

Trace map, Cayley transform and LS category of Lie groups

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Abstract The aim of this paper is to use the so-called Cayley transform in order to compute the LS category of Lie groups and homogeneous spaces by giving explicit categorical open coverings. When applied to $U(n)$, $U(2n)/Sp(n)$ and $U(n)/O(n)$ this method is simpler than those formerly known. We also show that the Cayley transform is related to height functions in Lie groups, allowing to give a local linear model of the set of critical points. As an application we give an explicit covering of $Sp(2)$ by categorical open sets. The obstacles to generalize these results to $Sp(n)$ are discussed.

Keywords LS category · Cayley transform · unitary group · symplectic group · Bott-Morse function · quaternionic left eigenvalue

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1 Introduction

Lusternik-Schnirelmann category is a homotopical invariant that has been widely studied [4, 11]. For a topological space X , the LS category $\text{cat}X$ is defined as the minimum number (minus one) of categorical open sets which are needed to cover X —an open set is categorical when it is contractible in X .

Unfortunately the LS category is very difficult to compute. For instance, while the result $\text{cat}Sp(2) = 3$ was proven by Schweitzer in 1965 [18], it was not until 2002 that $\text{cat}Sp(3) = 5$ appeared in [7], see also [10]. In general, the algebraic techniques involved are highly elaborated. It is then of interest to introduce more elementary methods.

The main idea of this paper is to compute the LS category of some Lie groups and homogeneous spaces of the orthogonal type by means of the so-called Cayley transformation. This will give proofs of $\text{cat}U(n) = n$, $\text{cat}U(n)/Sp(n) = n$ and $\text{cat}U(n)/O(n) = n$ which are simpler than the original ones by Singhof [19] and Mimura and Sugata [14].

Our method is closely related to Morse theory on Lie groups. Classically, the functions that use to be considered are “height” or “distance” as in [5, 21, 22]. On a matrix Lie group these functions are, up to a constant, of the form $h_X(A) = \Re \text{Tr}(XA)$, the real part of the trace, for some matrix X , a fact which allows to explicitly describe the Bott-Morse structure of these functions. We shall prove that the Cayley transform serves to linearize the gradient flow of h_X and to give local charts for the set of critical points. These results generalize those of Volchenko and Kozachko [22], see also [21].

Let us remember that in a compact manifold the LS category (plus one) is a lower bound for the number of critical points of any smooth function—be Morse or not. The reason is that—roughly speaking—for each critical point the gradient flow defines a categorical open set. What is nice in our setting is that this flow is given by the contraction associated to the Cayley map. Moreover, the Morse interpretation above allows us to give an explicit covering of $Sp(2)$ by four categorical open sets, a result that completes the abstract proof in [18].

At the end of the paper we explain how the generalization of our results to the symplectic group $Sp(n)$ depends on the computation of the so-called left eigenvalues of a quaternionic matrix [25], a topic about which very little is known, out of the case $n = 2$.

We hope that the ideas presented here will deserve further attention.

2 The Cayley transform

The classical Cayley transform was introduced in [3] as a way to express an orthogonal transformation by means of skew-symmetric coordinates. It is given by

$$c(X) = \frac{I - X}{I + X}.$$

This map is defined for all matrices having their eigenvalues different from -1 . It equals its own inverse, $c^2 = \text{id}$ and can be thought as a generalization of the stereographic projection. Its basic properties appear in [17], see also [23].

In order to obtain a categorical covering of the orthogonal Lie groups we shall introduce in the next paragraphs a convenient generalization of the classical Cayley map.

2.1 Preliminaries

Let the algebra \mathbb{K} be either \mathbb{R} (reals), \mathbb{C} (complex) or \mathbb{H} (quaternions). We say that the matrix $A \in \mathcal{M}(n, \mathbb{K})$ is *orthogonal* if $AA^* = \text{id}$, where $A^* = \bar{A}^t$ is the conjugate transpose. Such a matrix can be identified with a (right) \mathbb{K} -linear map $\mathbb{K}^n \rightarrow \mathbb{K}^n$ preserving the product $\langle v, w \rangle = v^*w$. Let us denote by $O(n, \mathbb{K})$ the Lie group of orthogonal matrices. Depending on \mathbb{K} this group corresponds to the orthogonal group $O(n)$, the unitary group $U(n)$ or the symplectic group $Sp(n)$.

