Dynamics and Cohomology of Foliations

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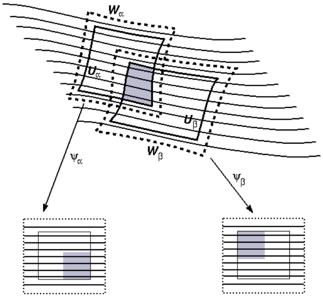
Definition of foliation

A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds of codimension q: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



Fundamental problems

Problem: "Classify" the foliations on a given manifold M

Two classification schemes have been developed since 1970:
using either "homotopy" or "dynamics".

Question: How are the cohomology invariants of a foliation related to its dynamical behavior?

Integrable homotopy equivalence

Let q denote the codimension of the foliation $\mathcal{F}$.

$q = m - p$ where p is the leaf dimension, $m = \dim M$

Assume throughout that $\mathcal{F}$ is transversally C^r for $r \geq 2$.

Definition: Two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of codimension q on M are *integrably homotopic* if there exists a foliation $\mathcal{F}$ of codimension q on $M \times \mathbb{R}$ which is transverse to the slices $M \times \{t\} \subset M \times \mathbb{R}$ for $t = 0, 1$, such that the restrictions $\mathcal{F}|M \times \{t\} = \mathcal{F}_t$ for $t = 0, 1$.

Integrable homotopy is a fairly weak notion of equivalence. For example, if M is an open contractible manifold then any two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on M are integrably homotopic.

Classifying spaces:

$B\Gamma_q$ denotes the "classifying space" of codimension-q foliations introduced by André Haefliger in 1970.

There is a natural map $\nu\colon B\Gamma_q \to BO_q$.

Theorem: (Haefliger [1970]) Each foliation $\mathcal{F}$ on M of codimension q determines a well-defined map $h_{\mathcal{F}}\colon M \to B\Gamma_q$ whose homotopy class in uniquely defined by $\mathcal{F}$, and depends only upon the integrable homotopy class of $\mathcal{F}$. The composition $\nu \circ h_{\mathcal{F}}\colon M \to BO_q$ classifies the normal bundle $Q \to M$ of $\mathcal{F}$.

Theorem: (Thurston [1975]) Let M be a closed manifold. A map $h\colon M \to B\Gamma_q \times BO_p$ for which the composition

$$(\nu \times Id) \circ h\colon M \to BO_q \times BO_p \to BO_m$$

classifies the tangent bundle TM, determines an integrable homotopy class of a codimension-q foliation $\mathcal{F}_h$ on M.

Primary characteristic classes

The Pontrjagin classes of the normal bundle $Q \to M$ factor through the map:

$$
\begin{tikzcd}
 & B\Gamma_q \arrow[d, "\nu"] \\
M \arrow[ur, "h_{\mathcal{F}}"] \arrow[r, "h_Q"] & BO_q
\end{tikzcd}
$$

Theorem: (Bott [1970])

$$h_Q^* \colon H^\ell(BO_q; \mathbb{R}) \to H^\ell(M; \mathbb{R}) \text{ is trivial for } \ell > 2q.$$

Theorem: (Bott-Heitsch [1972])

$$h_Q^* \colon H^\ell(BO_q; \mathbb{Z}) \to H^\ell(M; \mathbb{Z}) \text{ is injective for all } \ell.$$

Secondary classes

Theorem: (Godbillon-Vey [1971]) For $q \geq 1$, the Godbillon-Vey class $GV(\mathcal{F}) = \Delta(h_1 c_1^q) \in H^{2q+1}(M; \mathbb{R})$ is an integrable homotopy invariant.

$$WO_q \cong \Lambda(h_1, h_3, \ldots, h_{q/2}) \otimes \mathbb{R}[c_1, c_2, \ldots, c_q] \ , \ d_W h_i = c_i, d_W c_i = 0$$

Theorem: (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972]) For $q \geq 1$, there is a non-trivial space of secondary invariants $H^*(WO_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{tikzcd} & H^*(B\Gamma_q; \mathbb{R}) \arrow[d, "h_{\mathcal{F}}^*"] \\ H^*(WO_q) \arrow[ur, "\tilde{\Delta}"] \arrow[r, "\Delta_{\mathcal{F}}"] & H^*(M; \mathbb{R}) \end{tikzcd}$$

The study of the images of the maps $\Delta_{\mathcal{F}}$ has been the principle source of information about the (non-trivial) homotopy type of $B\Gamma_q$.

