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MORSE THEORY AND LUSTERNIK-SCHNIRELMANN CATEGORY OF QUATERNIONIC GRASSMANNIANS

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Abstract The Lusternik-Schnirelmann category of the quaternionic Grassmannian $G_{n,k} = \mathrm{Sp}(n)/(\mathrm{Sp}(k)\times \mathrm{Sp}(n-k))$ is known to be $k(n-k)$. In this note we show that this result can be deduced from Morse theory.

1. Introduction

The Lusternik-Schnirelmann category, $\operatorname{cat} X$, of a topological space X is defined as the least integer $m \geq 0$ such that X admits a covering by $m+1$ open sets which are contractible in X. This homotopy invariant is often difficult to compute, specially in the context of spaces of quaternionic matrices. For instance, in the case of the symplectic groups, $\mathrm{Sp}(n)$, we only know some low values, as $\operatorname{cat}\mathrm{Sp}(2) = 3$ [18], $\operatorname{cat}\mathrm{Sp}(3) = 5$ [4], and some bounds as $\operatorname{cat}\mathrm{Sp}(n) \geq n+2$ when $n \geq 3$ [8] or $\operatorname{cat}\mathrm{Sp}(n) \leq \binom{n+1}{2}$ [10]. In the case of quaternionic Stiefel manifolds, there is also a partial result, $\operatorname{cat} X_{n,k} = k$ when $n \geq 2k$ [14].

Still, we know the LS-category of the quaternionic Grassmann manifolds $G_{n,k} = \mathrm{Sp}(n)/(\mathrm{Sp}(k) \times \mathrm{Sp}(n-k))$. First, recall the existence of lower and upper bounds for the LS-category. By definition, the *cup length* of a space X, $\operatorname{cup} X$, is the largest integer ℓ such that there exists a product $x_1 \cdots x_\ell \neq 0$, with $x_i \in \tilde{H}^*(X;A)$, for any coefficient ring A. Then, if X is an $(n-1)$-connected CW-complex, we have (see [2])

$$\operatorname{cup} X \leq \operatorname{cat} X \leq (\dim X)/n.$$

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The cohomology ring with integer coefficients of $G_{n,k}$ is [13, Theorem 6.9.3]

$$H^*(G_{n,k};\mathbb{Z}) \cong (\mathbb{Z}[q_1,\dots,q_k]\otimes\mathbb{Z}[q_1,\dots,q_{n-k}])/\big(\sum_{i+j=l} q_i\otimes q_j; l\geq 1\big),$$

where q_i is the Pontrjagin class of degree $4i$. Then, its cup length is greater than, or equal to, $k(n-k)$, due to the non-nullity of the class $(q_1\cdots q_k)^{n-k}$. On the other hand, the dimension of $G_{n,k}$ is $4k(n-k)$, because $\dim \mathrm{Sp}(n) = 2n^2+n$. Also, $G_{n,k}$ is 3-connected [15, Paragraph 10.8], so finally we have

$$\operatorname{cat} G_{n,k} = \operatorname{cup} G_{n,k} = k(n-k).$$

One of the main features of the LS-category is its link with critical points of smooth functions. More precisely, if $\operatorname{Crit} X$ denotes the minimum number of critical points for (almost) any smooth function on a paracompact manifold X then we may write (see [2, Theorem 1.15] for more details)

$$\operatorname{cat} X + 1 \leq \operatorname{Crit} X.$$

In the case of a finite number of critical points, Y. Rudyak and F. Schlenk have given a better upper bound.

Theorem 1.1 ([17]). *Let X be a compact connected manifold and $f\colon X\to\mathbb{R}$ any smooth function with a finite number of critical points. Then $\operatorname{cat} X+1$ is a lower bound for the number of critical values of f.*

In Section 2, we prove that for the quaternionic Grassmannians the bound above is reached by a Morse function, induced from a height function on the symplectic group $\mathrm{Sp}(n)$, through the embedding $G_{n,k}\subset \mathrm{Sp}(n)$ which sends the k-dimensional vector subspace $V\subset\mathbb{H}^n$ into the orthogonal symmetry associated to $V^\perp$.