Remark 1 The Cayley transform maps a classic orthogonal Lie group like $U(n)$ or $Sp(n)$ into its Lie algebra \mathfrak{g} of skew-hermitian matrices. In fact, suppose that A is a unitary or symplectic matrix. It can be diagonalized, $A = UDU^*$, to a *complex* diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ [2]. Then

$$c(A) = U \text{diag}(\pi(\lambda_1), \dots, \pi(\lambda_n))U^*,$$

where π is the stereographic projection $\pi: S^1 \setminus \{-1\} \rightarrow i\mathbb{R}$.

2.2 Generalized Cayley transform

Let $A \in O(n, \mathbb{K})$ be an orthogonal matrix, where \mathbb{K} is \mathbb{R} , \mathbb{C} or \mathbb{H} .

Definition 1 Let us denote by $\Omega(A) \subset \mathcal{M}(n, \mathbb{K})$ the open set of matrices X such that $A + X$ is invertible. The *Cayley transform centered at A* is the map

$$c_A: \Omega(A) \rightarrow \Omega(A^*)$$

given by

$$c_A(X) = (I - A^*X)(A + X)^{-1}.$$

The classical Cayley map corresponds to $A = I$. As we shall see in the next Proposition, the application c_A is well defined and it is invertible, with $c_A^{-1} = c_{A^*}$.

Proposition 1 *If $X \in \Omega(A)$ then*

1. $c_A(X) = (A + X)^{-1}(I - XA^*)$;
2. *the inverse matrix of $A^* + c_A(X)$ is $(1/2)(A + X)$;*
3. *if $X \in \Omega(A)$ then $c_A(X) \in \Omega(A^*)$;*
4. c_A *is a diffeomorphism, with $c_A^{-1} = c_{A^*}$.*

Proof (1) It suffices to verify that $(A + X)(I - A^*X) = (I - XA^*)(A + X)$, which is immediate because $AA^* = \text{id}$. For (2) we compute

$$(A^* + (I - A^*X)(A + X)^{-1})(1/2)(A + X) = (1/2)(A^*A + A^*X + I - A^*X) = I.$$

Part (3) comes immediately from (2). Finally, by using (2) we obtain that

$$\begin{aligned} (c_{A^*} \circ c_A)(X) &= \\ (I - Ac_A(X))(1/2)(A + X) &= \\ (I - A(I - A^*X)(A + X)^{-1})(1/2)(A + X) &= \\ (1/2)((A + X) - A + AA^*X) &= X. \quad \square \end{aligned}$$

We shall need the following interesting properties, which are easy to prove:

Proposition 2 *Let $X \in \Omega(A)$. Then*

1. $X^* \in \Omega(A^*)$ *and $c_{A^*}(X^*) = c_A(X)^*$;*
2. $UXU^* \in \Omega(UAU^*)$ *for any matrix $U \in O(n, \mathbb{K})$ and*

$$c_{UAU^*}(UXU^*) = Uc_A(X)U^*;$$

3. *if the matrix X is invertible then $X^{-1} \in \Omega(A^*)$ because*

$$(A^* + X^{-1})^{-1} = A(A + X)^{-1}X.$$

Moreover

$$c_{A^*}(X^{-1}) = -Ac_A(X)A.$$

2.3 Categorical open sets

The results in this paragraph show that the domain of the Cayley transform in an orthogonal group is contractible.