Homotopy chaos

Theorem: (Bott–Heitsch [1972]) $B\Gamma_q^r$ does not have finite topological type for $q \geq 2$.

Theorem: (Thurston [1972]) $\pi_3(B\Gamma_1^r)$ surjects onto $\mathbb{R}$.

Theorem: (Heitsch [1978]) There are continuous families of foliations with non-trivial variations of their secondary classes for $q \geq 3$.

Theorem: (Rasmussen [1980]) There are continuous families of foliations with non-trivial variations of their secondary classes for $q = 2$.

Corollary: $B\Gamma_q^r$ has uncountable topological type for all $q \geq 1$.

Theorem: (Hurder [1980]) For $q \geq 2$, $\pi_n(B\Gamma_q^r) \to \mathbb{R}^{k_n} \to 0$ where $k_{2q+1} \neq 0$, and in general, k_n has a subsequence $k_{n_\ell} \to \infty$

Secondary classes measure some uncountable aspect of foliation geometry.

C^2 is essential

In contrast, Takashi Tsuboi proved the following amazing result:

Theorem: (Tsuboi [1989]) The classifying map of the normal bundle $\nu\colon B\Gamma_q^1 \to BO(q)$ for foliations of transverse differentiability class C^1 is a homotopy equivalence.

The proof is a technical tour-de-force, using Mather-Thurston type techniques for the study of $B\Gamma$.

When the C^1 and C^2 situations are radically different, one asks if there is some aspects of dynamical systems involved? (There are other reasons to ask this question, too.)

Foliation dynamics

- A continuous dynamical system on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x, t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M, and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.

- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the dynamics of $\mathcal{F}$ asks for properties of the aggregate and statistical behavior of the collection of its leaves.

Pseudogroups & Groupoids

Every foliation admits a discrete model by choosing a section $\mathcal{T} \subset M$, an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$.

Definition: A pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if there is

- a relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}$ such that $\langle \Gamma \rangle = \mathcal{G}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Definition: The groupoid of $\mathcal{G}$ is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \ \& \ x \in D(g)\} \ , \ \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: The normal cocycle $D\varphi\colon \Gamma_{\mathcal{G}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D\varphi[g]_x = D_x g\colon\ T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D[h]_y \cdot D[g]_x$$

Pseudogroup word length

Definition: For $g \in \Gamma_{\mathcal{G}}$, the word length $\|[g]\|_x$ of the germ $[g]_x$ of g at x is the least n such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_n}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another.

Asymptotic exponent

Definition: The transverse expansion rate function at x is

$$\lambda(\mathcal{G}, n, x) = \max_{\|[g]\|_x \leq n} \frac{\ln\left(\max\{\|D_x g\|, \|D_y g^{-1}\|\}\right)}{\|[g]\|_x} \ \geq \ 0$$

Definition: The asymptotic transverse growth rate at x is

$$\lambda(\mathcal{G}, x) = \limsup_{n \to \infty} \ \lambda(\mathcal{G}, n, x) \ \geq \ 0$$

This is essentially the maximum Lyapunov exponent for $\mathcal{G}$ at x.

