DE GRUYTER OPEN DOI 10.2478/spma-2014-0002 — Spec. Matrices 2014; 2:11–18

Research Article Open Access

E. Macías-Virgós and M.J. Pereira-Sáez

Cayley-Hamilton theorem for left eigenvalues of 3×3 quaternionic matrices

Abstract: We prove that any quaternionic matrix of order $n \leq 3$ admits a characteristic function, whose roots are the left eigenvalues, that satisfies Cayley-Hamilton theorem.

Keywords: Quaternionic matrix, left eigenvalue, characteristic function, Cayley-Hamilton theorem

MSC: 15B33, 15A18

E. Macías-Virgós: Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain, E-mail: quique.macias@usc.es
M.J. Pereira-Sáez : Facultade de Economía e Empresa, Universidade da Coruña, Spain, E-mail: maria.jose.pereira@udc.es

1 Introduction

Very little is known about left eigenvalues of quaternionic matrices, see Zhang's reviews [8, 9]. In a previous paper [6] we introduced the notion of (left) characteristic function for a quaternionic matrix, whose roots are the left eigenvalues. Explicitly, we say that $\mu\colon \mathbb{H} \to \mathbb{H}$ is a characteristic function of the matrix $A \in \mathbb{H}^{n\times n}$ if, up to a constant, its norm verifies that $|\mu(\lambda)| = \mathrm{Sdet}(A - \lambda I)$, for all $\lambda \in \mathbb{H}$, where $\mathrm{Sdet}\colon \mathbb{H}^{n\times n} \to [0, +\infty)$ is Study's determinant. In particular, λ is a left eigenvalue of A if and only if $\mu(\lambda) = 0$ (see definitions in Section 2). In the present paper we discuss Cayley-Hamilton theorem in this setting, for $n \leq 3$.

When $n = 2$, the characteristic function given in formula (2) is analogous to those considered by Wood [7] or Huang [5]. It is a polynomial $\mu(\lambda)$ for which it is easy to check that $\mu(A) = 0$.

Now, let $n = 3$. When the matrix has some zero entry outside the diagonal, it is known [6] that the characteristic function μ can be taken to be a (non unilateral) quaternionic polynomial of degree 3. Otherwise, μ will be a rational function of the form $P(\lambda) - Q(\lambda)(\lambda_0 - \lambda)^{-1}F(\lambda)$, where P, Q, F are polynomials, defined outside one point of discontinuity $\lambda = \lambda_0$ called the pole. Anyway, in both cases μ can be extended in a natural way to a matrix map $\mu\colon \mathbb{H}^{n\times n} \to \mathbb{H}^{n\times n}$ and we shall check that Cayley-Hamilton theorem always holds.

Then, our main result can be stated as follows.

Theorem A. *For any quaternionic matrix $A \in \mathbb{H}^{n\times n}$, $n \leq 3$, there exists a characteristic function μ_A such that $\mu_A(A) = 0$.*

We point out two difficulties. First, in the classical commutative case one has a characteristic polynomial with complex variable which naturally extends to a polynomial with a matrix variable. But in our setting there is not an obvious extension of any arbitrary characteristic function. Fortunately, we were able to find characteristic functions which are polynomials or rational functions, which naturally extend to matrices.

The second problem is that none of the known proofs of the commutative case seems to adapt to our setting, so we shall have to do a brute force computation.

We thank the remarks of two anonymous referees.

2 Preliminaries

If A is a quaternionic $n \times n$ matrix, we can write $A = X + \mathbf{j}Y$, with $X, Y \in \mathbb{C}^{n\times n}$, and let

$$c(A) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in \mathbb{C}^{2n\times 2n}.$$

Since $\det c(A) \geq 0$ is a nonnegative real number, we can define the *Study's determinant* of A as

$$\mathrm{Sdet}(A) = (\det c(A))^{1/2} \geq 0. \tag{1}$$

For a complex matrix X, $\mathrm{Sdet}(X)$ equals $|\det(X)|$, the absolute value of the complex determinant, see [1, 2] for a general discussion of quaternionic determinants.