Recall that the Lie algebra of $G = O(n, \mathbb{K})$ is formed by the skew-symmetric (resp. skew-hermitian) matrices,

$$\mathfrak{g} = \{X \in \mathcal{M}(n, \mathbb{K}) : X + X^* = 0\}.$$

As a vector space $\mathfrak{g} = T_I G$, so the tangent space at any other point $A \in G$ is

$$T_A G = L_A(T_I G) = \{Y \in \mathcal{M}(n, \mathbb{K}) : A^*Y + Y^*A = 0\}.$$

Proposition 3 *Let $X \in \mathfrak{g}$ be a skew-symmetric (resp. skew-hermitian) matrix. Then X has not real eigenvalues different from zero.*

Proof Suppose that there exists $t \in \mathbb{R}$ such that $Xv = vt$ for some $v \in \mathbb{K}^n$, $v \neq 0$. Then $v^*Xv = v^*vt = |v|^2t$ is a real number and, consequently,

$$v^*Xv = (v^*Xv)^* = v^*X^*v = v^*(-X)v = -v^*Xv.$$

Therefore v^*Xv is null. *i.e.* $|v|^2t = 0$ hence $t = 0$. \square

Corollary 1 *The real vector space $T_{A^*}G$ is contained in $\Omega(A^*)$.*

Proof If $A^* + Y$ is not invertible then there exists some $v \neq 0$ such that $Yv = -A^*v$ so $AYv = -v$. This means that the skew-symmetric matrix AY has -1 as an eigenvalue, contradicting Proposition 3. \square

Let $G = O(n, \mathbb{K})$. We shall denote by $\Omega_G(A)$ the open subset $\Omega(A) \cap G \subset G$.

Theorem 1 *The generalized Cayley transform c_A maps diffeomorphically $\Omega_G(A)$ onto $T_{A^*}G$, with $c_A(A) = 0$. As a consequence, the open set $\Omega_G(A)$ is contractible.*

The proof is an immediate consequence of Propositions 1, 2, 3 and Corollary 1.

3 Bott-Morse functions on Lie groups

There is a deep relationship between the Cayley transform and Morse theory in Lie groups. In this Section we prove (Proposition 4) that one can integrate the gradient flow of any height function $h_X: G \rightarrow \mathbb{R}$ by applying the Cayley transform c_{A^*} to a simple curve in $T_{A^*}G$, provided that $c_{A^*}(0) = A$ is a critical point. As a consequence we give a local model for the set Σ of critical points of h_X (Theorem 3).

3.1 Critical points of a height function

Let $G = O(n, \mathbb{K})$ be an orthogonal group embedded in the euclidean space $E = \mathcal{M}(n, \mathbb{K})$. The euclidean metric is given by $\langle A, B \rangle = \Re \operatorname{Tr}(A^*B)$, the real part of the trace. Height functions on G with respect to some hyperplane were studied in [21] and [22], while distance functions to a given point are studied in [5]. Both classes of functions are given, up to a constant, by the formula $h_X(A) = \Re \operatorname{Tr}(XA)$ for some matrix $X \in E$, as it is easy to prove.

Example 1 When $X = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a positive real diagonal matrix, with $\lambda_1 < \dots < \lambda_n$, the function h_X is a perfect Morse function, whose critical points are the diagonal matrices

$$\operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon_k = \pm 1.$$

This result is proven in [5] and [21], see also [16].

Example 2 On the other hand, the case $X = I$ (in fact equivalent to $X = tI$, $t \in \mathbb{R} \setminus \{0\}$), was first studied by Frankel [8]. This time, the height map is a Bott-Morse function, invariant by the adjoint action. The set $\Sigma(n)$ of critical points is formed by the matrices A such that $A^2 = I$.

The two preceding examples are particular cases of the next Theorem 2, that we suppose is more or less folk. It gives a general description of the set of critical points of an arbitrary height function h_X .

First, a direct computation shows that the gradient of h_X on G is given by

$$(\text{grad } h_X)_A = (1/2)(X^* - AXA).$$

Moreover, if $A \in G$ is a critical point, then the Hessian operator $(Hh_X)_A: T_A G \rightarrow T_A G$ is given by

$$(Hh_X)_A(U) = -(1/2)(AXU + UXA), \quad U \in T_A G.$$

We keep the notation of Example 2.