Expansion classification

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

where each are $\mathcal{F}$–saturated, Borel subsets of M, defined by:

1. Elliptic points: $\mathcal{E} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \forall\ n \geq 0,\ \lambda(\mathcal{G}, n, x) \leq \kappa(x)\}$
   i.e., "points of bounded expansion" – e.g., Riemannian foliations
2. Parabolic points: $\mathcal{P} \cap \mathcal{T} = \{x \in \mathcal{T} - (\mathcal{E} \cap \mathcal{T}) \mid \lambda(\mathcal{G}, x) = 0\}$
   i.e., "points of slow-growth expansion" – e.g., distal foliations
3. Partially Hyperbolic points: $\mathcal{H} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}, x) > 0\}$
   i.e., "points of exponential-growth expansion" – non-uniformly, partially hyperbolic foliations

Secondary classes and dynamics

A secondary class $h_I c_J \in H^*(WO_q)$ is *residual* if c_J has degree $2q$.

Theorem: (Hurder, 2006) Let $h_I c_J \in H^*(WO_q)$ be a residual secondary class (e.g., Godbillon-Vey type). Suppose that $\Delta_{\mathcal{F}}(h_I c_J) \in H^*(M; \mathbb{R})$ is non-zero. Then the hyperbolic component $\mathcal{H}$ has positive Lebesgue measure.

Moreover, the elliptic $\mathcal{E}$ and parabolic $\mathcal{P}$ components do not contribute to the secondary classes. (i.e., The Weil measure for h_I vanishes on these components, hence the restrictions of the residual secondary classes to these sets are trivial in cohomology.)

Understanding the "dynamical meaning of the residual secondary classes" in $H^*(WO_q)$ requires understanding the dynamics of foliations which have non-uniformly, partially hyperbolic behavior on a set of positive measure.

Framed foliations

But… is this a true picture of the relation between topology and dynamics?

Definition: $\mathcal{F}$ is framed if there is a framing $s \colon M \to \mathbf{Fr}(Q)$ of the normal bundle $Q \to M$. The classifying space $F\Gamma_q$ of framed foliations is the homotopy fiber

$$F\Gamma_q \to B\Gamma_q \to BO_q$$

The transgressions of the Pontrjagin classes $p_j = c_{2j}$ are now defined:

$$W_q \cong \Lambda(h_1, h_2, \ldots, h_{q/2}) \otimes \mathbb{R}[c_1, c_2, \ldots, c_q] \ , \ d_W h_i = c_i, d_W c_i = 0$$

Theorem': (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972]) There is a functorial characteristic map

$$\Delta^s \colon H^*(W_q) \to H^*(F\Gamma_q; \mathbb{R})$$

Classes involving the terms h_{2i} can also vary in examples.

Minimal sets

Introduce another basic idea of dynamics:

Definition: A closed, saturated subset $K \subset M$ is *minimal* if every leaf $L \subset K$ is dense in K.

A minimal set K can be one of three types:
- $K = L$ is a compact leaf of $\mathcal{F}$
- K has interior, hence M connected implies $K = M$
- K is not a leaf, and has no interior, hence K is a perfect subset.

The latter case is called an *exceptional* minimal set for historical reasons.

An essential exceptional parabolic minimal set

Theorem: (Hurder, 2008) For $q \geq 2$, there exists a framed foliation $\mathcal{F}$ with exceptional minimal set $\mathcal{S}$ such that:

• $\mathcal{F}$ is a parabolic foliation – $\mathcal{S}$ has no transverse hyperbolicity
• For every open neighborhood $\mathcal{S} \subset U$, the classifying map $h_{\mathcal{F}} \colon U \to F\Gamma_q$ is not homotopically trivial.

$\mathcal{S}$ is a generalized solenoid, which is transversally a Cantor set $\mathcal{C}$, and the holonomy of $\mathcal{F}$ restricted to $\mathcal{C}$ is equivalent to an "adding machine".

Bott-Heitsch revisited

For the construction of $\mathcal{S}$, we go back to the beginning:

Theorem: (Bott-Heitsch [1972])

$$h_Q^* \colon H^*(BO_q; \mathbb{Z}) \to H^*(M; \mathbb{Z}) \text{ is injective for all } *.$$

We recall the proof for the case of oriented normal bundles and $q = 2$.