Theorem 1.2. *The minimum number of critical values of Morse height functions on $G_{n,k}$ is $\operatorname{cat} G_{n,k}+1$.*

With the same method, two of the authors gave in [10] an upper bound for the LS-category of the symplectic group $\mathrm{Sp}(n)$.

Notice that the critical set of Morse height functions on the complex Grassmannian have been determined in [3], where it is indicated that an analogous result holds for the quaternionic case (see the remarks after Theorem 3). Since an explicit proof is not given in [3], we provide one in Proposition 2.2.

The remaining contents of this paper are as follows. In Section 3, we take again the height functions on $\mathrm{Sp}(n)$, but replacing the previous embedding by the Cartan model of the symmetric space $G_{n,k}$. In this case, we prove that the previous lower bound cannot be reached, except in the particular case of projective quaternionic spaces. Finally, in Section 4, we use a generalized Cayley transform to provide an explicit covering of $G_{n,k}$, made of contractible open sets associated to the critical points of a Morse height function.

2. Height functions

(a) Height functions on $\mathrm{Sp}(n)$

The symplectic group $\mathrm{Sp}(n) = \{A \in \mathbb{H}^{n\times n} \mid AA^* = I\}$ is embedded in the space $\mathbb{H}^{n\times n}$ of quaternionic matrices of order n. (Here A^* means the conjugate transpose of A.)

A complete description of the different height functions on $\mathrm{Sp}(n)$ appears in [7]. Any of them can be reduced to the form $h_X(A) = \Re\,\mathrm{Tr}(XA)$, where $\Re\,\mathrm{Tr}$ denotes the real part of the trace, for some matrix $X \in \mathbb{H}^{n\times n}$. For instance, the function $h_I(A) = \Re\,\mathrm{Tr}(A)$ was studied by Frankel in [6]. It is a Morse-Bott function, whose gradient is given by

$$(\mathrm{grad}\, h_I)_A = I - A^2.$$

Then the critical set is

$$\Sigma(h_I) = \{A \in \mathrm{Sp}(n) \mid A = A^*\}.$$

By considering the conjugation action of $\mathrm{Sp}(n)$ onto itself, this set is the disjoint union of the orbits

$$\mathcal{F}_k = \{BJ_kB^* \mid B \in \mathrm{Sp}(n)\}$$

of the matrices

$$J_k = \mathrm{diag}(-I_k, I_{n-k}), \quad 0 \le k \le n.$$

Since the action of $\mathrm{Sp}(n)$ on $\mathcal{F}_k$ is transitive, with isotropy subgroup $\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)$, it follows that $\mathcal{F}_k$ is diffeomorphic to the quaternionic Grassmann manifold $G_{n,k}$.

Another height function, $h_D \colon \mathrm{Sp}(n) \to \mathbb{R}$, is defined by $h_D(A) = \Re\,\mathrm{Tr}(DA)$, where D is the real diagonal matrix $D = \mathrm{diag}(1, 2, \dots, n)$. This is a Morse function. Its gradient is given by

$$(\mathrm{grad}\, h_D)_A = \frac{1}{2}(D - ADA),$$

and its critical set $\Sigma(h_D)$ is formed of the matrices $A = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$.

In the next paragraph, we restrict this Morse function h_D to each of the critical orbits of Frankel's function h_I and count its critical levels. This will give us a proof of Theorem 1.2.

(b) Height functions on the Grassmannian $G_{n,k}$

From now on, we fix some integer $k \le n$ and consider the orbit $\mathcal{F} = \mathcal{F}_k \subset \mathrm{Sp}(n)$ of $J = J_k$, which is diffeomorphic to the Grassmannian $G_{n,k}$ of k-planes in $\mathbb{H}^n$. In other words, we are considering the embedding $G_{n,k} \subset \mathrm{Sp}(n)$ which sends the k-dimensional subspace V onto the linear map which is equal to $-\mathrm{id}$ on V and to $+\mathrm{id}$ on $V^\perp$.

