$$u(x;t) = \mathbb{P}_x[T_K \leq t]. \tag{31}$$

Since K_1 and K_2 are disjoint we have by (4) and (31)

$$\begin{aligned}
&E_{K_1\cup K_2}(t)\\
&=\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1\cup K_2}\le t]-|K_1\cup K_2|\\
&=\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_2}\le t]+\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1}\le t]-\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1}\le t, T_{K_2}\le t]\\
&-|K_1\cup K_2|\\
&=E_{K_1}(t)+E_{K_2}(t)-\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1}\le t, T_{K_2}\le t].
\end{aligned}\tag{32}$$

By (5) and (32)

$$H_{K_1,K_2}(t)=\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1}\le t, T_{K_2}\le t].\tag{33}$$

Since K_1, K_2 are non-polar, (33) is strictly increasing on $[0,\infty)$. To prove (11) we first obtain an upper bound for $H_{K_1,K_2}(t)$. By (33)

$$H_{K_1,K_2}(t)=\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1}<T_{K_2}\le t]+\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_2}<T_{K_1}\le t],\tag{34}$$

since the set of Brownian paths for which $T_{K_1}=T_{K_2}$ has Wiener measure zero. By the strong Markov property

$$\begin{aligned}
\mathbb{P}_x[T_{K_1}<T_{K_2}\le t]&=\mathbb{E}_x[\mathbb{1}_{T_{K_1}<t}\ \mathbb{P}_{B(T_{K_1})}[T_{K_2}<t-T_{K_1}]]\\
&\le \mathbb{E}_x[\mathbb{1}_{T_{K_1}<t}\ \mathbb{P}_{B(T_{K_1})}[T_{K_2}<t]].
\end{aligned}\tag{35}$$

It follows from Lemma 1.2 on p.2 in [8] that if C is a closed set then

$$\mathbb{P}_x[T_C\le t]\le 2\ \mathbb{P}_x[|x-B(t)|\ge d(x,C)],\tag{36}$$

where $d(x,C)$ is the distance from x to C. Putting $B(T_{K_1})=x$ and $C=K_2$ we obtain by (36)

$$\mathbb{P}_{B(T_{K_1})}[T_{K_2}<t]\le 2(4\pi t)^{-m/2}\int_{\{y:\ |y|\ge d\}} e^{-|y|^2/(4t)}dy.\tag{37}$$

Hence by (35), (37)

$$\mathbb{P}_x[T_{K_1}<T_{K_2}\le t]\le 2\mathbb{P}_x[T_{K_1}<t](4\pi t)^{-m/2}\int_{\{y:|y|>d\}} e^{-|y|^2/(4t)}dy.\tag{38}$$

Similarly we obtain that

$$\mathbb{P}_x[T_{K_2} < T_{K_1} \le t] \le 2\mathbb{P}_x[T_{K_2} < t](4\pi t)^{-m/2} \int_{\{y:|y|>d\}} e^{-|y|^2/(4t)} dy. \tag{39}$$

Integrating both sides of the inequalities (38), (39) with respect to x over R^m we obtain by (34)

$$\begin{aligned} &H_{K_1,K_2}(t) \\ &\le 2(|K_1| + |K_2| + E_{K_1}(t) + E_{K_2}(t))(4\pi t)^{-m/2} \int_{\{y:|y|>d\}} e^{-|y|^2/(4t)} dy. \end{aligned} \tag{40}$$

Let $\epsilon \in (0,1)$ be arbitrary. Then

$$\begin{aligned} \int_{\{y:|y|>d\}} e^{-|y|^2/(4t)} dy &\le e^{-d^2(1-\epsilon)/(4t)} \int_{\{y:|y|>d\}} e^{-\epsilon|y|^2/(4t)} dy \\ &\le e^{-d^2(1-\epsilon)/(4t)} \int_{\mathbb{R}^m} e^{-\epsilon|y|^2/(4t)} dy \\ &= (4\pi t/\epsilon)^{m/2} e^{-d^2(1-\epsilon)/(4t)}. \end{aligned} \tag{41}$$

Let $t_0 = d^2$. Since K_1, K_2 are non-polar, $E_{K_1}(t), E_{K_2}(t)$ are strictly increasing on $[0, t_0]$. Hence by (39), (40) we have for $0 < t \le t_0$

$$H_{K_1,K_2}(t) \le 2(|K_1| + |K_2| + E_{K_1}(t_0) + E_{K_2}(t_0))\epsilon^{-m/2} e^{-d^2(1-\epsilon)/(4t)}. \tag{42}$$

Hence

$$\limsup_{t\to 0} t \log H_{K_1,K_2}(t) \le -d^2(1-\epsilon)/4. \tag{43}$$

Since $\epsilon \in (0,1)$ was arbitrary we obtain

$$\limsup_{t\to 0} t \log H_{K_1,K_2}(t) \le -d^2/4. \tag{44}$$

This proves the upper bound in (11).

