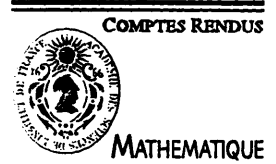




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Differential Geometry

Non-unimodular Lie foliations

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Abstract

Let \mathcal{F} be a G -Lie foliation on a compact manifold M . If \mathcal{F} is not unimodular then either M or the closures of the leaves fiber over S^1 . *To cite this article: E. Macias-Virgós, P. Martín-Méndez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

Feuilletages de Lie non unimodulaires. Soit \mathcal{F} un G -feuilletage de Lie sur une variété compacte M . Si \mathcal{F} n'est pas unimodulaire alors ou bien M ou bien les adhérences des feuilles fibrent sur S^1 . *Pour citer cet article : E. Macias-Virgós, P. Martín-Méndez, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Soit M une variété compacte, G un groupe de Lie connexe simplement connexe. Un G -feuilletage de Lie sur M [8] est défini par une 1-forme de rang maximal sur M à valeurs dans l'algèbre de Lie de G , telle que $d\omega = (-1/2)[\omega, \omega]$. Le feuilletage \mathcal{F} est *unimodulaire* si sa cohomologie basique vérifie la dualité de Poincaré. Rappelons que la cohomologie basique $H(M/\mathcal{F})$ d'une variété feuilletée est celle du complexe des formes basiques, c'est-à-dire des formes différentielles α sur M qui vérifient $i_X\alpha = 0$ et $i_X d\alpha = 0$ pour tout champ de vecteurs X tangent à \mathcal{F} .

Carrière [1] a donné le premier exemple de feuilletage de Lie non unimodulaire. Plus tard Masa [7] a montré que l'unimodularité d'un feuilletage riemannien (et a fortiori de Lie) équivaut à l'existence sur la variété d'une métrique riemannienne pour laquelle les feuilles sont des sous-variétés minimales. Dans l'exemple de Carrière,

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la variété $M = T_A^3$ fibre sur le cercle. Nous nous proposons de démontrer le résultat général suivant : si un G -feuilletage de Lie (avec ou sans feuilles denses) n'est pas unimodulaire alors ou bien la variété M ou bien les adhérences des feuilles fibrent sur S^1 .

Les ingrédients essentiels de la preuve sont le théorème de Tischler [9] et le résultat suivant obtenu par El Kacimi et Nicolau [3] : un G -feuilletage de Lie est unimodulaire si et seulement si G et l'adhérence K du groupe d'holonomie (qui n'est pas connexe en général) sont des groupes de Lie unimodulaires.

Il faut rappeler que l'unimodularité d'un groupe de Lie *connexe* équivaut à celle de son algèbre de Lie, mais ceci n'est pas le cas pour le sous-groupe K .

1. Introduction

Let M be a compact manifold and G a connected simply connected Lie group. A G -Lie foliation on M [8] is that defined by a maximal rank 1-form ω on M with values in the Lie algebra of G , such that $d\omega = (-1/2)[\omega, \omega]$. The foliation \mathcal{F} is said to be *unimodular* if the basic cohomology verifies Poincaré duality. Recall that the basic cohomology $H(M/\mathcal{F})$ of a foliated manifold is that of the complex of basic forms, i.e. differential forms α on M verifying $i_X\alpha = 0$ and $i_X d\alpha = 0$ for every vector field X tangent to \mathcal{F} .

The first example of a non unimodular Lie foliation was given by Carrière in [1]. Later Masa [7] proved the long-standing conjecture that unimodularity of a Riemannian foliation is equivalent to the existence of a Riemannian metric for which the leaves are minimal submanifolds.

In Carrière's example, the ambient manifold $M = T_A^3$ fibers over S^1 . The aim of this paper is to prove the following general property of non-unimodular Lie foliations, even when the leaves are not dense.

Theorem 1.1. *If a G -Lie foliation on a compact manifold M is not unimodular then either M or the closures of the leaves fiber over S^1 .*

2. Non-unimodular Lie foliations

The following result about the structure of Lie foliations is due to Fédida [4]. The G -Lie foliation \mathcal{F} is completely determined by the holonomy morphism $h : \pi_1(M) \rightarrow G$ and the developing map $D : \tilde{M} \rightarrow G$, where \tilde{M} is the covering space of M associated to the kernel of h . The map D is an h -equivariant fiber bundle. Let Γ be the image of h , and K its closure in G . Let K_e be the connected component of the identity in K . When the leaves of \mathcal{F} are not dense in M (that is $K \neq G$), then the closures of the leaves are the fibers of a fiber bundle $M \rightarrow W$ over the basic manifold $W = G/K$. In this case \mathcal{F} induces on each fiber another Lie foliation modeled by (the universal covering of) K_e (for the usual notations see also [6]).