Lemma 2.1. *If A is a point of the orbit $\mathcal{F}$, then the tangent space is*

$$T_A\mathcal{F} = \{Y \in \mathbb{H}^{n\times n} \mid AY + YA = 0,\ Y = Y^*\}.$$

Proof. Consider first the point $A = J$. The Lie algebra of $G = \mathrm{Sp}(n)$ is

$$\mathfrak{g} = \{X \in \mathbb{H}^{n\times n} \mid X + X^* = 0\},$$

and $Y \in T_J G$ if and only if $JY \in \mathfrak{g} = T_I G$. If we consider the fundamental vectors fields of the action of G onto $\mathcal{F}$, we have the curve $\exp(tX)J\exp(-tX)$ in $\mathcal{F}$, whose tangent vector at $t = 0$ is $Y = XJ - JX$. It verifies $Y = Y^*$ and $YJ + JY = 0$. Due to dimensional reasons, these two conditions exhaust all the vectors in $T_J\mathcal{F}$. Finally, if $A = BJB^*$ then $T_A\mathcal{F} = BT_JB^*$ and the result follows. □

Let $h_D^{\mathcal{F}} \colon \mathcal{F} \to \mathbb{R}$ be the restriction of the height function h_D to $\mathcal{F}$. The relationship between the critical sets of h_D and $h_D^{\mathcal{F}}$ is as follows. (This is not always true for an arbitrary height function, see [11].)

Proposition 2.2 ([3])**.** $\Sigma(h_D^{\mathcal{F}}) = \Sigma(h_D) \cap \mathcal{F}$.

Proof. The critical points of $h_D^{\mathcal{F}}$ are the points $A \in \mathcal{F}$ where the gradient of h_D is orthogonal to $\mathcal{F}$. Thus, we have $\Sigma(h_D^{\mathcal{F}}) \subset \Sigma(h_D) \cap \mathcal{F}$.

Conversely, let $A \in \mathcal{F}$. The gradient $(\operatorname{grad} h_D^{\mathcal{F}})_A$ is the projection of $(\operatorname{grad} h_D)_A$ onto $T_A\mathcal{F}$. But $Y = (\operatorname{grad} h_D)_A = (1/2)(D - ADA)$ verifies $Y = Y^*$ and $AY + YA = 0$, because $A = A^*$. Then $(\operatorname{grad} h_D^{\mathcal{F}})_A$ actually equals $(\operatorname{grad} h_D)_A$. We have proved $\Sigma(h_D) \cap \mathcal{F} \subset \Sigma(h_D^{\mathcal{F}})$. □

In [19] it is proven that a Morse function in $G_{n,k}$ is a perfect Morse function, without computing the critical set.

Theorem 1.2 is a direct consequence of the following result.

Proposition 2.3. *The Morse function,* $h_D^{\mathcal{F}} \colon \mathcal{F}_k \cong G_{n,k} \to \mathbb{R}$, *has exactly* $k(n-k)+1$ *critical levels.*

Proof. Recall that the critical points of h_D are the matrices $A = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$. Moreover, if $A \in \mathcal{F}$, then A has exactly k negative entries and $n-k$ positive entries, so there are $\binom{n}{k}$ critical points of h_D in $\mathcal{F}$. For a given $A \in \Sigma(h_D^{\mathcal{F}})$, we have

$$h_D^{\mathcal{F}}(A) = \Re \operatorname{Tr}(DA) = 1\varepsilon_1 + 2\varepsilon_2 + \cdots + n\varepsilon_n.$$

Taking into account the number of positive and negative diagonal entries, the maximum value of $h_D^{\mathcal{F}}$ at a critical point is

$$M = (-1 - \cdots - k) + (k + 1 + \cdots + n),$$

and the minimum value is

$$m = 1 + \cdots + (n-k) - (n-k+1) - \cdots - n.$$

Let c be any critical value, associated to a matrix A with the positive diagonal entries in ranges $(i_1, \ldots, i_{n-k})$ and the negative ones in ranges $(j_1, \ldots, j_k)$. The difference $c - m$ can be computed as,

$$c - m = (i_1 + \cdots + i_{n-k}) - (j_1 + \cdots + j_k)$$

$$
\begin{aligned}
&-(1+\cdots+(n-k))+((n-k+1)+\cdots+n)\\
&=(i_1+\cdots+i_{n-k})-(j_1+\cdots+j_k)-2(1+\cdots+n-k)+(1+\cdots+n)\\
&=2(i_1+\cdots+i_{n-k})-2(1+\cdots+n-k). \qquad (2.1)
\end{aligned}
$$

