HAUSDORFFIZED LEAF SPACES

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We introduce the notion of the Hausdorffized leaf space (shortly HLS) for an arbitrary foliation $\mathcal{F}$ on a compact Riemannian manifold (M, g). Next, we discuss the connection between HLS, warped foliations, and Gromov-Hausdorff topology. Finally, some topological properties of HLS are presented.

1. Hausdorffized leaf space

Let $(M, \mathcal{F}, g)$ be a compact foliated manifold. Let us set

$$\rho(L, L') = \inf\{\sum_{i=1}^{n-1} \operatorname{dist}(L_i, L_{i+1})\}$$

where the infimum is taken over all finite sequences of leaves beginning at $L_1 = L$ and ending at $L_n = L'$. Let $\sim$ be an equivalence relation in $\mathcal{L}$ given by:

$$L \sim L' \Leftrightarrow \rho(L, L') = 0, \quad L, L' \in \mathcal{F}. \tag{1}$$

Let $\tilde{\mathcal{L}} = \mathcal{L}/_\sim$. Put

$$\tilde{\rho}([L], [L']) = \rho(L, L'),$$

where $[L], [L'] \in \tilde{\mathcal{L}}$. A metric space $(\tilde{\mathcal{L}}, \tilde{\rho})$ is called *the Hausdorffized leaf space* of the foliation $\mathcal{F}$, and is denoted by $HLS(\mathcal{F})$.

Now, let f be a basic function on M. We modify the Riemannian structure g to g_f in the following way: $g_f(v, w) = f^2 g(v, w)$ while both v, w are tangent to the foliation $\mathcal{F}$, but if at least one of vectors v, w is perpendicular to $\mathcal{F}$ then we set $g_f(v, w) = g(v, w)$. Foliated Riemannian manifold $(M, \mathcal{F}, g_f)$ is called *the warped foliation* and denoted by M_f. The function f is called *the warping function*.

Theorem 1. *For an arbitrary compact foliated manifold $(M, \mathcal{F}, g)$ and any sequence (f_n) of warping functions on M converging to zero on a dense subset $G \subset M$, the Gromov-Hausdorff limit of a sequence of warped foliations is coincides with $HLS(\mathcal{F})$.*

2. Foliations with compact leaf

Let $(M, \mathcal{F})$ be a compact foliated manifold with a foliation of co-dimension q. Due to [3] one can consider a bad set B of a foliation $\mathcal{F}$. It is defined as follows:

Let $x \in M$ and let T be a q-dimensional transverse submanifold through x. Let us define $\sec_T : M \to \mathbb{N} \cup \{\infty\}$ as a map which assigns to $y \in M$ the cardinal of the set $L_y \cap T$, where L_y denotes the leaf through y. Now, let B denote the set of all $x \in M$ such that for any transversal T with $x \in \operatorname{int} T$ we have

$$\sup_{y \in M} \{\sec_T(y)\} = \infty.$$

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The set B is called the bad set of $\mathcal{F}$, while the set $G = M \setminus B$ is called the good set of $\mathcal{F}$.

Theorem 2. *Let $(M, \mathcal{F}, g)$ be a compact foliated Riemannian manifold. Let the good set G of $\mathcal{F}$ contains a compact leaf. Then the Hausdorff dimension*

$$\dim_H HLS(\mathcal{F}) = \operatorname{codim}(\mathcal{F}).$$

Corollary 1. *Let $(M, \mathcal{F}, g)$ be a foliated Riemannian manifold and $\mathcal{F}$ a foliation with all leaves compact. The Hausdorff dimension* $\dim HLS(\mathcal{F}) = \operatorname{codim}(\mathcal{F})$.

3. Co-dimension one foliations

Let $(M, \mathcal{F}, g)$ be a compact foliated Riemannian manifold carrying an arbitrary co-dimension one foliation. Then we have the following characterization of the HLS($\mathcal{F}$).

Theorem 3. *If good set G of $\mathcal{F}$ does not contains any compact leaf then* HLS($\mathcal{F}$) *is a singleton, otherwise it is locally homeomorphic to an interval.*

References

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