Variational formulae for the total mean curvatures of a codimension-one distribution

Vladimir Y. Rovenski$^1$ and Paweł G. Walczak$^2$

$^1$University of Haifa, Haifa, 31905, Israel
$^2$Uniwersytet Łódzki, ul. Banacha 22, 90-238 Łódź, Poland

Given quadratic matrices $B_1, \ldots, B_m$ of order $n$ over $\mathbb{R}$ and a unit matrix $I_n$ one can consider the determinant $\det(I_n + t_1 B_1 + \ldots + t_m B_m)$ and express it as a polynomial of real variables $t = (t_1, \ldots, t_m)$. Namely, $\sigma(\lambda_1, \ldots, \lambda_m)$ are defined by the formula [2]

$$\det(I_n + t_1 B_1 + \ldots + t_m B_m) = \sum_{|\lambda| \leq n} \sigma_{\lambda}(B_1, \ldots, B_m) t^\lambda,$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$, a sequence of nonnegative integers with $|\lambda| = \lambda_1 + \ldots + \lambda_m \leq n$. In particular, $\sigma_j(A)$ is the coefficient at $t^j$ of the determinant $\det(I_n + t A) = \sum_j \sigma_j(A) t^j$.

A problem we are interested in is to develop variational formulae for the functionals

$$I_m : D \mapsto \int_M \sigma_m(D) \, d\text{vol},$$

where $(M^n, g)$ is a fixed compact oriented $n$-dimensional Riemannian manifold with the curvature tensor $R$, $D$ ranges over the space of all codimension-one distributions (plane fields) on $M$, $m = 1, 2, \ldots, n-1$ (especially, $m$ even), $N$ the unit normal of $D$, and $\sigma_m(D)$ is the $m$-symmetric function of the shape operator $A_N : D \to D$. For example, the minimal value of $I_2(D)$ can be used for estimation from below of the energy $\mathcal{E}(N)$ of $N$, because [1],

$$\mathcal{E}(N) \geq \frac{1}{2n-2} \int_M \text{Ric}(N, N) \, d\text{vol} + \frac{n+1}{2} \text{vol}(M).$$

The Ricci curvature in direction $N$ is $\text{Ric}(N, N) = \sum_i (R(e_i, N)N, e_i)$, where $\{e_i\}$ is a local orthonormal basis of $D$. Denote by $R_N = R(\cdot, N)N$ (the Jacobi operator).

In order to develop variational formulae for (2), we use the integral formulae for a codimension-one distribution in our work [2]

$$\int_M g_m \, d\text{vol} = 0, \quad g_m = \sum_{|\lambda| = m} \sigma_{\lambda}(B_1, \ldots, B_m),$$

where $|\lambda| = \lambda_1 + 2\lambda_2 + \ldots + m\lambda_m$ and for locally symmetric spaces

$$B_{2k} = \frac{(-1)^k}{(2k)!!} R_N^k, \quad B_{2k+1} = \frac{(-1)^k}{(2k+1)!!} R_N^k A_N \quad (k \geq 0).$$

For example, $B_2 = -(1/2) R_N$, $B_3 = -(1/6) R_N A_N$, $B_4 = -(1/24) R_N^2$, and

$$g_2 = \sigma_2(A_N) + \sigma_1(B_2), \quad g_3 = \sigma_3(A_N) + \sigma_{(1,1)}(A_N, B_2) + \sigma_1(B_3), \quad g_4 = \sigma_4(A_N) + \sigma_{(2,1)}(A_N, B_2) + \sigma_{(1,1)}(A_N, B_3) + \sigma_2(B_2) + \sigma_1(B_4), \quad \text{and so on.}$$
Denote by \( \text{div} K = \sum_i (\nabla e_i K) e_i \) the divergence of a field of linear operators \( K : TM \to TM \), where \( \{e_i\} \) is a local base of \( TM \), orthonormal at a point under consideration. Denote by

\[
\text{Ric}(N) = \sum_i R(N, e_i) e_i - \text{the vector field dual via } g \text{ to the linear form } X \mapsto \text{Ric}(N, X) = \sum_i \langle R(e_i, N)X, e_i \rangle \text{ for any } X \in D. \]

Denote by

\[
R_1(S) \text{ the vector valued bilinear form } (X, Y) \mapsto R(SY, N)X, \\
R_2(S) \text{ the vector valued bilinear form } (X, Y) \mapsto R(X, SY)N, \\
R_3(S) \text{ the vector valued bilinear form } (X, Y) \mapsto R(X, N)SY,
\]

where \( S \) is a linear operator on \( D \). By the first Bianchi identity, we have \( R_3 = R_1 + R_2 \).

For \( \lambda = (\lambda_1, \ldots, \lambda_k) \) denote by \( \lambda_i = \lambda - e_i = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k) \). The range of indices \( i_1, \ldots, i_j \) in what follows is \( 1 \ldots m \). Set for further convenience \( R_N^{-1} = 0 \).

The main result of the present work is the following.

**Theorem 1.** Let \( D \) be a codimension one distribution (with the unit normal \( N \)) on a compact locally symmetric space \( M^{n+1} \). The distribution \( D \) is a critical point of the functional (2) if and only if the following Euler-Lagrange equation holds:

\[
\sum ||\lambda||=m, \sum_{j \geq 1} (-1)^{j-1} \sum_{i_2, \ldots, i_j} \sum_{k \geq 0} \left[ \frac{(-1)^k}{(2k+1)!} (\tilde{\text{BR}}^k_N)^{i} \nabla \sigma_{2k+1, i_2 \ldots i_j} (\tilde{B}) \right] \\
+ \frac{(-1)^k}{(2k+1)!} \sigma_{2k+1, i_2 \ldots i_j} (\tilde{B}) \left[ \text{div}((\tilde{BR}_N^k)^i) + k \text{ Tr} ((R_1 + R_3) A_N \tilde{BR}_N^{k-1}) \right] \\
+ \frac{(-1)^k}{(2k+1)!} k \sigma_{2k, i_2 \ldots i_j} (\tilde{B}) \text{ Tr} ((R_1 + R_3) \tilde{BR}_N^{k-1}) \right] = 0,
\]

where \( \tilde{B} = (B_1, \ldots B_m) \), see (5), \( \tilde{B} = B_{i_2} \ldots B_{i_j} \), and \( [\cdot]^\perp \) denotes \( N \)-orthogonal component.

**Corollary 1.** A distribution \( D \) orthogonal to a unit vector field \( N \) is a critical point for the functional \( I_2 \) if and only if \( \text{Ric}(N)^\perp = 0 \), \( N \) being the unit normal for \( D \) and \( (\cdot)^\perp \) the orthogonal projection onto \( D \). It is a point of local minimum if the form \( \mathcal{I}_{2,N} \) is positive definite on the space of sections of \( D \).

**Remark 1.** The method can be extended for \( m \geq 2 \) to calculate the variations of the total mixed scalar curvature of a distribution. The total mixed scalar curvature can be applied to the problem of minimization the energy and bending of distributions.

**Corollary 2.** A distribution \( D \) orthogonal to a unit vector field \( N \) is a critical point of the functional \( I_3 \) if and only if

\[
\left[ \text{Tr}(A_N) \tilde{\text{Ric}}(N) + \frac{1}{2} \nabla \text{Ric}(N, N) + \text{Tr} (R_3(A_N)) \right]^\perp = 0.
\]

**References**