This is an even number. Moreover, if A is a critical point with critical value $c=\Re\operatorname{Tr}(DA)$ different from the maximum M, there exists some diagonal entry $\varepsilon_i=1$ in A followed by an entry $\varepsilon_{i+1}=-1$. Thus, if we flip these two entries, we obtain a new critical point, B, such that

$$\Re\operatorname{Tr}(DB)-c=(-i+i+1)-(i-(i+1))=2. \qquad (2.2)$$

The combination of (2.1) and (2.2) shows that the set of critical levels of the function $h_D^{\mathcal{F}}$ is exactly $\{m,m+2,\dots,M\}$, whose cardinality equals

$$1+(M-m)/2=1+(2kn-2k^2)/2=k(n-k)+1.$$

□

From Theorem 1.1, we recover an upper bound of the LS-category of the Grassmannian,

$$\operatorname{cat} G_{n,k}\le k(n-k),$$

which is sharp because it coincides with its cup-length.

3. Cartan model

As before, we fix some $k\le n$ and consider the matrix $J=J_k=\operatorname{diag}(-I_k,I_{n-k})$. In this section, we modify the embedding of the Grassmann manifold $G_{n,k}$ into $\operatorname{Sp}(n)$ by using its structure of symmetric space. The subgroup $\operatorname{Sp}(k)\times\operatorname{Sp}(n-k)$ is the fixed point set of the automorphism $\sigma\colon\operatorname{Sp}(n)\to\operatorname{Sp}(n)$ defined by $\sigma(B)=JBJ$. By definition, the *Cartan embedding*

$$\gamma\colon G_{n,k}\cong\operatorname{Sp}(n)/(\operatorname{Sp}(k)\times\operatorname{Sp}(n-k))\to\operatorname{Sp}(n)$$

is given by $\gamma([B])=B\sigma(B)^*=BJB^*J$ and the *Cartan model* M is the image of γ. This image is [5, Theorem 15.1] the connected component of the identity of

$$N=\{B\in\operatorname{Sp}(n)\mid\sigma(B)=B^*\}.$$

In our case MJ equals the orbit $\mathcal{F}\subset\operatorname{Sp}(n)$ of J by the conjugation action.

Height functions on symmetric spaces have been studied in [11]. As a Morse function on M, we choose the restriction to M of the height function $h_D\colon\operatorname{Sp}(n)\to\mathbb{R}$. We denote it by $h_D^M\colon M\to\mathbb{R}$.

Proposition 3.1. *The critical set of the Morse function, $h_D^M\colon M\cong G_{n,k}\to\mathbb{R}$, is formed by the matrices*

$$A=\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_k,\varepsilon_{k+1},\dots,\varepsilon_n),\quad \varepsilon_i=\pm1,$$

such that the numbers of -1's, in the blocks $\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_k)$ and $\operatorname{diag}(\varepsilon_{k+1},\dots,\varepsilon_n)$, are the same.

Proof. First, as $\sigma(D) = D = D^*$, the result [11, Corollary 2.3.4] implies that the critical set of $\Sigma(h_D^M)$ equals the intersection $\Sigma(h_D) \cap M$.

Let $A = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n) \in M$ be a critical point of h_D. We write it as $A = \mathrm{diag}(E_k, E_{n-k})$ with two boxes, $E_k = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_k)$ and $E_{n-k} = \mathrm{diag}(\varepsilon_{k+1}, \dots, \varepsilon_n)$, of size k and $n-k$ respectively. Let p be the number of occurrences of -1 in the box E_k and q be the number of occurrences of -1 in E_{n-k}. Then, the product

$$AJ = \mathrm{diag}(-\varepsilon_1, \dots, -\varepsilon_k, \varepsilon_{k+1}, \dots, \varepsilon_n)$$

has $(k-p)+q$ entries equal to -1. Now, according to the definition of M, there exists $B \in \mathrm{Sp}(n)$ such that $AJ = BJB^*$ and the sets of eigenvalues of AJ and J coincide. This implies $k - p + q = k$, hence $p = q$. □

In general, these functions supply too many critical points and levels. For instance, with this method, we find 8 critical values in $G_{5,3}$, while $\mathrm{cat}\, G_{5,3} = 6$. An exception is the projective space $\mathbb{H}P^{n-1} = G_{n,1}$ that we detail now.