To prove the lower bound we first consider the case $m \ge 3$. Let $x_1 \in K_1, x_2 \in K_2$ be such that $|x_1 - x_2| = d$. Let $\epsilon \in (0,1)$ be arbitrary. Then by

(34) and the strong Markov property

$$
\begin{aligned}
H_{K_1,K_2}(t) &\geq \int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1} < T_{K_2} \leq t] \\
&\geq \int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1\cap B(x_1;\epsilon d)} < T_{K_2} \leq t] \\
&\geq \int_{\mathbb{R}^m} dx\ \mathbb{E}_x[\mathbb{1}_{T_{K_1\cap B(x_1;\epsilon d)}<t}\mathbb{P}_{B(T_{K_1\cap B(x_1;\epsilon d)})}[T_{K_2} < t - T_{K_1\cap B(x_1;\epsilon d)}]] \\
&\geq \int_{\mathbb{R}^m} dx\ \mathbb{E}_x[\mathbb{1}_{T_{K_1\cap B(x_1;\epsilon d)}<\epsilon t}\mathbb{P}_{B(T_{K_1\cap B(x_1;\epsilon d)})}[T_{K_2} < t - T_{K_1\cap B(x_1;\epsilon d)}]] \\
&\geq \int_{\mathbb{R}^m} dx\ \mathbb{E}_x[\mathbb{1}_{T_{K_1\cap B(x_1;\epsilon d)}<\epsilon t}\mathbb{P}_{B(T_{K_1\cap B(x_1;\epsilon d)})}[T_{K_2} < t(1-\epsilon)]] \\
&\geq \int_{\mathbb{R}^m} dx\ \mathbb{E}_x[\mathbb{1}_{T_{K_1\cap B(x_1;\epsilon d)}<\epsilon t}\mathbb{P}_{B(T_{K_1\cap B(x_1;\epsilon d)})}[T_{K_2\cap B(x_2;\epsilon d)} < t(1-\epsilon)]].
\end{aligned}
\tag{45}
$$

Define the last exit time from the compact set K by

$$L_K = \sup\{s \geq 0 : B(s) \in K\}. \tag{46}$$

By the results of Section 2 in Chapter 3 of [8]

$$\mathbb{P}_z[0 < L_K < t] = \int_{[0,t]} ds \int_K p(z,y;s)\mu_K(dy), \tag{47}$$

where

$$p(z,y;s) = (4\pi s)^{-m/2}\ e^{-|z-y|^2/(4s)}, \tag{48}$$

and where μ_K is the equilibrium measure for the compact set K. Recall that

$$\int \mu_K(dy) = Cap(K). \tag{49}$$

For any $0 < \epsilon < 1/2$ we have that

$$
\begin{aligned}
\mathbb{P}_z[T_{K_2\cap B(x_2;\epsilon d)} < t(1-\epsilon)] &\geq \mathbb{P}_z[t(1-2\epsilon) < L_{K_2\cap B(x_2;\epsilon d)} < t(1-\epsilon)] \\
&\geq \int_{[t-2\epsilon t, t-\epsilon t]} ds \int p(z,y;s)\mu_{K_2\cap B(x_2;\epsilon d)}(dy).
\end{aligned}
\tag{50}
$$

For $z \in B(x_1;\epsilon d), y \in B(x_2;\epsilon d)$ and $s \in [t-2\epsilon t, t-\epsilon t]$

$$p(z,y;s) \geq (4\pi t)^{-m/2} e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))}. \tag{51}$$

By (49-51) we have for $z \in B(x_1; \epsilon d)$

$$\mathbb{P}_z[T_{K_2 \cap B(x_2;\epsilon d)} < t(1-\epsilon)] \\ \geq \epsilon t (4\pi t)^{-m/2} e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))} Cap(K_2 \cap B(x_2; \epsilon d)). \tag{52}$$

By (45) and (52) we conclude that

$$H_{K_1,K_2}(t) \geq \int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1 \cap B(x_1;\epsilon d)} < \epsilon t] \\ \times \epsilon t (4\pi t)^{-m/2} e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))} Cap(K_2 \cap B(x_2; \epsilon d)). \tag{53}$$

It remains to find a lower bound for the integral in (53). By (46), (47) and Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^m} dx\ \mathbb{P}_x[T_{K_1 \cap B(x,;\epsilon d)} < \epsilon t] &\geq \int_{\mathbb{R}^m} dx\ \mathbb{P}_x[L_{K_1 \cap B(x_1;\epsilon d)} < \epsilon t] \\ &= \int_{\mathbb{R}^m} dx \int_{[0,\epsilon t]} ds \int p(x,y;s) \mu_{K_1 \cap B(x_1;\epsilon d)}(dy) \\ &= \epsilon t\ Cap(K_1 \cap B(x_1; \epsilon d)).\end{aligned} \tag{54}$$

