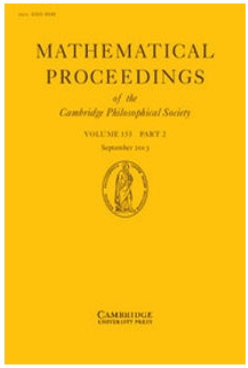

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An upper bound for the Lusternik–Schnirelmann category of the symplectic group

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Abstract

We prove that the LS category of the symplectic group $Sp(n)$ is bounded above by $\binom{n+1}{2}$. This is achieved by computing the number of critical levels of a height function.

1. *Introduction*

The Lusternik–Schnirelmann category of a space G is defined as the least integer $k \geqslant 0$ such that G admits a covering by $k+1$ open subsets that are contractible in G (*categorical* open sets). A classical problem raised by T. Ganea [**3**] is to compute this invariant for Lie groups. However only a small number of calculations have been carried out, mainly due to the difficulty of the homotopical techniques involved. This is particularly true for the symplectic group where the only known results are cat $Sp(2) = 3$, obtained by P. Schweitzer in 1965 [**11**] and cat $Sp(3) = 5$ proved by L. Fernández, A. G. Tato, J. Strom and D. Tanré in 2001 [**1**] and by N. Iwase and M. Mimura [**6**]. The latter authors also proved that cat $Sp(n) \geqslant n+2$ for $n \geqslant 3$.

In 2009, M. Hunziker and M. R. Sepanski [**5**] showed that the LS category of a simple, simply connected, compact Lie group G is bounded above by the sum of the relative categories of the conjugacy classes $\mathcal{O}_k$ of exp v_k, $0 \leqslant k \leqslant n = \dim G$, where $\{v_0, \ldots, v_n\}$ are the vertices of the fundamental alcove for the action of the affine Weyl group on the Lie algebra of a maximal torus of G. These ideas had been sketched out by J. Oprea and D. Tanré in an unpublished paper (2004). For $G = SU(n)$ the computation retrieves W. Singhof's result cat $SU(n) = n-1$ [**12**]. For the symplectic group $G = Sp(n)$, Hunzinker–Sepanski's formula is

$$\text{cat}\, Sp(n) + 1 \leqslant \sum_{k=0}^{n} \big(\text{cat}_{Sp(n)}\, \mathcal{O}_k + 1\big). \qquad (1{\cdot}1)$$

Moreover they conjectured that $\operatorname{cat}_{Sp(n)} \mathcal{O}_k \leqslant \min\{k, n-k\}$, which should imply that

$$\operatorname{cat} Sp(n) \leqslant \lfloor \frac{(n+2)^2}{4} \rfloor - 1.$$

The submanifolds $\mathcal{O}_k$ are Grassmannians, but the computation of their relative category seems to rely in a subtle way on the difference between left and right eigenvalues of quaternionic matrices, a subject about which very little is known (see for instance the authors' results [**7**, **8**] about left eigenvalues of symplectic matrices).

In this paper we shall prove that formula (1·1) is a very particular case of our Theorem 3·3, which compares the category of the ambient manifold G with the category of the critical subset $\Sigma(h_X)$ of any height function h_X. As a consequence we obtain the upper bound

$$\operatorname{cat} Sp(n) \leqslant \binom{n+1}{2}.$$

2. *Height functions on Lie groups*

In [**4**] the authors studied arbitrary height functions on a Lie group G of orthogonal type. Let us briefly recall some results.

Let $\mathbb{K}$ be either $\mathbb{R}$ (reals), $\mathbb{C}$ (complex) or $\mathbb{H}$ (quaternions). The orthogonal group (resp. unitary, symplectic)

$$G = O(n, \mathbb{K}) = \{A \in \mathcal{M}_{n\times n}(\mathbb{K}) \colon AA^* = I\}$$

is embedded in the Euclidean space $\mathcal{M}_{n\times n}(\mathbb{K})$, where the usual inner product is given by $\langle A, B\rangle = \Re \operatorname{Tr}(A^*B)$ (we denote by $\Re \operatorname{Tr}$ the real part of the trace and $*$ the conjugate transpose). Consequently, the height function $h_X \colon G \to \mathbb{R}$ with respect to the hyperplane perpendicular to the vector $X^* \in \mathcal{M}_{n\times n}(\mathbb{K})$, $X \neq 0$, is given, up to a constant, by the formula

$$h_X(A) = \Re \operatorname{Tr}(XA).$$

Depending on the singular values of X this function will be Morse (non-degenerate critical points) or just Morse–Bott (non-degenerate critical *submanifolds*). In fact, we have the following structure theorem for the critical set $\Sigma(h_X)$ of h_X.