Example 3.2. When $k = 1$, the projective space $G_{n,1} = \mathbb{H}P^{n-1}$ is diffeomorphic to the orbit $\mathcal{F} = \mathcal{F}_1$ of the diagonal matrix $J = \mathrm{diag}(-1, I_{n-1}) \in \mathrm{Sp}(n)$. We apply the two methods in this situation.

- *First, we consider the height function $h_D^{\mathcal{F}}$ of Section 2.* Its critical set is determined by $\Sigma(h_D^{\mathcal{F}}) = \{A_1, \dots, A_n\}$, with

  $$A_i = \mathrm{diag}(1, \dots, 1, -1^{(i)}, 1, \dots, 1), \text{ for } 1 \le i \le n.$$

  The corresponding set of critical values is

  $$\{n(n+1)/2 - 2i \mid 1 \le i \le n\}.$$

- *Secondly, let h_D^M be the Morse function induced on the Cartan model.* Its critical set is given by $\Sigma(h_D^M) = \{E_1, E_2, \dots, E_n\}$ with $E_1 = I$ and

  $$E_i = \mathrm{diag}(-1, 1, \dots, 1, -1^{(i)}, 1, \dots, 1), \text{ for } 2 \le i \le n.$$

  The corresponding set of critical values is

  $$\{n(n+1)/2 - 2j, \mid 0 \le j \le n-1\}.$$

For projective spaces, the sets of critical values obtained by the two methods have the same cardinal, n, and $\mathrm{cat}\, \mathbb{H}P^{n-1} = n - 1$.

4. Explicit coverings

We fix some $k \le n$ and consider the orbit $\mathcal{F} = \mathcal{F}_k \subset \mathrm{Sp}(n)$ of $J = J_k$ as above. Recall from [11] that one can build an explicit covering of the Cartan model M, by contractible open sets obtained from a generalized Cayley transform [7]. Since $MJ = \mathcal{F}$, this covering can be transformed in order to obtain a covering of the orbit $\mathcal{F}$. We detail now this construction.

Theorem 4.1. *Set* $\Omega(J) = \{A \in \mathbb{H}^{n\times n} \mid A + J \text{ is invertible}\}$. *The map* $c_J \colon \Omega(J) \to \Omega(J)$ *given by*

$$c_J(A) = (I - JA)(A + J)^{-1}$$

verifies $c_J \circ c_J = \mathrm{id}$ *and induces a diffeomorphism,* $c_J \colon T_J\mathcal{F} \xrightarrow{\cong} \Omega(J) \cap \mathcal{F}$.

Proof. The equality $c_J \circ c_J = \mathrm{id}$ comes from an easy computation. It is also known that $c_J \colon \Omega(J) \cap \mathrm{Sp}(n) \to T_J\mathrm{Sp}(n)$ is a diffeomorphism, see [7, Theorem 3].

Let $Y \in T_J\mathcal{F}$, that is, $Y = Y^*$ and $YJ + JY = 0$. Then we have $c_J(Y)^* = (Y + J)^{-1}(I - YJ)$. As $(Y + J)(I - JY) = (I - YJ)(Y + J)$, we get $c_J(Y) = c_J(Y)^*$ and $c_J(Y)$ is a Hermitian quaternionic matrix. Therefore, $c_J(Y)$ belongs to the disjoint union $\mathcal{F}_0 \sqcup \cdots \sqcup \mathcal{F}_n$ of orbits. Since $c_J(0) = J$, we deduce $c_J(T_J\mathcal{F}) \subset \mathcal{F}$ by connectedness.

Conversely, if $A \in \Omega(J) \cap \mathcal{F}$ we know that $Y = c_J(A) \in T_J\mathrm{Sp}(n)$ because $A \in \Omega(J) \cap \mathrm{Sp}(n)$ and $c_J^{-1} = c_J$. Thus the matrix $JY \in T_I\mathrm{Sp}(n)$ is skew-Hermitian. Moreover, as $A \in \mathcal{F}$ we have also $A = A^*$, hence $Y = Y^*$. This shows, with Lemma 2.1, that $Y \in T_J\mathcal{F}$. □

By translating the contractible open set $\Omega_{\mathcal{F}}(J) := \Omega(J) \cap \mathcal{F}$ we obtain a finite covering of the orbit $\mathcal{F} = \mathcal{F}_k \cong G_{n,k}$, associated to the critical set of $h_D^{\mathcal{F}}$, as follows.

