Genuine deformations of submanifolds

Abstract

The isometric (conformal) deformation problem for a given Euclidean submanifold $f\colon M^n \to \mathbb{R}^{n+p}$ with dimension n and codimension p and a given positive integer q is to describe all possible isometric (conformal) deformations $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ of f, that is, all immersions $\hat{f}\colon M^n \to \mathbb{R}^{n+q}$ whose induced metric coincides with (is conformal to) that induced by f. A satisfactory answer to the local version of the problem in both isometric and conformal settings is known only in the case $p = 1 = q$, and goes back almost a century to the works of Sbrana and Cartan.

In higher codimensions, the problem becomes much harder. A basic observation is that a submanifold of a (conformally or isometrically) deformable one has also that property. This has led M. Dajczer and L. Florit to introduce the notion of a *genuine* isometric deformation of a submanifold. That an isometric deformation is genuine means that the submanifold can not be included into a submanifold of higher dimension in such a way that the deformation of the former is induced by a deformation of the latter. They proved that an Euclidean submanifold together with a genuine deformation of it in low (but not necessarily equal) codimensions must be mutually ruled, and they obtained a sharp estimate for the dimension of the rulings.

After discussing such results and their relation to several previous articles, our aim is to report on recent work with L. Florit, in which we extend the notion of a genuine deformation of a submanifold to the conformal setting, and describe the geometric structure of a submanifold together with a genuine conformal deformation of it. We explain the unifying character of that result by showing how it implies some knew and old conformal rigidity theorems.