A quaternion λ is said to be a *left eigenvalue* of the matrix $A \in \mathbb{H}^{n\times n}$ if $Av = \lambda v$ for some nonzero vector $v \in \mathbb{H}^n$. Equivalently, the matrix $A - \lambda I$ is not invertible, that is $\mathrm{Sdet}(A - \lambda I) = 0$.

Definition 2.1. [6] A map $\mu\colon \mathbb{H} \to \mathbb{H}$ is a *characteristic function* of the matrix $A \in \mathbb{H}^{n\times n}$ if there exists a nonzero real constant C such that $|\mu(\lambda)| = C \cdot \mathrm{Sdet}(A - \lambda I)$ for all $\lambda \in \mathbb{H}$.

Example 2.2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{H}^{2\times 2}$. If $b = 0$ then $\mathrm{Sdet}(A) = |da|$ and the map $\mu(\lambda) = (d - \lambda)(a - \lambda)$ is a characteristic function. If $b \neq 0$ we have

$$\mathrm{Sdet}(A) = \mathrm{Sdet}\begin{bmatrix} 0 & b \\ c - db^{-1}a & d \end{bmatrix} = |b||c - db^{-1}a|$$

because Sdet does not change when a (right) multiple of one column is added to another column (see Corollary 2.10 in [6]); and we consider the characteristic function

$$\mu(\lambda) = c - (d - \lambda)b^{-1}(a - \lambda). \tag{2}$$

Remark 2.3. The relationship between characteristic functions and the theory of quasideterminants from Gelfand *et al.* [4] has been discussed in [6, Section 3.2]. In particular, none of the quasideterminants of $A - \lambda I$ gives the complete left spectrum. A version of Cayley-Hamilton theorem in the non-commutative setting which involves certain polynomials related to quasideterminants is given in [3].

Remark 2.4. In contrast with left eigenvalues, *right* eigenvalues of a quaternionic matrix $A \in \mathbb{H}^{n\times n}$ are well understood. Actually, they are the quaternions similar to the (complex) eigenvalues of the matrix $c(A)$ defined above. In particular the following Cayley-Hamilton theorem was proved by Zhang [8]: let $p(z) = \det(c(A) - zI) = \sum_{k=0}^{2n} c_k z^k$ be the (complex) characteristic polynomial of $c(A)$, then $p(A) = \sum_{k=0}^{2n} c_k A^k = 0$.

3 Characteristic functions of 3 × 3 matrices

Let $A = \begin{bmatrix} a & b & c \\ f & g & h \\ p & q & r \end{bmatrix}$ be a quaternionic matrix of order 3. The following characteristic functions were defined in [6].

3.1 Case $n = 3$, $c = 0$

First, if both $b, h = 0$ we have a triangular matrix, and we can take

$$\mu(\lambda) = (r - \lambda)(g - \lambda)(a - \lambda). \tag{3}$$

If $b = 0$ but $h \neq 0$ we can reduce to the 2×2 case by elementary transformations, so

$$\mu(\lambda) = \left(q - (r-\lambda)h^{-1}(g-\lambda)\right)(a-\lambda). \tag{4}$$

Finally, if $b \neq 0$ we can make a zero in the left top corner of $A - \lambda I$ and then permute the second and last column, in order to reduce the matrix to the 2×2 case. Explicitly we have

$$\operatorname{Sdet}(A-\lambda I) = \operatorname{Sdet}\begin{bmatrix} 0 & 0 & b \\ f-(g-\lambda)b^{-1}(a-\lambda) & h & g-\lambda \\ p-qb^{-1}(a-\lambda) & r-\lambda & q \end{bmatrix} = |b| \operatorname{Sdet}\begin{bmatrix} f-(g-\lambda)b^{-1}(a-\lambda) & h \\ p-qb^{-1}(a-\lambda) & r-\lambda \end{bmatrix}. \tag{5}$$

The corresponding characteristic functions then follow from the 2×2 situation. Namely, when $h = 0$ we define

$$\mu(\lambda) = (r-\lambda)\left(f-(g-\lambda)b^{-1}(a-\lambda)\right). \tag{6}$$

Otherwise, for $h \neq 0$ we take

$$\mu(\lambda) = p - qb^{-1}(a-\lambda) - (r-\lambda)h^{-1}\left(f-(g-\lambda)b^{-1}(a-\lambda)\right). \tag{7}$$

As explained at the end of Subsection 4.2, the case $c = 0$ also applies to any matrix with a nonzero entry outside the diagonal.