Let $P = P_\sigma \in \mathbb{R}^{n\times n}$ be any real matrix obtained from the identity matrix by a permutation $\sigma \in \mathcal{S}_n$ of rows. Any critical point of $h_D^{\mathcal{F}}$ is of the form $A_P = PJP^*$, by Proposition 2.2. We define

$$\Omega_{\mathcal{F}}(A_P) := P\Omega_{\mathcal{F}}(J)P^* = \{A \in \mathcal{F} \mid A + A_P \text{ is invertible}\}.$$

Proposition 4.2. *The family* $(\Omega_{\mathcal{F}}(A_{P_\sigma}))_{\sigma\in\mathcal{S}_n}$ *is a finite covering of the orbit* $\mathcal{F} \cong G_{n,k}$ *by contractible open sets.*

Before giving the proof, we establish a technical point.

Lemma 4.3. *Let* $A = BJB^*$ *be a point of the orbit* $\mathcal{F}$*, with* $B \in \mathrm{Sp}(n)$*, and let* $P = P_\sigma$ *be a permutation matrix as above. If we write* $PB = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$*, we have that* $A \in \Omega_{\mathcal{F}}(A_P)$ *if, and only if, the matrix* α *is invertible.*

Proof. By definition, the matrix $A \in \Omega_{\mathcal{F}}(A_P)$ if, and only if, the matrix $BJB^* + PJP$ is invertible, that is, if, and only if, the matrix

$$PBJ + JPB = \begin{pmatrix} -2\alpha & 0 \\ 0 & -2\delta \end{pmatrix}$$

is invertible. Therefore, we have only to prove that α invertible implies δ invertible. Since $PB \in \mathrm{Sp}(n)$ we have the following relations:

$$\alpha\alpha^* + \gamma\gamma^* = I, \tag{4.1}$$

$$\delta\delta^* + \beta\beta^* = I, \tag{4.2}$$

$$\alpha\beta^* + \gamma\delta^* = 0. \tag{4.3}$$

From (4.3), we obtain $\beta^* = -\alpha^{-1}\gamma\delta^*$ and $\beta\beta^* = (\delta\gamma^*\alpha^{*-1})(\alpha^{-1}\gamma\delta^*)$. Using (4.1), we deduce

$$\beta\beta^* = \delta\gamma^*(\alpha\alpha^*)^{-1}\gamma\delta^* = \delta\gamma^*(I - \gamma\gamma^*)^{-1}\gamma\delta^*.$$

Finally, the equality (4.2) gives

$$\delta\delta^* = I - \beta\beta^* = I - \delta\gamma^*(I - \gamma\gamma^*)^{-1}\gamma\delta^*,$$

from which we deduce

$$\delta\left(I + \gamma^*(I - \gamma\gamma^*)^{-1}\gamma\right)\delta^* = I. \tag{4.4}$$

Recall from [16, Theorem 5.9.2] or [1, 12], the existence of a "determinant" (with real values) for quaternionic square matrices, whose nullity characterizes the non-invertible matrices and such that $\det(UV) = \det(U)\det(V)$. The equality (4.4) implies $\det\delta \neq 0$ and therefore, the matrix δ is invertible. □

Proof of Proposition 4.2. According to Lemma 4.3, if $A = BJB^*$ we have only to exhibit a permutation matrix $P = P_\sigma$ such that the box α in PB is invertible. As the first k columns of B are orthogonal, the submatrix formed by the first k columns of B has rank k. For quaternionic matrix, the rank given by the columns (as right $\mathbb{H}$-vector space) equals the rank given by the rows (as left $\mathbb{H}$-vector space), see [9]. Therefore, there are some k rows which are independent. By permuting the rows of B, that is, by multiplying B on the left by some permutation matrix P, we can suppose that the first k rows and k columns of some PB are independent. Thus the submatrix α is invertible as required. □

Notice that the number of open sets in the covering of $G_{n,k}$ equals that of critical points (the permutations of the matrix J), that is, $\binom{n}{k}$. As we wrote above, this number is in general strictly greater than the category of $G_{n,k}$, except if $k = 1$ where we recover the situation of Example 3.2.

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