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Non-Closed Lie Subgroups of Lie Groups

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Abstract: We obtain several homotopy obstructions to the existence of non-closed connected Lie subgroups H in a connected Lie group G.

First we show that the foliation $F(G, H)$ on G determined by H is transversely complete [4]; moreover, for K the closure of H in G, $F(K, H)$ is an abelian Lie foliation [2].

Then we prove that $\pi_1(K)$ and $\pi_1(H)$ have the same torsion subgroup, $\pi_n(K) = \pi_n(H)$ for all $n \geq 2$, and $\text{rank}\pi_1(K) - \text{rank}\pi_1(H) > \text{codim } F(K, H)$. This implies, for instance, that a contractible (e.g. simply connected solvable) Lie subgroup of a compact Lie group must be abelian. Also, if $\text{rank}\pi_1(G) \leq 1$ then any connected invariant Lie subgroup of G is closed; this generalizes a well-known theorem of Mal'cev [3] for simply connected Lie groups.

Finally, we show that the results of Van Est on (CA) Lie groups [6], [7] provide many interesting examples of such foliations. Actually, any Lie group with non-compact centre is the (dense) leaf of a foliation defined by a closed 1-form. Conversely, when the centre is compact, the latter is true only for (CA) Lie groups (e.g. nilpotent or semisimple).
Key words: *Lie foliation, dense Lie subgroup, homotopy group*
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1. Introduction

Let G be a connected Lie group. Each (immersed, not necessarily closed) connected Lie subgroup H of G determines a foliation $F = F(G, H)$ on G, whose leaves are the cosets gH, $g \in G$. We remark that F is a transversely complete foliation with abelian structural Lie algebra [4].

Let K be the closure of H in G. Then the foliation $F(K, H)$ induced by F on each fibre of the fibration $G \to G/K$ is an abelian Lie foliation, and the results of Molino and Fédida [2] apply. This is proved in Section 2. We refer the reader to the book of Molino [4] for details about transversely complete and Lie foliations.

This enables us to easily prove in Section 3 several obstructions to the existence of non-closed connected Lie subgroups H in a Lie group G.

First, since $\pi_1(K)/\pi_1(H)$ must be isomorphic to a dense subgroup of $\mathbb{R}^n$, $n = \text{codim}F(K, H)$, neither simply connected nor compact semisimple Lie groups can have proper dense connected Lie subgroups.

Also, we shall prove that when G is a connected Lie group with $\text{rank}\pi_1(G) \leq 1$, then any invariant connected Lie subgroup of G must be closed. For G simply connected this is a classic theorem of Mal'cev [3]. Our generalization includes finite or cyclic fundamental groups, or products of these.

Finally, we prove that a non-abelian contractible Lie subgroup cannot have a compact closure in any Lie group. This is to compare with the well known result that in all Lie groups the closure of a non-closed 1-parameter subgroup must be a torus.

One might suppose that is not easy to have explicit examples of non-closed connected Lie subgroups. For instance, on $K = T^2 \times S^3$ the only one is $F(K,H) = F \times S^3$ for F a dense irrational flow on the torus. However, in Section 4 we show that the work of Van Est on (CA) Lie algebras [6], [7] provides many examples of the type of foliations we are interested in.

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2. The Foliation Defined by a Lie Subgroup

Most of the known results about non-closed Lie subgroups of Lie groups are due to Mal'cev [3] and Van Est [6], [7].

We shall only consider connected Lie groups, but not impose compactness.

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra of G. The Lie algebra $\mathfrak{h}$ of any Lie subgroup H defines on G a completely integrable distribution whose maximal integral manifold at e is H; that is to say, we have a foliation $F(G,H)$ on G, whose leaves are the images of H by left translations L_g, $g \in G$. Since H is not necessarily closed, the space of leaves G/H may not be Hausdorff.