3.2 Case $n = 3, c \neq 0$

When $c \neq 0$ we call $\lambda_0 = g - hc^{-1}b$ the *pole* of A. We define:
(i) if $\lambda = \lambda_0$,

$$\mu(\lambda_0) = \left(q-(r-\lambda_0)c^{-1}b\right)\left(f-hc^{-1}(a-\lambda_0)\right);$$

(ii) otherwise,

$$\mu(\lambda) = (\lambda_0-\lambda)\left[\left(p-(r-\lambda)c^{-1}(a-\lambda)\right) - \left(q-(r-\lambda)c^{-1}b\right)(\lambda_0-\lambda)^{-1}\left(f-hc^{-1}(a-\lambda)\right)\right].$$

This is a rational function which may not be continuous at λ_0 (see Theorem 5.6 of [6]). We shall extend it to a map in the space of matrices in the following natural way. Let

$$f_0 = f - hc^{-1}(a-\lambda_0),$$
$$q_0 = q - (r-\lambda_0)c^{-1}b.$$

Lemma 3.1. *The matrix $\lambda_0 I - A$ is invertible if and only if $f_0, q_0 \neq 0$.*

Proof. Take the matrix $A - \lambda_0 I = \begin{bmatrix} a-\lambda_0 & b & c \\ f & g-\lambda_0 & h \\ p & q & r-\lambda_0 \end{bmatrix}$ and make elementary transformations to obtain the matrix

$$\begin{bmatrix} 0 & 0 & c \\ f_0 & 0 & h \\ p-(r-\lambda_0)c^{-1}(a-\lambda_0) & q_0 & r-\lambda_0 \end{bmatrix}.$$

Then $\operatorname{Sdet}(\lambda_0 I - A) = |c| \cdot |q_0 f_0|$. □

Now we associate to the matrix A the map $\mu_A : \mathbb{H}^{n\times n} \to \mathbb{H}^{n\times n}$ given by
(i) if $\lambda_0 I - B$ is not invertible,

$$\mu_A(B) = q_0 f_0 I; \tag{8}$$

(ii) otherwise,

$$\mu_A(B) = (\lambda_0 I - B)\left[\left(pI-(rI-B)c^{-1}(aI-B)\right) - \left(qI-(rI-B)c^{-1}b\right)(\lambda_0 I-B)^{-1}\left(fI-hc^{-1}(aI-B)\right)\right]. \tag{9}$$

4 Cayley-Hamilton theorem

We now discuss Cayley-Hamilton Theorem A.

4.1 Case $n = 2$

Proposition 4.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a* 2×2 *quaternionic matrix with* $b \neq 0$*. Let* $\mu(\lambda)$ *be the characteristic function defined in* (2)*. Then* $\mu(A) = 0$.

Proof. We have

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} - \begin{bmatrix} d-a & -b \\ -c & 0 \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & -b \\ -c & a-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

Corollary 4.2. $Ab^{-1}A = Ab^{-1}a + db^{-1}A + (c - db^{-1}a)I$.

This is a generalization of the usual formula $A^2 = (\mathrm{tr}A)A - (\det A)I$ for 2×2 matrices in the commutative setting. As it is well-known this kind of relationships can be used to reduce the order of a polynomial in A and to compute the exponential $\exp(A)$, or more generally, to determine analytic functions on the matrix.