$H^*(BSO_2; \mathbb{Z}) \cong \mathbb{Z}[e]$

Let $n > 2$, and set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Embed $\mathbb{Z}_n \subset SO_2$, acts isometrically on $\mathbb{R}^2$

$\mathbb{Z}_n$ acts freely on $\mathbb{S}^{2k+1}$ for $k > 0$.

$$\mathbb{E}_{n,k} = \mathbb{S}^{2k} \times \mathbb{R}^2/\mathbb{Z}_n \ , \ \mathcal{F}_{n,k} = \text{ flat bundle foliation}$$

For $* \leq 2k$ have injection:

$$\mathbb{Z}_n[e] \to H^*(BSO_2; \mathbb{Z}_n) \to H^*(B\Gamma_2; \mathbb{Z}_n) \to H^*(\mathbb{E}_{n,\ell}; \mathbb{Z}_n)$$

So $H^*(BSO_2; \mathbb{Z}_n) \to H^*(B\Gamma_2; \mathbb{Z}_n)$ is injective for all $*$ and all $n \to \infty$.

$H^*(BSO_2; \mathbb{Z}) \to H^*(B\Gamma_2; \mathbb{Z})$ injective follows from this.

General case for $q > 2$ uses splitting principle, for torsion subgroups of maximal torus, $\mathbb{Z}_n^k \subset \mathbb{T}^k \subset SO_{2k}$

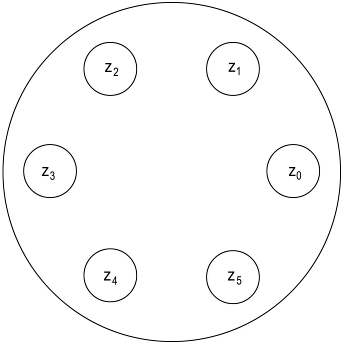
Question: Can we realize this limit process $(n, k) \to \infty$ with foliation?

Dynamics of flat bundles

Switch to groupoid model: $\mathbb{Z}_n$ acting on disk $\mathbb{D}^2 \subset \mathbb{R}^2$ via rotations.

Action is free except at center point of disk.

Pick $0 \neq z_1 \in \mathbb{D}^2$, with orbit $\mathbb{Z}_n \cdot z_1 = \{z_{1,0}, \ldots z_{1,n-1}\}$.
Consider disks $\mathbb{D}^2_{1,i}(z_{1,i}, \epsilon_1) \subset \mathbb{D}^2$ for $\epsilon_1 > 0$ sufficiently small. Here is illustration in case of $n = 6$:



Semi-simplicial realization of flat bundles

Let $\Gamma_{2,n} = (\mathbb{D}^2, \mathbb{Z}_n)$ denote the associated groupoid.

$|\Gamma_{2,n}|$ is the semi-simplicial space realizing the groupoid.

Then the classifying map factors:

$$\mathbb{E}_{n,k} \longrightarrow |\Gamma_{2,n}| \longrightarrow B\Gamma_2$$

Corollary: $H^*(BSO_2; \mathbb{Z}_n) \to H^*(B\Gamma_2; \mathbb{Z}_n) \to H^*(|\Gamma_{2,n}|; \mathbb{Z}_n)$
is injective for all $*$ and all $n \to \infty$.

Construction of solenoids

Choose $n_1 < n_2 < \cdots$ tending to infinity rapidly. Example: $n_k = 3^{k!}$

Choose $\epsilon_k \to 0$ rapidly, but slower than $1/n_k$. Example: $\epsilon_n = \epsilon_0 \cdot (3^n d_n)^{-1}$

Restriction of $\Gamma_{2,n_1} = (\mathbb{D}^2, \mathbb{Z}_{n_1})$ to the invariant set

$$\mathcal{S}_1 = \mathbb{D}^2_{1,0}(z_{1,0}, \epsilon_1) \cup \mathbb{D}^2_{1,1}(z_{1,1}, \epsilon_1) \cup \cdots \cup \mathbb{D}^2_{n-1}(z_{1,n_1-1}, \epsilon_1)$$

is free, so we can repeat this construction of an action on $\mathcal{S}_1$.