Let K be the closure of H in G. Now the foliation $F(G,K)$ is actually defined by the (*basic*) fibration $G \to G/K$ because K is a closed Lie subgroup of G. In view of Molino's results [4], we are interested in the foliation $F(K,H)$ induced by $F(G,H)$ in each fibre L_gK of the basic fibration. The key result is the following

Lemma 2.1. [3]. *Let H be a connected dense Lie subgroup of a Lie group K. Then*
(i) *H is an invariant subgroup of K.*
(ii) *The Lie algebra $\mathfrak{h}$ of H is an ideal of the Lie algebra $\mathfrak{k}$ of K, and the quotient Lie algebra $\mathfrak{k} / \mathfrak{h}$ is abelian.*

Finally, it is easy to check that the right invariant vector fields on G are foliate for $F(G,H)$. Then we have proved

Proposition 2.2. *Let H be a connected Lie subgroup of a connected Lie group G. Then*
(i) *The foliation $F(G,H)$ is transversely complete.*
(ii) *Its basic fibration is $F(G,K)$, for K the closure of H in G.*
(iii) *The structural Lie algebra of the foliation $F(G,H)$ is abelian.*

We now explain Fédida's construction [2] in our setting.

Let us consider the universal covering $\tilde{K}$ of K. The covering map $\tilde{K} \to K$ is a morphism of Lie groups whose kernel is $\pi_1(K)$. The same holds for H, $\tilde{H}$ and $\pi_1(H)$. Let $p : \tilde{K} \to \mathbb{R}^n$ be the morphism of simply connected Lie groups associated to the projection $p_* : \mathfrak{k} \to \mathfrak{k}/\mathfrak{h}$. Then there is a (*holonomy*) morphism $h : \pi_1(K) \to \mathbb{R}^n$ given by the restriction of p to $\pi_1(K)$. The image of h is a dense subgroup Γ of $\mathbb{R}^n$, isomorphic to $\pi_1(K)/\pi_1(H)$. Since $K = \tilde{K}/\pi_1(K)$, the covering space of K with group Γ of deck transformations is $K_\Gamma = \tilde{K}/\pi_1(H)$.

Thus, when lifting the foliation $F(K,H)$ to K_Γ we actually obtain a (*developing*) fibration, given by the well defined morphism of Lie groups $D : K_\Gamma \to \mathbb{R}^n$ induced by p. Furthermore we have the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(H) & \subset & \pi_1(K) & \to & \Gamma \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{H} & \subset & \tilde{K} & \to & \mathbb{R}^n \\ \downarrow & & \downarrow & & \downarrow \\ H & \subset & K & \to & \underline{L} \end{array}$$

where rows and columns are exact sequences of *group* morphisms. Note that the space of leaves of $F(K,H)$, that is $\underline{L} = K/H = \mathbb{R}^n/\Gamma$, *is not* a Lie group.

Remark 1. $F(K,H)$ is the Lie foliation defined by the closed vectorial 1-form $\omega_g : T_gK = L_{g*}\,\mathfrak{k} \cong \mathfrak{k} \to \mathfrak{k}\,/\,\mathfrak{h}$. As for all Lie groups, K is homeomorphic to $K_0 \times \mathbb{R}^p$ with K_0 any maximal compact Lie subgroup of K. Then $[\omega] \in H^1(K,\mathbb{R}^n) = H^1(K_0,\mathbb{R}^n)$. If ω_g restricted to $T_gK_0 = \mathfrak{k}_0$ defines a foliation F_0 in K_0, then, by a result of Tischler [5], F_0 can be arbitrarily approached by a fibration over a torus, so K fibres over a torus too.

Example 1. A classical example of a non-closed Lie subgroup of the torus T^2 is given by the line $(e^{2\pi it}, e^{2\pi i\alpha t})$ with $\alpha \notin \mathbb{Q}$. This subgroup defines the irrational flow associated to the closed 1-form $\omega = dy - \alpha dx$. Thus we have a $\mathbb{R}$-Lie foliation with dense leaves. Its holonomy morphism $\pi_1(T^2) = Z \oplus Z \to \mathbb{R}$ is given by $h(n,m) = n + \alpha m$.

3. Homotopy Obstructions

We shall now state some restrictions to the existence of H dense in K in terms of their homotopy groups π_n for $n \geq 1$.

From the exact homotopy sequence of $D : K_\Gamma \to \mathbb{R}^n$ we obtain that $\pi_1(K_\Gamma)$ $(= ker h)$ equals $\pi_1(H)$, because $\pi_2(\mathbb{R}^n) = 0$. Thus, $\pi_1(K)/\pi_1(H)$ is isomorphic to a dense subgroup $\Gamma = im h$ of $\mathbb{R}^n$, with $n = \mathrm{codim} F(K,H)$.