It follows that the basic cohomology $H(M/\mathcal{F})$ is isomorphic to $H_K(G)$, the De Rham cohomology of differential forms on G invariant by K . It is finite dimensional [2]; hence duality is equivalent to $H_K^n(G) \neq 0$, $n = \text{codim } \mathcal{F}$. We shall exploit the following result from El Kacimi and Nicolau.

Theorem 2.1 [3]. *$H_K^n(G) \neq 0$ if and only if the Lie groups G and K are unimodular.*

Recall that the modular function of the Lie group G is given by $m_G(x) = \det \text{Ad}_G(x)$. Due to connectedness, the unimodularity of G (analogously K_e) is equivalent to that of its Lie algebra (i.e. $\text{trace ad}_X = 0$), but this is not the case for K .

Proof of Theorem 1.1. Let the foliation \mathcal{F} be non-unimodular. If the Lie group G is not unimodular, we can consider the non-trivial multiplicative modular map $m_G : G \rightarrow \mathbb{R}^+$. Then the compositions $\log m_G \circ D$ and

$\log m_G \circ h$ can be viewed respectively as a developing map and a holonomy morphism defining a codimension one Lie foliation on M . In other words we have a foliation defined by a closed form. By Tischler’s theorem [9], M fibers over S^1 .

On the other hand, if G is unimodular, K can not be unimodular by Theorem 2.1, while K_e may be unimodular or not (see Remark 1 below).

- (i) If K_e is not unimodular, by taking into account its modular function the argument above shows that the closures of the leaves (where there is an induced \tilde{K}_e -Lie foliation) fiber over S^1 .
- (ii) If K_e is unimodular, let m_K be the non-trivial multiplicative map $K \rightarrow \mathbb{R}^+$ given by $m_K(x) = |\det Ad_K(x)|$. We shall prove the existence of a map $m : G \rightarrow \mathbb{R}^+$ such that $m|_K = m_K$ and $m(xy) = m(x) \cdot m(y)$ for all $x \in G, y \in K$. Then, by considering the compositions $f \circ D$ and $f \circ h$, where $f = \log m$, Tischler’s theorem allows us again to ensure that M fibers over S^1 . Notice that the map $f : G \rightarrow \mathbb{R}$ is onto, because the map $m|_K = m_K \neq 1$ is not bounded, and the Lie group G is connected. The equivariance follows from the properties of m .

Let us show the existence of the map m . Let $W = G/K$ be the basic manifold of the foliation. Its universal covering is $\tilde{W} = G/K_e$ and $\pi_1(W) = K/K_e$. Since $m_K(x) = 1$ for $x \in K_e$, the morphism

$$\bar{m}_K : K/K_e \rightarrow \mathbb{R}^+$$

is well defined. Then

$$\log \bar{m}_K : K/K_e = \pi_1(W) \rightarrow \mathbb{R}$$

belongs to $\text{Hom}(\pi_1(W), \mathbb{R})$. Since W is compact, we can identify it with a cohomology class $[\omega] \in H^1_{DR}(W)$ such that

$$\log \bar{m}_K([\alpha]) = \int_{\alpha} \omega \quad \text{for all } [\alpha] \in \pi_1(W).$$

Let $\pi : G \rightarrow W = G/K$ be the projection. Then $\pi^*\omega \in \Omega^1(G)$ is a closed form. Since $H^1_{DR}(G) = 0$ (G is simply connected), there exists $f : G \rightarrow \mathbb{R}$ such that $df = \pi^*\omega$. It is obvious that we can suppose $f(e) = 0$. Then the function we are looking for is $m = e^f$. In fact we have:

- (a) $f(xy) = f(x) + f(y)$ for all $x \in G, y \in K$:

Let us consider the composition $f \circ R_y : G \rightarrow \mathbb{R}$, where R_y denotes the right traslation by the element $y \in G$. Then

$$\begin{aligned} (d(f \circ R_y))_x(v) &= (df)_{xy}(dR_y)_x(v) \\ &= (\pi^*\omega)_{xy}((dR_y)_x(v)) = (R_y^*\pi^*\omega)_x(v) \\ &= (\pi^*\omega)_x(v) = (df)_x(v). \end{aligned}$$

Then $d(f \circ R_y) = df$ for all $y \in K$ so, because G is connected, $f \circ R_y = f + c(y)$. Since $f(e) = 0$ we have $c(y) = f(y)$.