4.2 Case $n = 3$, $c = 0$

For $n = 3$, a direct computation will show that Cayley-Hamilton theorem is true in the case $c = 0$.

Proposition 4.3. *Let* $\mu(\lambda)$ *be the characteristic function defined in Subsection 3.1. Then* $\mu(A) = 0$.

Proof. If $b, h = 0$, we take formula (3), so $\mu(A)$ equals

$$\begin{bmatrix} r-a & 0 & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} \begin{bmatrix} g-a & 0 & 0 \\ -f & 0 & 0 \\ -p & -q & g-r \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & 0 \\ -p & -q & a-r \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $b = 0$, $h \neq 0$, we take formula (4). We check

$$\begin{bmatrix} r-a & 0 & 0 \\ -f & r-g & -h \\ -p & -q & 0 \end{bmatrix} h^{-1} \begin{bmatrix} g-a & 0 & 0 \\ -f & 0 & -h \\ -p & -q & (g-r) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix} = q \begin{bmatrix} 0 & 0 & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix},$$

that is, $(rI - A)h^{-1}(gI - A)(aI - A) = q(aI - A)$, hence $\mu(A) = 0$.

If $b \neq 0$ and $h = 0$ we take formula (6). We check

$$\begin{bmatrix} r-a & -b & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} \begin{bmatrix} g-a & -b & 0 \\ -f & 0 & 0 \\ -p & -q & g-r \end{bmatrix} b^{-1} \begin{bmatrix} 0 & -b & 0 \\ -f & a-g & 0 \\ -p & -q & a-r \end{bmatrix} = \begin{bmatrix} r-a & -b & 0 \\ -f & r-g & 0 \\ -p & -q & 0 \end{bmatrix} f,$$

that is, $(rI - A)(gI - A)b^{-1}(aI - A) = (rI - A)f$, hence $\mu(A) = 0$.

On the other hand, if $b, h \neq 0$ we take formula (7). Then we compute

$$pI - qb^{-1}(aI - A) - (rI - A)h^{-1}f =$$

$$\begin{bmatrix} p-(r-a)h^{-1}f & q+bh^{-1}f & 0 \\ qb^{-1}f+fh^{-1}f & p-qb^{-1}(a-g)-(r-g)h^{-1}f & qb^{-1}h+f \\ qb^{-1}p+ph^{-1}f & qb^{-1}q+qh^{-1}f & p-qb^{-1}(a-r) \end{bmatrix}$$

and we check it equals

$$-\begin{bmatrix} r-a & -b & 0 \\ -f & r-g & -h \\ -p & -q & 0 \end{bmatrix} h^{-1} \begin{bmatrix} g-a & -b & 0 \\ -f & 0 & -h \\ -p & -q & g-r \end{bmatrix} b^{-1} \begin{bmatrix} 0 & -b & 0 \\ -f & a-g & -h \\ -p & -q & a-r \end{bmatrix},$$

that is, $-(rI - A)h^{-1}(gI - A)b^{-1}(aI - A)$, hence $\mu(A) = 0$. □

Let B be a 3×3 quaternionic matrix with some zero entry outside the diagonal. By consecutively permuting rows and columns we can transform it into a matrix $A = PBP^{-1}$, with P an invertible real matrix, where the zero entry of A is in the corner position c. In fact, the matrices A and B have the same characteristic functions, because $\mathrm{Sdet}(B - \lambda I) = \mathrm{Sdet}(A - \lambda I)$. Then from formulae (6) and (7) it follows that there exists a polynomial characteristic function μ for B (cf. Theorem 5.1 in [6]). The next Lemma proves that $\mu(B) = 0$.