Theorem 2 *Let $X \in E = \mathcal{M}(n, \mathbb{K})$ be an arbitrary matrix. Let n_0 be the dimension of its kernel and let $0 < t_1 < \dots < t_k$ be the non-null (real) eigenvalues of XX^* , with multiplicities n_1, \dots, n_k . Then the set of critical points of the height function h_X is*

$$\Sigma(h_X) \cong O(n_0, \mathbb{K}) \times \Sigma(n_1) \times \dots \times \Sigma(n_k) \text{ (diffeomorphism), } n_0 + n_1 + \dots + n_k = n.$$

Proof First, if $Y = UDU^*$, then $\Sigma(h_Y) = U\Sigma(h_D)U^*$. On the other hand, if $X = US$ is a polar decomposition of X (U orthogonal, S hermitian) then $\Sigma(h_X) = \Sigma(h_S)U^*$. Finally, if $C = \text{diag}(-I_p, +I_q)$ then $\Sigma(h_Y) = C\Sigma(h_{YC})$. These properties prove that we can restrict ourselves to the case where X is given by diagonal blocks

$$\begin{pmatrix} 0 & & & & \\ & t_1 I & & & \\ & & \ddots & & \\ & & & & t_k I \end{pmatrix}, \quad 0 < t_1 < \dots < t_k,$$

of size n_0, n_1, \dots, n_k . The rest of the proof is a direct computation. \square

The transformations in the proof of Theorem 2 preserve the non-degenerate critical points, so we have the following corollary which completely characterizes the height functions which are Morse functions.

Corollary 2 *The height function h_X is Morse if and only if the matrix XX^* is invertible and has n different eigenvalues.*

3.2 Gradient flow

Our next Proposition is a generalization of the same result for the classical Cayley transform c_I by Volchenko and Kozachko [22]. Following the terminology of these authors we shall call *linearization* the process of transforming the gradient flow of h_X in G to a flow in the Lie algebra.

Proposition 4 Let h_X be an arbitrary height function on $G = O(n, \mathbb{K})$ and let A be a critical point. The solution to the gradient equation

$$\alpha' = \frac{1}{2}(X^* - \alpha X \alpha)$$

passing through $\alpha(0) \in \Omega_G(A)$ is the image by the generalized Cayley transform c_{A^*} of the curve in $T_{A^*}G$ defined as

$$\beta(t) = \exp(-XAt/2) \cdot \beta_0 \cdot \exp(-AXt/2), \quad \beta_0 = c_A(\alpha(0)).$$

Proof First, from $Ae^{XA} = e^{AX}A$ it follows that $A\beta + (A\beta)^* = 0$, that is $\beta \in T_{A^*}G$. Notice that the matrices XA and AX are symmetric (hermitian) because $X^* = AXA$ and $A^*A = \text{id}$.

Now, from the definition of β , it is

$$\beta' = (-1/2)(XA\beta + \beta AX). \quad (1)$$

Let

$$\alpha = c_{A^*} \circ \beta = (I - A\beta)(A^* + \beta)^{-1},$$

hence

$$\alpha(A^* + \beta) = I - A\beta.$$

Differentiation gives

$$\alpha'(A^* + \beta) + \alpha\beta' = -A\beta'$$

that is

$$\alpha'(A^* + \beta) = -(A + \alpha)\beta'.$$

By Proposition 1, the inverse of $A^* + \beta = A^* + c_A(\alpha)$ is $(1/2)(A + \alpha)$, so

$$\begin{aligned} \alpha' &= \\ (-1/2)(A + \alpha)\beta'(A + \alpha) &= \\ (+1/4)(A + \alpha)(XA\beta + \beta AX)(A + \alpha). & \end{aligned} \quad (2)$$

Now

$$\beta = c_A(\alpha) = (I - A^*\alpha)(A + \alpha)^{-1} = (A + \alpha)^{-1}(I - \alpha A^*)$$

implies that

$$\beta(A + \alpha) = I - A^*\alpha$$

and

$$(A + \alpha)\beta = I - \alpha A^*$$

so from Equation (2) we have

$$\begin{aligned} 4\alpha' &= \\ (A + \alpha)XA(I - A^*\alpha) + (I - \alpha A^*)AX(A + \alpha) &= \\ 2AXA - 2\alpha X \alpha &= \\ 2(X^* - \alpha X \alpha). & \quad \square \end{aligned}$$

Remark 2 Indeed, when $X = A = I$, $\beta(t) = \exp(-t)\beta_0$ is the radial contraction to β_0 .

3.3 Local model of the critical point set

We now show how the Cayley transform serves to give a local chart for the set of critical points. This result is completely new.