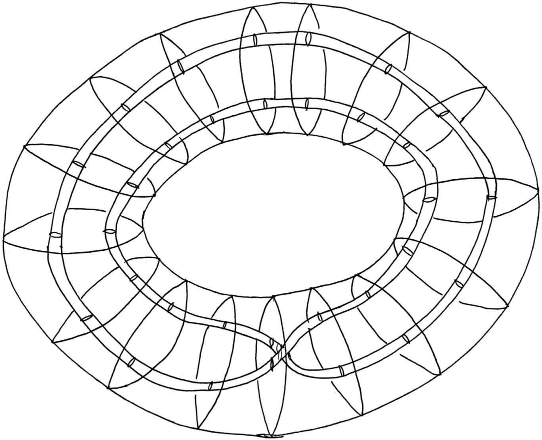
Choose $z_{2,0} \in \mathbb{D}^2_{1,0}(z_{1,0}, \epsilon_1)$ which is not on center.

Repeat above construction for $\mathbb{Z}_{n_2}$ on disks $\mathbb{D}^2_{2,0}(z_{2,0}, \epsilon_2)$.

There is one catch! Cannot just insert this action into Γ_{2,n_1}. The plug will not be smooth.

Need to make deformation of action from identity on boundary of V_1 to rotation by $2\pi/n_2$ on boundary of $\mathbb{D}^2_{2,0}(z_{2,0}, \epsilon_2)$.

Picture of stage 1: $n_1 = 2$



Limit solenoid

Let $\Gamma_{2,\infty}$ the smooth groupoid resulting from the limit of this construction. The action on $\mathbb{D}^2$ is distal!

Proposition: The dynamics of $\Gamma_{2,\infty}$ contains a solenoidal minimal set

$$\mathcal{S} = \bigcap_{k=1}^{\infty} |\mathcal{S}_k|$$

Proposition: For every open neighborhood $\mathcal{S} \subset U \subset |\Gamma_{2,\infty}|$ there exists some $k \gg 0$ such that $|\mathcal{S}_k| \subset U$

Corollary: For $k \gg 0$ there is an inclusion $|\Gamma_{2,k}| \subset |\Gamma_{2,\infty}|$.

Homotopical consequences

Let U be an open neighborhood, $\mathcal{S} \subset U \subset |\Gamma_{2,\infty}|$.

Proposition: $H^*(BSO_2;\mathbb{Z}) \to H^*(B\Gamma_2;\mathbb{Z}) \to H^*(U;\mathbb{Z})$ is injective.

Corollary: The image of the classifying map $U \to B\Gamma_2$ cannot have finite type in all odd dimensions > 4.

One obtains framed foliations by considering the frame bundle $\widehat{U} \to U$ of the normal bundle on U.

The foliation $\mathcal{F}$ on U lifts to a foliation $\widehat{\mathcal{F}}$ on $\widehat{U}$.

By finite-type considerations, we obtain

Theorem: The image of the classifying map $\widehat{U} \to F\Gamma_q$ cannot have finite type in all odd dimensions > 4.

Chern-Simons invariants

Theorem: The Chern-Simons invariants in $H^{2*-1}(B\Gamma_2; \mathbb{R}/\mathbb{Z})$ are non-trivial on the image of $|\Gamma_{2,\infty}| \to B\Gamma_q$ in all odd dimensions > 4.

Remark 1: Apparently, the transgression classes of the Pontrjagin classes $H^{4*}(BSO_q; \mathbb{R})$ do not depend on dynamics in the same way as before.

Remark 2: The above construction admits many generalizations to embedded braid diagrams. Unclear what cohomology theories will be needed to detect them.