Putting (53) and (54) together, we see that

$$H_{K_1,K_2}(t) \geq (\epsilon t)^2 (4\pi t)^{-m/2} \\ \times Cap(K_1 \cap B(x_1; \epsilon d)) Cap(K_2 \cap B(x_2; \epsilon d)) e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))}. \tag{55}$$

By the regularity hypotheses in Theorem 1 we have that both capacities in the right hand side of (55) are strictly positive for any $\epsilon > 0$. It follows that

$$\liminf_{t \to 0} t \log H_{K_1,K_2}(t) \geq -\frac{d^2(1+2\epsilon)^2}{4(1-2\epsilon)}. \tag{56}$$

Since $\epsilon \in (0, \frac{1}{2})$ was arbitrary we conclude that

$$\liminf_{t \to 0} t \log H_{K_1,K_2}(t) \geq -d^2/4. \tag{57}$$

Theorem 1 follows for $m \geq 3$ by (44) and (57).

To complete the proof of Theorem 1 we consider the case $m = 2$. Let $z = B(T_{K_1 \cap B(x_1;\epsilon d)})$ in (45). For any $0 < \varepsilon < 1/2$ we have that

$$\mathbb{P}_z[T_{K_2 \cap B(x_2;\epsilon d)} < t(1-\epsilon)] \geq \mathbb{P}_z[B(t(1-2\epsilon)) \in K_2 \cap B(x_2; \epsilon d)] \\ = (4\pi t)^{-1} \int_{K_2 \cap B(x_2;\epsilon d)} e^{-|z-y|^2/(4t(1-2\epsilon))} dy. \tag{58}$$

For $y \in K_2 \cap B(x_2; \epsilon d), |z - y| \le d(1 + 2\epsilon)$. Hence the right hand side of (58) is bounded from below by

$$(4\pi t)^{-1}|K_2 \cap B(x_2; \epsilon d)| e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))}. \tag{59}$$

By (45)

$$\begin{aligned} &H_{K_1,K_2}(t) \\ &\ge (4\pi t)^{-1}|K_2 \cap B(x_2; \epsilon d)| e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon))} \int_{\mathbb{R}^2} dx\ \mathbb{P}_x[T_{K_1 \cap B(x_1;\epsilon d)} < \epsilon t]. \end{aligned} \tag{60}$$

But

$$\begin{aligned} \int_{\mathbb{R}^2} dx \mathbb{P}_x[T_{K_1 \cap B(x_1;\epsilon d)} < \epsilon t] &\ge \int_{K_1 \cap B(x_1;\epsilon d)} \mathbb{P}_x[T_{K_1 \cap B(x_1;\epsilon d)} < \epsilon t] dx \\ &\ge \int_{K_1 \cap B(x_1;\epsilon d)} \mathbb{P}_x[B(\epsilon t/2) \in K_1 \cap B(x_1; \epsilon d)] dx \\ &= \int_{K_1 \cap B(x_1;\epsilon d)} dx \int_{K_1 \cap B(x_1;\epsilon d)} dy (4\pi t)^{-1} e^{-|x-y|^2/(2\epsilon t)} \\ &\ge \int_{K_1 \cap B(x_1;\epsilon d)} dx \int_{K_1 \cap B(x_1;\epsilon d)} dy (4\pi t)^{-1} e^{-2d^2\epsilon/t} \\ &= (4\pi t)^{-1}|K_1 \cap B(x_1; \epsilon d)|^2 e^{-2d^2\epsilon/t}. \end{aligned} \tag{61}$$

By (60) and (61)

$$H_{K_1,K_2}(t) \ge (4\pi t)^{-2}|K_1 \cap B(x_1; \epsilon d)|^2 |K_2 \cap B(x_2; \epsilon d)| e^{-d^2(1+2\epsilon)^2/(4t(1-2\epsilon)) - 2d^2\epsilon/t}. \tag{62}$$

Hence

$$\liminf_{t \to 0} t \log H_{K_1,K_2}(t) \ge -d^2(1 + 2\epsilon)^2/(4(1 - 2\epsilon)) - 2d^2\epsilon. \tag{63}$$