Lemma 4.4. *Let A be a quaternionic matrix such that $\mu(A) = 0$ for some quaternionic polynomial $\mu(\lambda)$. Let $B = PAP^{-1}$, with P an invertible real matrix. Then $\mu(B) = 0$.*

Proof. If $\nu(\lambda) = q_1\lambda q_2\lambda \cdots q_k\lambda q_{k+1}$ is a monomial then $\nu(B) = \nu(PAP^{-1}) = P\nu(A)P^{-1}$. So for the polynomial μ it is $\mu(B) = P\mu(A)P^{-1} = 0$. □

Example 4.5. The matrix $A = \begin{bmatrix} 1 & \mathbf{i} & \mathbf{i} \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & \mathbf{j} \end{bmatrix}$ is real similar to $\begin{bmatrix} \mathbf{j} & -1 & 0 \\ \mathbf{k} & \mathbf{j} & \mathbf{i} \\ \mathbf{i} & \mathbf{i} & 1 \end{bmatrix}$, whose characteristic function is given by formula (7), that is

$$\mu(\lambda) = \mathbf{i} + \mathbf{i}(\mathbf{j} - \lambda) + (1 - \lambda)\mathbf{i}\left(\mathbf{k} + (\mathbf{j} - \lambda)^2\right).$$

Then the following equation holds:

$$A\mathbf{i}A^2 = A\mathbf{i}A\mathbf{j} + A\mathbf{k}A + \mathbf{i}A^2 - \mathbf{i}A\mathbf{j} + A(\mathbf{i} + \mathbf{j}) - (\mathbf{i} + \mathbf{k})A + (\mathbf{k} - \mathbf{j})I.$$

4.3 Case $n = 3$, $c \neq 0$

When $c \neq 0$, the characteristic function of the matrix A is a rational function with a pole, as defined in Section 3.2. The following Proposition completes the proof of Theorem A.

Proposition 4.6. *The map μ_A in* (8) *and* (9) *satisfies $\mu_A(A) = 0$.*

Proof. If $\lambda_0 I - A$ is not invertible, then $\mu(A) = q_0 f_0 I = 0$ by Lemma 3.1. Otherwise it suffices to prove that

$$pI - (rI - A)c^{-1}(aI - A) = \left(qI - (rI - A)c^{-1}b\right)(\lambda_0 I - A)^{-1}\left(fI - hc^{-1}(aI - A)\right). \tag{10}$$

A direct computation shows that the first term of (10) equals

$$P = \begin{bmatrix} -bc^{-1}f & -q + (r-a)c^{-1}b + bc^{-1}(a-g) & \vdots \\ (r-g)c^{-1}f - hc^{-1}p & p - fc^{-1}b - hc^{-1}q - (r-g)c^{-1}(a-g) & \vdots \\ -qc^{-1}f & -pc^{-1}b + qc^{-1}(a-g) & \vdots \end{bmatrix}$$

$$\begin{matrix} \vdots & -bc^{-1}h \\ \vdots & -f+(r-g)c^{-1}h+hc^{-1}(a-r) \\ \vdots & -qc^{-1}h \end{bmatrix}.$$

We now want to compute the second term in (10).

We start by computing $(\lambda_0 I - A)^{-1}$ by Gaussian elimination. Let

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c^{-1}(\lambda_0 - a) & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c^{-1}b & 1 \end{bmatrix}.$$

Then

$$(\lambda_0 I - A)P_1P_2 = \begin{bmatrix} 0 & 0 & -c \\ -f_0 & 0 & -h \\ -p_0 & -q_0 & \lambda_0 - r \end{bmatrix}, \tag{11}$$

where

$$p_0 = p - (\lambda_0 - r)c^{-1}(\lambda_0 - a).$$

The inverse of the matrix $(\lambda_0 I - A)P_1P_2$ can be computed by hand; it is

$$M = \begin{bmatrix} f_0^{-1}hc^{-1} & -f_0^{-1} & 0 \\ -q_0^{-1}\left(p_0f_0^{-1}hc^{-1} + (\lambda_0 - r)c^{-1}\right) & q_0^{-1}p_0f_0^{-1} & -q_0^{-1} \\ -c^{-1} & 0 & 0 \end{bmatrix}.$$