- (b) $f|_K = \log \bar{m}_K$:

Let β be a path in G joining the identity e with a point $y \in K$. If we project through π we have a loop $\alpha = \pi\beta$ in W . Then,

$$\int_{\pi\beta} \omega = \log \bar{m}_K([\pi\beta]).$$

Now, by considering the isomorphism between $\pi_1(W)$ and K/K_e which sends each loop into the final point of any lifted path, it is easy to check that $\bar{m}_K([\pi\beta]) = m_K(y)$.

On the other side,

$$\begin{aligned} \int_{\pi\beta} \omega &= \int_{[0,1]} (\pi\beta)^* \omega = \int_{[0,1]} \beta^* \pi^* \omega = \int_{[0,1]} \beta^*(df) = \int_{[0,1]} d(\beta^* f) \\ &= \int_{[0,1]} d(f \circ \beta) = (f\beta)(1) - (f\beta)(0) = f(y). \end{aligned}$$

This proves that $\log \bar{m}_K(y) = f(y)$ for all $y \in K$. \square

Remark 1. When K is not unimodular, it may happen that the connected component K_e was unimodular, even if G/K is a compact manifold. Take for instance the linear action of $G = \mathrm{SL}(2, \mathbb{R})$ on the cylinder $C = \mathbb{R}^2 - \{0\}$. Fix a real number $\lambda > 0$. The quotient of C by the equivalence relation $v \equiv \lambda v$ is a torus. The isotropy of the induced action of G on T^2 is the Lie subgroup $K = K_\lambda$ of matrices

$$\begin{pmatrix} \lambda^n & t \\ 0 & \lambda^{-n} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad n \in \mathbb{Z}, t \in \mathbb{R}.$$

Then $K_e = \mathbb{R}$ and the modular function of K is $m_K(n, t) = \lambda^{2n}$.

Remark 2. The proof of Theorem 2.1 in [3] implicitly assumes that the manifold W is orientable, but this may not be true. We sketch how to correct their argument.

The universal covering $\tilde{W} = G/K_e$ is simply connected, hence orientable. Let K_2 be the group of elements $y \in K$ such that $R_y : \tilde{W} \rightarrow \tilde{W}$ preserves a fixed orientation. Then W is orientable if and only if $K = K_2$. If this is not the case, K_2 is a subgroup of index 2 of K such that $\Gamma_2 = \Gamma \cap K_2$ is dense in K_2 . Take the compact connected double covering $M_2 \rightarrow M$ corresponding to $h^{-1}(\Gamma_2) \subset \pi_1(M)$. Then M_2 is an orientable manifold, and the lifted foliation \mathcal{F}_2 is a G -Lie foliation with holonomy Γ_2 and basic manifold $W_2 = G/K_2$.

Now we must verify that \mathcal{F} is unimodular if and only if \mathcal{F}_2 is unimodular. The basic forms for \mathcal{F} are the basic forms for \mathcal{F}_2 which are \mathbb{Z}_2 -invariant. Moreover to any basic form α in \mathcal{F}_2 we can associate the \mathbb{Z}_2 -invariant basic form $\alpha^+ = (1/2)(\alpha + L_{-1}^* \alpha)$, thus proving that the morphism $H(M/\mathcal{F}) \rightarrow H(M_2/\mathcal{F}_2)$ between the basic cohomologies is injective. Hence \mathcal{F} unimodular implies \mathcal{F}_2 unimodular. Conversely, suppose that \mathcal{F}_2 is unimodular. Then the morphism $H_G(G) \rightarrow H(M_2/\mathcal{F}_2)$ is injective, as shown in the same paper [3] (and previously by M. Llabrés and A. Reventós in [5], who also suppose the basic manifold to be orientable). Then $H_G(G) \rightarrow H(M/\mathcal{F})$ is injective too. Moreover the Lie group G is unimodular, hence $H_G^n(G) \neq 0$. This proves that \mathcal{F} is unimodular.

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