Proposition 3.1. *Let H be a connected dense (proper) Lie subgroup of a Lie group K. Then*

(i) *$\pi_1(H)$ is a subgroup of $\pi_1(K)$ such that $\pi_1(K)/\pi_1(H)$ is isomorphic to Z^s, for $s = \mathrm{rank}\pi_1(K) - \mathrm{rank}\pi_1(H)$. Moreover $s > n = \dim K - \dim H$.*

(ii) *$\pi_1(H)$ and $\pi_1(K)$ have the same torsion subgroup.*

(iii) *$\pi_n(H) = \pi_n(K)$ for all $n \geq 2$.*

Proof. (i), (ii). $\pi_1(K)$ is of finite type because it only depends on a maximal compact subgroup of K. Since Γ is a subgroup of $\mathbb{R}^n$, it has no torsion. Then $\Gamma = \pi_1(K)/\pi_1(H)$ is isomorphic to the direct sum of s copies of Z. Let $\pi_1(K) = F + T$, $\pi_1(H) = F' + T'$ be decompositions in (abelian) free and torsion components. It is easy to verify that $T = T'$, then $F/F' = \pi_1(K)/\pi_1(H)$. But $\dim F/F' = \dim F - \dim F'$ because F/F' has no torsion.

The condition $s > n$ follows from the density of Γ.

(iii). The homotopy sequence of the fibration $D : K_\Gamma \to \mathbb{R}^n$ with fibre H gives

$\pi_n(H) = \pi_n(K_\Gamma)$. For $n \geq 2$ the latter is $\pi_n(K)$ because $K_\Gamma \to K$ is a covering map. □

Remark 2. Let $(1, x_1, ..., x_n)$ be a vector in $\mathbb{R}^{n+1}$ with $\mathbb{Q}$-independent coordinates. Then its canonical image generates a dense cyclic subgroup of T^{n+1}. This shows that one can not improve the condition $s > n$.

It is rather obvious that a simply connected Lie group cannot have (proper) dense connected Lie subgroups. Having in mind the Example 1 we obtain the

Corollary 3.2. *Let K be a (not necessarily compact) Lie group with a dense connected (proper) Lie subgroup H.*
(i) *If $\pi_1(K) = Z \oplus Z$ then H is simply connected of codimension* 1.
(ii) *If H is simply connected then $\pi_1(K) = Z^s$ with $s > \dim K - \dim H$.*

Now we generalize Mal'cev's theorem we cited in the Introduction.

Theorem 3.3. *Let G be a connected Lie group with* $\operatorname{rank}\pi_1(G) \leq 1$. *If H is a connected invariant Lie subgroup of G, then H is closed in G.*

Proof. Let H be invariant in G. Then its closure K is invariant too. Thus G/K is a (Hausdorff) Lie group, hence $\pi_2(G/K) = 0$. The homotopy sequence of $G \to G/K$ shows that $\pi_1(K)$ is contained in $\pi_1(G)$, then $\operatorname{rank}\pi_1(K) \leq 1$. By Proposition 3.1 we have $1 - \operatorname{rank}\pi_1(H) > \dim K - \dim H$, then $K = H$. □

To obtain further consequences of Proposition 3.1 we must review some known facts about the homotopy groups of a Lie group. We follow the article of Borel [1].

For any connected Lie group G, $\pi_n(G)$ equals π_n of a maximal compact Lie subgroup ($n \geq 0$). Then we can suppose G compact. In this case, G has a finite covering G^* which is a direct product of a torus and simple non-abelian Lie groups. Then $\pi_n(G) = \pi_n(G^*)$ for $n \geq 2$, and $\pi_1(G^*)$ is a subgroup of finite order of $\pi_1(G)$.

For a compact connected semisimple Lie group G, $\pi_2(G) = 0$ and $\pi_1(G)$ is finite. Then $\pi_2 = 0$ for all Lie groups. Also recall that π_1 is an abelian group for all H-spaces. Finally, $\pi_3(G) = Z$ for G compact connected, simply connected, simple and non-abelian.

