The problem of a surface reconstruction by principal curvatures and directions

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We formulate and study a problem of reconstruction of a C^2-regular surface whose curvature lines form a regular net. Such investigations can also be motivated by possible applications in optics. We study the case of a surface-graph $z = f(x, y)$ over a rectangle, assuming that the projection of one of the principal directions is given in the rectangle and both principal curvatures are partially given on its sides. We formulate a boundary value problem for hyperbolic system of equations consisting of two coupled blocks: parametric Goursat Type Problem and so called Principal Curvature Transport Problem. We prove the existence and uniqueness of a solution $f(x, y)$ for "small" boundary values of principal curvatures. Also we prove a stability of the solution w.r. to some variations of principal curvatures. We introduce a class of evolvent surfaces that have the property $\alpha(x, y) = const > 0$. In this case the problem can be solved by a simple procedure. Some illustrative solutions of the problem are obtained using appropriate implicit difference scheme.

Denote by $\Pi = [0, a]\times[0, b] \subset \mathbb{R}^2$ a rectangle, $A_- = \min\limits_{(x,y)\in\Pi} \alpha$, $\bar{A} = \max\limits_{(x,y)\in\Pi} \alpha$,

$$r_A = \min\{A_-,\, 1/\bar{A}\}, \qquad \rho_A = \frac{1}{16(a+b)}\left(1-(1+2\,r_A)^{-2}\right). \tag{1}$$

In the sequel we denote by k_1, k_2 the principal curvatures of a surface $M_f : z = f(x, y),\ (x, y) \in \Pi$, and assume that the function $\alpha(x, y) \in C^2(\Pi)$ is positive.

Problem 1. Given $\alpha \in C^2(\Pi)$ and $\varphi_1, \varphi_2 \in C^1([0, a])$, $\varphi_3, \phi_4 \in C^1([0, b])$, reconstruct a C^2-surface $M : z = f(x, y),\ (x, y) \in \Pi$ by the data:

(i) the field of projections $\vec{l}_1(x, y) = (\alpha(x, y), 1)$ on the plane XY of directions of the first principal curvature $k_1(x, y)$;

(ii) the principal curvatures k_1, k_2 on a part of the boundary $\partial\Pi$:

$$k_1(x, 0) = \varphi_1(x),\ k_1(a, y) = \varphi_3(y), \quad k_2(x, 0) = \varphi_2(x),\ k_2(0, y) = \varphi_4(y), \tag{2}$$

(iii) the zero values of $f,\ f_{,x}, f_{,y}$ at the origin $(0, 0)$.

Definition 1. For positive reals N, N', a, b denote by $\mathcal{B}_{N,N',a,b}$ the set of all sets of functions $\varphi_1, \varphi_2 \in C^1([0,a])$, $\varphi_3, \phi_4 \in C^1([0,b])$, satisfying the conditions:

$$\varphi_1(a) = \varphi_3(0), \quad \varphi_2(0) = \varphi_4(0),$$

$$\max\{\|\varphi_1\|_{[0,a]}, \|\varphi_2\|_{[0,a]}, \|\varphi_3\|_{[0,b]}, \|\varphi_4\|_{[0,b]}\} < N,$$
$$\max\{\|\varphi_1'\|_{[0,a]}, \|\varphi_2'\|_{[0,a]}, \|\varphi_3'\|_{[0,b]}, \|\varphi_4'\|_{[0,b]}\} < N'.$$

Definition 2. Denote by $\mathcal{F}_{\alpha,a,b}$ the set of all C^2-regular surfaces $M_f : \ z = f(x,y)$ satisfying the conditions:

(a) $k_i \in C^{0,1}(\Pi)$ and $\|k_i\|_\Pi \le \rho_A$ for $i = 1, 2$;

(b) the field of projections on XY of directions of k_1 is given by $(\alpha(x,y), 1)$;

(c) $f|_{\Omega_j(f)} \in C^3(\Omega_j(f))$ $(j = 1, 2, \dots, l(f))$, where $\Omega_j(f)$ are the closures of connected components of the set $\Pi \setminus (\Gamma_1(f) \cup \Gamma_2(f))$, and $\Gamma_1(f)$, $\Gamma_2(f)$ are projections on XY of the (first and second) curvature lines of M_f passing through the points $(0,0)$ and $(0,a)$, respectively.

Theorem 1. *(Existence and uniqueness of a solution) There exist positive numbers N, N' and L, M such that if a boundary data $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ of Problem 1 belongs to the class $\mathcal{B}_{N,N',a,b}$, then Problem 1 has a unique solution $M_f : \ z = f(x,y)$ in the class $\mathcal{F}_{\alpha,a,b}$ and the following holds:*

$$k_1, k_2 \in C^{0,1}_L(\Pi),$$

$$\nabla f \in C^{0,1}_M(\Pi, \mathbb{R}^2), \qquad \|\nabla f\|_\Pi \le r_A.$$

Theorem 2. *(Stability of solution) Let N, N' be the positive constants from the assertion of Theorem 1. The solution of Problem 1 depends continuously on the boundary data in the following sense: for any $\epsilon > 0$ there exists $\delta > 0$ such that if the boundary data $(\varphi_{1,i}, \varphi_{2,i}, \varphi_{3,i}, \varphi_{4,i})$ $(i = 1, 2)$ belong to the class $\mathcal{B}_{N,N',a,b}$ and satisfy the conditions*

$$\|\varphi_{1,1} - \varphi_{1,2}\|_{[0,a]} < \delta, \quad \|\varphi_{2,1} - \varphi_{2,2}\|_{[0,a]} < \delta,$$

$$\|\varphi_{3,1} - \varphi_{3,2}\|_{[0,b]} < \delta, \quad \|\varphi_{4,1} - \varphi_{4,2}\|_{[0,b]} < \delta,$$

then the corresponding solutions $M_{f_i} : \ z = f_i(x,y)$ $(i = 1, 2)$ of Problem 1 (which exist in $\mathcal{F}_{\alpha,a,b}$ and are unique there by Theorem 1) have the properties:

$$\|f_1 - f_2\|_\Pi < \epsilon, \quad \|\nabla f_1 - \nabla f_2\|_\Pi < \epsilon,$$

$$\|k_{1,i} - k_{2,i}\|_\Pi < \epsilon \quad (i = 1, 2),$$

where $k_{1,i}$ and $k_{2,i}$ are the principal curvatures of the surface M_{f_i}.