It follows that $(\lambda_0 I - A)^{-1}$ equals P_1P_2M, that is,

$$\begin{bmatrix} f_0^{-1}hc^{-1} & \vdots \\ -q_0^{-1}\left(p_0f_0^{-1}hc^{-1} + (\lambda_0 - r)c^{-1}\right) & \vdots \\ -c^{-1} + c^{-1}(\lambda_0 - a)f_0^{-1}hc^{-1} + c^{-1}bq_0^{-1}\left(p_0f_0^{-1}hc^{-1} + (\lambda_0 - r)c^{-1}\right) & \vdots \end{matrix}$$

$$\begin{matrix} \vdots & -f_0^{-1} & 0 \\ \vdots & q_0^{-1}p_0f_0^{-1} & -q_0^{-1} \\ \vdots & -c^{-1}(\lambda_0 - a)f_0^{-1} - c^{-1}bq_0^{-1}p_0f_0^{-1} & c^{-1}bq_0^{-1} \end{bmatrix}.$$

Moreover

$$F = fI - hc^{-1}(aI - A) = \begin{bmatrix} f & hc^{-1}b & h \\ hc^{-1}f & f - hc^{-1}(a-g) & hc^{-1}h \\ hc^{-1}p & hc^{-1}q & f - hc^{-1}(a-r) \end{bmatrix}$$

while

$$Q = qI - (rI - A)c^{-1}b = \begin{bmatrix} q - (r-a)c^{-1}b & bc^{-1}b & b \\ fc^{-1}b & q - (r-g)c^{-1}b & hc^{-1}b \\ pc^{-1}b & qc^{-1}b & q \end{bmatrix}.$$

Now we compute $(P_1P_2M)F$, which equals

$$\begin{bmatrix} 0 & \vdots \\ -q_0^{-1}\left((\lambda_0 - r)c^{-1}f + hc^{-1}p\right) & \vdots \\ c^{-1}\left(bq_0^{-1}\left((\lambda_0 - r)c^{-1}f + hc^{-1}p\right) - f\right) & \vdots \end{matrix}$$

$$\begin{matrix} \vdots & -1 & \vdots \\ \vdots & q_0^{-1}\left(p_0-(\lambda_0-r)c^{-1}hc^{-1}b-hc^{-1}q\right) & \vdots \\ \vdots & -c^{-1}bq_0^{-1}\left(p-(\lambda_0-r)c^{-1}(g-a)-hc^{-1}q\right)+c^{-1}(a-g) & \vdots \end{matrix}$$

$$\left.\begin{matrix} \vdots & 0 \\ \vdots & -q_0^{-1}\left((\lambda_0-r)c^{-1}h+f-hc^{-1}(a-r)\right) \\ \vdots & -c^{-1}h+c^{-1}bq_0^{-1}\left((\lambda_0-r)c^{-1}h+f-hc^{-1}(a-r)\right) \end{matrix}\right] .$$

Finally, the second in term (10) is $Q(P_1P_2MF)$, which gives the desired result. □

Example 4.7. Let $A = \begin{bmatrix} 1 & \mathbf{i} & -\mathbf{j} \\ \mathbf{i} & -1 & \mathbf{k} \\ 1 & -1 & \mathbf{j} \end{bmatrix}$. The pole is $\lambda_0 = -2$ and $\mu(\lambda_0) = -4-4\mathbf{i}+8\mathbf{j}$. For $\lambda \neq -2$, the characteristic function is

$$\mu(\lambda) = -(2+\lambda)\left(2+\lambda(-1+\mathbf{j})-\lambda\mathbf{j}\lambda+(-1+\mathbf{i}-\lambda\mathbf{k})(2+\lambda)^{-1}\mathbf{i}(2-\lambda)\right). \tag{12}$$