Since $\epsilon \in (0, 1/2)$ was arbitrary we conclude that

$$\liminf_{t \to 0} t \log H_{K_1,K_2}(t) \ge -d^2/4. \tag{64}$$

This completes the proof of Theorem 1. $\square$

3 Heat content for regions with an obstacle

In this section we compare the heat contents of regions D in $\mathbb{R}^m$ with compact isometric obstacles C and $\widetilde{C}$ respectively. Let C be a compact subset of an open, bounded and connected set D in $\mathbb{R}^m$. Let $K = (\partial D) \cup C$, and let u be a solution of (1-3). We define the heat content of $D \backslash C$ by

$$Q_{D\backslash C}(t) = \int_{D\backslash C} u(x;t)dx. \tag{65}$$

Let $\widetilde{C}$ be a subset of D where $\widetilde{C}$ is isometric to C, and put

$$d = \min\{|x_1 - x_2| : x_1 \in \partial D,\ x_2 \in C\}, \tag{66}$$

$$\tilde{d} = \min\{|x_1 - x_2| : x_1 \in \partial D,\ x_2 \in \widetilde{C}\} \tag{67}$$

Theorem 4. *Suppose* $(\mathbb{R}^m \backslash D) \cup C$ *satisfies* (9) *if* $m = 2$ *or* (10) *if* $m \geq 3$. *Suppose that* $d < \tilde{d}$. *Then for all* t *sufficiently small*

$$Q_{D\backslash \widetilde{C}}(t) > Q_{D\backslash C}(t), \tag{68}$$

and

$$\lim_{t\to 0} t \log(Q_{D\backslash \widetilde{C}}(t) - Q_{D\backslash C}(t)) = -d^2/4. \tag{69}$$

Proof. We apply Theorem 1 with

$$K_1 = \{x \in \mathbb{R}^m \backslash D : d(x, D) \leq 1\}, \tag{70}$$

and $K_2 = C$ or $K_2 = \widetilde{C}$ respectively. Since C and $\widetilde{C}$ are isometric we have that $E_C(t) = E_{\widetilde{C}}(t)$. Hence

$$E_{K_1\cup C}(t) - E_{K_1\cup \widetilde{C}}(t) = H_{K_1,\widetilde{C}}(t) - H_{K_1,C}(t). \tag{71}$$

Since

$$E_{K_1\cup \widetilde{C}}(t) - E_{K_1\cup C}(t) = Q_{D\backslash \widetilde{C}}(t) - Q_{D\backslash C}(t), \tag{72}$$

we have that

$$Q_{D\backslash \widetilde{C}}(t) - Q_{D\backslash C}(t) = H_{K_1,C}(t)\left(1 - \frac{H_{K_1,\widetilde{C}}(t)}{H_{K_1,C}(t)}\right). \tag{73}$$

By Theorem 1 we have

$$\lim_{t\to 0} t \log \frac{H_{K_1,C}(t)}{H_{K_1,\widetilde{C}}(t)} = (\tilde{d}^2 - d^2)/4. \tag{74}$$

Hence there exists $t_0 > 0$ such that for $0 < t \le t_0$,

$$H_{K_1,\widetilde{C}}(t)/H_{K_1,C}(t) \le e^{(d^2-\tilde{d}^2)/8t}. \tag{75}$$

Hence for $0 < t \le t_0$

$$\left(1 - e^{(d^2-\tilde{d}^2)/(8t)}\right) H_{K_1,C}(t) \le Q_{D\backslash\widetilde{C}}(t) \le H_{K_1,C}(t). \tag{76}$$

It follows that (68) holds for all $0 < t \le t_0$. Furthermore, by Theorem 1

$$\limsup_{t\to 0} t\log(Q_{D\backslash\widetilde{C}}(t) - Q_{D\backslash C}(t)) \le \limsup_{t\to 0} t\log H_{K_1,C}(t) = -d^2/4, \tag{77}$$

and

$$\begin{aligned}\liminf_{t\to 0} t\log(Q_{D\backslash\widetilde{C}}(t) - Q_{D\backslash C}(t)) \ge \liminf_{t\to 0} t\log H_{K_1,C}(t) \\ + \liminf_{t\to 0} t\log(1 - e^{(d^2-\tilde{d}^2)/(8t)}) = -d^2/4.\end{aligned} \tag{78}$$

This completes the proof of Theorem 4. □

It is possible to prove a result analogous to Theorem 4 for the trace of the Dirichlet heat semigroup. We omit the proof of the following.

Theorem 5. *Suppose D, C and $\widetilde{C}$ satisfy the conditions of Theorem 4. Then for all t sufficiently small*

$$0 < \mathrm{trace}(e^{t\Delta_{D\backslash\tilde{C}}}) < \mathrm{trace}(e^{t\Delta_{D\backslash C}}) < \infty, \tag{79}$$

where $\Delta_{D\backslash C}, \Delta_{D\backslash\widetilde{C}}$ are the Dirichlet Laplacians for the open sets $D\backslash C, D\backslash\widetilde{C}$ respectively. Furthermore

$$\lim_{t\to 0} t\log(\mathrm{trace}(e^{t\Delta_{D\backslash C}}) - \mathrm{trace}(e^{t\Delta_{D\backslash\tilde{C}}})) = -d^2. \tag{80}$$

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