It follows that a connected Lie group which is simple or compact semisimple does not contain (proper) dense connected Lie subgroups. Now, we improve part (ii) of Corollary 3.2 in the compact case:

Theorem 3.4. *Any contractible Lie subgroup of a compact Lie group is abelian.*

Proof. Let K be the closure of the subgroup H. Since $\pi_1(K) = Z^s$ with $s > \dim K - \dim H$, $\pi_1(K^*)$ has no torsion. Then K^* is a product of a torus and *simply connected* compact simple Lie groups. Therefore, for $n \geq 2$, the homotopy groups $\pi_n(H) = \pi_n(K)$ equal those of the finite covering K^*. But $\pi_3(K^*)$ is not zero except for K^* a torus. Then K is an abelian compact Lie group, hence a torus too. □

Remark 3. The proof still works for $\pi_1(H)$ without torsion (or $\pi_3(H)$ finite, but the conclusion would imply $\pi_3(H) = 0$ and then H is contractible).

Corollary 3.5. *Let H be a simply connected, non-abelian solvable Lie group. Then its closure in any Lie group is not compact.*

4. Examples

Theorem 4.1. *Any connected Lie group H with non-compact centre is the (dense) leaf of a foliation $F(G, H)$ defined by a closed 1-form.*

Proof. If $Z(H)$ is not connected and its identity component $Z(H)_e$ is compact, then $Z(H)/Z(H)_e$ is not finite. Thus we can take $x \in Z(H)$ without torsion. In other case, $Z(H)_e$ is abelian non-compact, then isomorphic to the product of a torus and some $\mathbb{R}^m, m > 0$, and we take $x \in \mathbb{R}^m, x \neq 0$.

Let us choose $\alpha \notin \mathbb{Q}$. Let K be the quotient $(S^1 \times H)/D$ where D is the discrete central subgroup of $S^1 \times H$ generated by $(e^{2\pi i\alpha}, x)$. Then H is a Lie subgroup of K by means of the canonical map $f(h) = (1, h)D$. To prove the density of $f(H)$ in K, let $z \in K, z = (t, h)D$. Since the rotation α is not rational, there exists a sequence $\{n_i\alpha\} \to t$ in S^1, $n_i \in Z$. Then $\{f(hx^{-n_i})\} \to z$ because $(n_i\alpha, h)D = (1, hx^{-n_i})D$. □

Remark 4. Then any connected Lie group H with non-compact centre $Z(H)$ is the (dense) leaf of some $\mathbb{R}$-Lie foliation $F(K, H)$, with $K = (S^1 \times H)/D$. Since $\pi_1(K) = \pi_1(H) \oplus Z^2$, the holonomy morphism $\pi_1(K) \to \mathbb{R}$ is given by $h(\phi, n, m) = n + \alpha m$ and the developing map D is the trivial fibering $\mathbb{R} \times H \to \mathbb{R}$. For $H = \mathbb{R}$ this is the Example 1.

The latter result is implicit in the work of Van Est on (CA) Lie algebras [6]. Recall that a connected Lie group H with centre $Z(H)$ is a *(CA) Lie group* if $\mathrm{Ad}(H) = H/Z(H)$ is closed in $GL(\mathfrak{h})$. All nilpotent or semisimple Lie groups are (CA). In [7] it is also proved that any non-(CA) Lie group is dense in some (CA) Lie group. Also, in the proof of Theorem 4.1, one verifies that if H is (CA) then $K = (S^1 \times H)/D$ is (CA).

Conversely, let H be a Lie group with compact centre. Then H is the dense leaf of some Lie foliation $F(G, H)$ if and only if H is not (CA). This partial converse of Theorem 4.1 was first proved by Van Est in a different way.

Proposition 4.2. *Let H be a connected (CA) Lie group. If H is a non-closed Lie subgroup of some Lie group then its centre $Z(H)$ is not compact.*

Proof. Let K be the closure of H in some Lie group G. Then the (CA) condition implies that $Z(H)$ is dense in Z(K). When $Z(H)$ is compact we have $Z(H) = Z(K)$, then $\mathrm{Ad}(H)$ is a dense Lie subgroup of $\mathrm{Ad}(K)$. But $\pi_1(\mathrm{Ad}(H)) = Z(H)/Z(H)_e$ equals $\pi_1(K)$, then by Proposition 3.1 $\mathrm{Ad}(H) = \mathrm{Ad}(K)$, that is $H = K$. □

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