With the notations of the proof of Proposition 4.6, it is

$$(\lambda_0 I - A)^{-1} = (1/12)\begin{bmatrix} -3 & 3\mathbf{i} & 0 \\ 2\mathbf{i}-\mathbf{j}-\mathbf{k} & -8+2\mathbf{i}+\mathbf{j}+3\mathbf{k} & 2+2\mathbf{i}+4\mathbf{k} \\ 1+\mathbf{i}-\mathbf{j} & -3-\mathbf{i}+2\mathbf{j}+\mathbf{k} & -4+2\mathbf{j}+2\mathbf{k} \end{bmatrix},$$

$$P = \begin{bmatrix} -\mathbf{j} & 1-\mathbf{i}+3\mathbf{k} & 1 \\ -\mathbf{k} & 3-\mathbf{i}-3\mathbf{j} & -\mathbf{i} \\ -\mathbf{k} & -2\mathbf{j}+\mathbf{k} & \mathbf{i} \end{bmatrix},$$

$$Q = \begin{bmatrix} -1+\mathbf{i}-\mathbf{k} & \mathbf{j} & \mathbf{i} \\ \mathbf{j} & -1+\mathbf{i}+\mathbf{k} & 1 \\ -\mathbf{k} & \mathbf{k} & -1 \end{bmatrix},$$

$$F = \begin{bmatrix} \mathbf{i} & 1 & \mathbf{k} \\ 1 & 3\mathbf{i} & \mathbf{j} \\ -\mathbf{i} & \mathbf{i} & 2\mathbf{i}-\mathbf{k} \end{bmatrix}.$$

We check $P - Q(\lambda_0 I - A)^{-1}F = 0$.

Now, the characteristic function of $\lambda_0 I - A$ is $\mu(-2-\lambda)$. Accordingly to Theorem 5.10 in [6], since λ_0 is not a left eigenvalue of A, the matrix $\lambda_0 I - A$ is invertible and its inverse has a polynomial characteristic function, given by formula (7), that is,

$$\nu(\lambda) = -12\mathbf{j} + \frac{1}{2}\lambda(2-3\mathbf{j}-\mathbf{k}) - 3\mathbf{j}\lambda - \mathbf{k}\lambda\mathbf{i} - \frac{1}{3}\mathbf{k}\lambda\mathbf{i}\lambda + \frac{1}{3}\lambda^2 - \frac{1}{2}\lambda\mathbf{j}\lambda - 12\lambda(-1+\mathbf{i}+2\mathbf{k})\lambda\mathbf{i} - \frac{1}{36}\lambda(-1+\mathbf{i}+2\mathbf{k})\lambda\mathbf{i}\lambda.$$

Then, as a consequence of Cayley-Hamilton theorem, we have the relationships

$$\mu(-2I - A) = 0,$$

$$\nu((-2I - A)^{-1}) = 0.$$

References

[1] Aslaksen, H. Quaternionic determinants. *Math. Intell.* 18, No. 3, 57–65 (1996).

[2] Cohen, N.; De Leo, S. The quaternionic determinant. *Electron. J. Linear Algebra* 7, 100–111 (2000).
[3] Gelfand, I.M.; Krob, D.; Lascoux, A.; Leclerc, B.; Retakh, V.; Thibon, J.-Y. Noncommutative symmetric functions. *Adv. Math.*, 112, No. 2, 218–348 (1995).
[4] Gelfand, I.; Gelfand, S.; Retakh, V.; Lee Wilson R. Quasideterminants. *Adv. Math.* 193, No. 1, 56–141 (2005).
[5] Huang, L. On two questions about quaternion matrices. *Linear Algebra Appl.* 318, No. 1-3, 79–86 (2000).
[6] Macías-Virgós, E.; Pereira-Sáez, M.J. A topological approach to left eigenvalues of quaternionic matrices, *Linear Multilinear Algebra* 62, No. 2, 139–158 (2014).
[7] Wood, R.M.W. Quaternionic eigenvalues. *Bull. Lond. Math. Soc.* 17, 137–138 (1985).
[8] Zhang, F. Quaternions and matrices of quaternions. *Linear Algebra Appl.* 251, 21–57 (1997).
[9] Zhang, F. Geršgorin type theorems for quaternionic matrices. *Linear Algebra Appl.* 424, No. 1, 139–153 (2007).

Received September 3, 2013; accepted February 11, 2014.