Let $h_X(A) = \Re \operatorname{Tr}(XA)$ be an arbitrary height function on the Lie group $G = O(n, \mathbb{K})$. Let Σ be the set of critical points of h_X . If $A \in \Sigma$ is a critical point we denote by $S(A)$ the real vector space

$$S(A) = \{\beta_0 \in T_{A^*}G : XA\beta_0 + \beta_0AX = 0\}.$$

Theorem 3 *The Cayley map $c_{A^*}: S(A) \rightarrow \Sigma \cap \Omega_G(A)$ is a diffeomorphism.*

Proof We need to prove that the curve $\beta(t)$ in Proposition 4 is constant if and only if $\beta_0 \in S(A)$. This can be achieved by using Equation (1). Then, by Proposition 4, $c_{A^*}(\beta_0)$ is a critical point if and only if $\beta'(t) = 0$ for all t , which is equivalent to

$$XA\beta_0 + \beta_0AX = 0. \quad \square$$

Example 3 Suppose $X = I$ and $\mathbb{K} = \mathbb{C}$. Then the critical points of h_I are the matrices $A \in U(n)$ such that $A^2 = I$. Such a matrix $A = A^*$ can be diagonalized to $D = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_k = \pm 1$. On the other hand, $\beta_0 \in T_A G$ if and only if $A\beta_0$ is skew-symmetric, $A\beta_0 = -\beta_0^*A$, while $\beta_0 \in S(A)$ if and only if $A\beta_0 + \beta_0A = 0$. It follows that $\beta_0 = \beta_0^*$.

So, for instance, the identity I and its opposite $-I$ are critical points that are isolated because $S(\pm I) = 0$. On the other hand, let $A = \operatorname{diag}(I_p, -I_q)$. Then $\beta_0 \in T_A G$ must be of the form

$$\beta_0 = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix},$$

which implies $\dim S(A) = 2pq$. This is in fact the dimension of the (critical) orbit of A , which is diffeomorphic to the Grassmannian $U(p+q)/(U(p) \times U(q))$.

Example 4 Let $X = \operatorname{diag}(q_1, \dots, q_n)$ be a diagonal matrix, with $q_k \neq 0$. Assume that $|q_1| < \dots < |q_n|$. This time the gradient condition $A^*X^* = XA$ implies that a critical point has the form

$$A = \operatorname{diag}(\pm|q_1|/q_1, \dots, \pm|q_n|/q_n).$$

Since

$$XA = AX = \operatorname{diag}(\varepsilon_1|q_1|, \dots, \varepsilon_n|q_n|), \quad \varepsilon_k = \pm 1,$$

it follows that $S(A) = 0$. So all critical points are isolated.

4 Applications to LS category

4.1 The unitary group $U(n)$

Singhof [19] proved that the LS category of the special unitary group $SU(n)$ is $n - 1$, hence that of $U(n) \cong S^1 \times SU(n)$ (diffeomorphism) is n . Although he obtained an explicit categorical covering by using the exponential map, his method has the

inconvenience that a logarithm branch has to be chosen, hence introducing some technical complexities.

With the Cayley transform we obtain at each point $A \in G = U(n)$ a contractible open set $\Omega_G(A)$ diffeomorphic to the Lie algebra $\mathfrak{u}(n)$ (see Theorem 1). It is then very easy to find an explicit categorical covering of the group by $n + 1$ open sets.

Theorem 4 $\text{cat}U(n) = n$.

Proof Let $X \in U(n)$ be a unitary matrix and let $z \cdot I \in U(n)$ be the diagonal matrix $\text{diag}(z, \dots, z)$, where $z \in \mathbb{C}$ is any complex number with $|z| = 1$. Remember that $X \in \Omega(z \cdot I)$ iff the matrix $z \cdot I + X$ is invertible.

Let us take $n + 1$ different complex numbers z_0, \dots, z_n , with $|z_k| = 1$. Since any matrix $X \in U(n)$ has at most n different eigenvalues there is always some $-z_k$ which is not an eigenvalue of X , that is $X \in \Omega(z_k \cdot I)$. This proves that $U(n) = \cup_{k=0}^n \Omega_G(z_k \cdot I)$, hence $\text{cat}U(n) \leq n$. On the other hand, as it is well known [11], a lower bound for the LS category is given by the length of the cup product. The cohomology of the unitary group being

$$H(U(n)) = \Lambda(x_1, x_3, \dots, x_{2n-1})$$

—see [4, p. 273], the longest non-null cup product is $x_1 \wedge x_3 \wedge \dots \wedge x_{2n-1} \in H^{n^2}$, hence $n = \text{l.c.p.} \leq \text{cat}U(n)$. Equality follows. \square

4.2 The symmetric spaces $U(2n)/Sp(n)$ and $U(n)/O(n)$

Theorem 5 $\text{cat}U(2n)/Sp(n) = n$.

The following proof is also a simplification of the original one [14].

Proof Following Mimura and Sugata, we consider the action $U \cdot X = UXU^T$ of $U(2n)$ on the manifold

$$M = \{X \in U(2n) : X + X^T = 0\}.$$

This action turns out to be transitive with isotropy $Sp(n)$. Then $U(2n)/Sp(n) \cong M$. Here the symplectic group is identified (via complexification) with the subgroup of matrices $U \in U(2n)$ such that $UJU^T = J$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Now, consider the manifold

$$M' = JM = \{Y \in U(2n) : Y^T = -JYJ\}$$

which is diffeomorphic to M because $J^2 = -I$.

Let $Y \in M'$, let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $Y \in \Omega(\lambda I)$ if and only if $-\lambda$ is an eigenvalue of Y . But as remarked in [14], $Yv = -v\lambda$ implies $Y(J\bar{v}) = -(J\bar{v})\lambda$, which means that Y can be diagonalized as

$$Y = U \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} U^*, \quad D = \text{diag}(\lambda_1, \dots, \lambda_n),$$

so

$$Y + \lambda I = U \begin{pmatrix} D + \lambda I & 0 \\ 0 & D + \lambda I \end{pmatrix} U^*,$$

showing that the maximum number of different eigenvalues of Y is n . This implies that if we take $n + 1$ different complex numbers z_0, \dots, z_n , with $|z_k| = 1$, the open sets $\Omega(z_k I)$ will cover M' .

It only remains to show that the Cayley contraction remains inside M' . More explicitly, for any $z \in \mathbb{C}$, $|z| = 1$, let

$$c_{zI}: \Omega(zI) \rightarrow T_{zI}G$$

be the Cayley map; let $Y \in \Omega(zI) \cap M'$ and take the radial contraction $tc_{zI}(Y)$, $t \in [0, 1]$. In the same way of Proposition 2 it is easy to prove that

$$c_{zI}(Y^T) = -c_{zI}(Y)^T$$

and that

$$c_{zI}(-JYJ) = -Jc_{zI}(Y)J.$$

This implies that the image $c_{zI}(tc_{zI}(Y))$ of the contraction by the inverse Cayley map is contained in M' .

Hence $\text{cat}U(2n)/Sp(n) \leq n$. On the other hand [15, p. 149],

$$H(U(2n)/Sp(n)) = \Lambda(x_1, x_5, x_9, \dots, x_{4n-3})$$

so $n = \text{l.c.p.} \leq \text{cat}U(2n)/Sp(n)$. Equality follows. \square

Theorem 6 $\text{cat}U(n)/O(n) = n$.

The proof is completely analogous to the preceding one.

4.3 The symplectic group $Sp(2)$

The LS category of $G = Sp(2)$ was computed for the first time by Schweitzer [18], who proved that $\text{cat}Sp(2) = 3$.

By using the Morse theory explained above we shall easily obtain an explicit covering by four categorical open sets.

Let us consider the four critical points of a height function h_X as in Example 1 of Section 3, namely: the identity $I = \text{diag}(1, 1)$ and the matrices $P = \text{diag}(-1, 1)$, $-P$ and $-I$.

Theorem 7 $\{\pm\Omega_G(I), \pm\Omega_G(P)\}$ is a categorical covering of $G = Sp(2)$.

Proof Remember that $X \in \Omega_G(A)$ means that $A + X$ is invertible. First we see that the open sets $\Omega_G(I)$ and $\Omega_G(-I)$ cover the whole group excepting the orbit UPU^* of the matrix P by the adjoint action. In fact, by diagonalization, this orbit is formed by the matrices having 1 and -1 as eigenvalues and is diffeomorphic to the sphere S^4 , so it can be covered by stereographic projection.

Explicitly, we must prove that given $X = (x_{ij}) = UPU^*$ in the orbit of P , either $P+X$ or $-P+X$ is invertible.

Since $P^2 = X^2 = I$ we have

$$(P+X)^2 = 2I + PX + XP = 2 \operatorname{diag}(1 - x_{11}, 1 + x_{22}).$$

Then $P+X$ is invertible if and only if $x_{11} \neq 1$ and $x_{22} \neq -1$. Suppose $x_{11} = 1$. The condition $X^*X = I$ means that the columns of X form an orthonormal basis of \mathbb{H}^2 for the hermitian product $\langle v, w \rangle = v^*w$. Then, $x_{11} = 1$ implies $x_{12} = x_{21} = 0$. But since X is in the orbit of P , it must be $x_{22} = -1$. Hence $X = -P$. The same conclusion is obtained from $x_{22} = -1$. So in fact $\Omega_G(P)$ covers all the orbit of P , excepting $-P$. Since $-P \in \Omega_G(-P)$, the proof is done. \square

Remark 3 The cohomology of the symplectic group is [15, p. 119]

$$H(Sp(n)) = \Lambda(x_3, x_7, \dots, x_{4n-1})$$

so the longest non-null product is $x_3 \wedge x_7 \wedge \dots \wedge x_{4n-1}$ and l.c.p. $Sp(n) = n$. However Singhof [20] was able to prove that $\operatorname{cat} Sp(n) \geq n + 1$ for $n \geq 2$.

4.4 Left eigenvalues of symplectic matrices

Obviously it is not worthy to apply the method of critical points above to the symplectic group $Sp(n)$, $n > 2$ (for instance it is known that $\operatorname{cat} Sp(3) = 5$ [7]). Instead, in this section we discuss the possibility of extending the eigenvalue method of paragraphs 4.1 and 4.2.

We shall briefly explain the underlying difficulties. First, it is necessary to endow the quaternionic space \mathbb{H}^n with the structure of a *right* \mathbb{H} -vector space, in order to obtain the usual results for the matrix associated to a linear map. Second, the theory of right eigenvalues is well established, including diagonalization of symplectic matrices [1, 2, 6, 24]. However, for a matrix $A \in Sp(n)$ and a quaternion $\sigma \in \mathbb{H}$, the condition that $A - \sigma I$ be invertible is *not* related to σ being a right eigenvalue. Instead we must consider *left* eigenvalues.

Definition 2 A quaternion $\sigma \in \mathbb{H}$ is a *left eigenvalue* of the matrix $A \in \mathcal{M}(n, \mathbb{H})$ if and only if there exists $v \in \mathbb{H}^n$, $v \neq 0$, such that $Av = \sigma v$. Equivalently, the matrix $A - \sigma I$ is not invertible.

Unfortunately, very little is known about left eigenvalues of quaternionic matrices. Their existence, number, and methods for computing them are only partially understood, see [25] for a recent review. For $n = 2$ two of the authors were able to prove the following Theorem, based on previous results by Huang and So [9].

Theorem 8 [12] *A symplectic matrix $A \in Sp(2)$ has either one, two or infinite left eigenvalues. The latter case can only occur when*

$$A = L_q \circ R_\theta = \begin{pmatrix} q \cos \theta & -q \sin \theta \\ q \sin \theta & q \cos \theta \end{pmatrix}, \quad q \in \mathbb{H}, |q| = 1, \theta \in \mathbb{R}, \sin \theta \neq 0.$$

As a corollary it is possible to obtain the following deceptive result.

Corollary 3 [13] *Let $\sigma_0, \dots, \sigma_3$ be four arbitrary quaternions with norm 1. Then the categorical open sets $\Omega_G(\sigma_0 \cdot I), \dots, \Omega_G(\sigma_3 \cdot I)$ do not cover the group $G = Sp(2)$.*

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