For $p \geqslant 1$ let us denote $\Sigma(p)$ the disjoint union $G_0^p \sqcup \cdots \sqcup G_p^p$ of the Grassmannians

$$G_q^p = \frac{O(p)}{O(q)\times O(p-q)}.$$

Then $\Sigma(p)$ is the critical set of the function h_I, where $I = I_p$ is the identity matrix of order p. The function h_I was first studied by T. Frankel [**2**]. Notice that $\Sigma(1) = S^0$ are two points.

THEOREM 2·1. *Let h_X be an arbitrary height function on G. Let $0 < t_1 < \cdots < t_k$ be the singular values of X, with multiplicities $n_0, n_1, \ldots, n_k$. Then*

$$\Sigma(h_X) \cong O(n_o) \times \Sigma(n_1) \times \cdots \times \Sigma(n_k).$$

Proof. The gradient of h_X at the point $A \in G$ is the projection of X^* onto the tangent space T_AG, that is,

$$(\operatorname{grad} h_X)_A = \frac{1}{2}(X^* - AXA).$$

Let $X = UDV^*$ be the singular value decomposition (SVD) of X, where

$$D = \mathrm{diag}\,(0, \overset{n_0)}{\ldots}, 0, t_1, \overset{n_1)}{\ldots}, t_1, \ldots, t_k, \overset{n_k)}{\ldots}, t_k)$$

is a diagonal matrix with blocks of size $n_0, n_1, \ldots, n_k$. Then

$$(\mathrm{grad}\; h_X)_A = V(\mathrm{grad}\; h_D)_{V^*AU}U^*.$$

Analogously, the Hessian $(Hh_X)_A\colon T_AG \to T_AG$ is given by the projection of the derivative of the gradient, that is,

$$(Hh_X)_A(Y) = (-1/2)(AXY + YXA) = V(Hh_D)_{V^*AU}(V^*YU)U^*.$$

Hence the problem reduces to study the function h_D, whose critical set is the product of the critical sets of functions h_X with $X = t_qI$, that is, the critical sets $\Sigma(n_q)$.

Example 1. Let $G = U(2)$ and $X = \mathrm{diag}\,(0, 1)$, that is, $n_0 = 1, n_1 = 1$. Then $\Sigma(h_X) \cong U(1) \times \Sigma(1) = S^1 \sqcup S^1$, two disjoint circles. On the other hand, if $X = \mathrm{diag}\,(1, 1)$ then $n_0 = 0$ and $n_1 = 2$, hence $\Sigma(h_X) = \Sigma(2)$, the union of two points and a sphere $S^3/S^1 \cong S^2$. Finally, when $X = \mathrm{diag}\,(1, 2)$, that is, $n_0 = 0, n_1 = 1, n_2 = 1$, the critical set $\Sigma(1) \times \Sigma(1)$ is formed by four points.

3. *LS category and critical points*

The *relative* LS category of a submanifold $A \subset G$ is the minimum number $k \geqslant 0$ such that A can be covered by $k + 1$ categorical open sets of G. Clearly if $A, B \subset G$ are disjoint compact submanifolds then

$$\mathrm{cat}_G(A \cup B) \leqslant \mathrm{cat}_G\, A + \mathrm{cat}_G\, B + 1. \tag{3·1}$$

For G a closed manifold it is well known that cat $G + 1$ is a lower bound for the number of critical points of any smooth function $f\colon G \to \mathbb{R}$. When the critical set $\Sigma(f)$ is not finite we still have the following formula (for a complete discussion see the paper by Y. B. Rudyak and F. Schlenk [**10**], also [**9**]):

THEOREM 3·1. *Let $c_1, \ldots, c_p$ be the critical values of the function f. Then*

$$\mathrm{cat}\; G + 1 \leqslant \sum_{q=1}^{p} \big(\mathrm{cat}_G\, \big(\Sigma(f) \cap f^{-1}(c_q)\big) + 1\big).$$

Taking into account that in a path-connected manifold a finite collection of points has category 0 we have:

COROLLARY 3·2. *Assume G is connected. When all the critical points are isolated,* cat $G + 1$ *is bounded above by the number p of critical* levels *of f.*

We start by proving in an easy way the result of Hunzinker and Sepanski cited in the Introduction.

Example 2. Let $G = Sp(n)$ be the symplectic (quaternionic) group and consider the height function h_I, which is invariant by the adjoint action. The critical set is $\Sigma(n) = G_0^n \sqcup \cdots \sqcup G_n^n$, where the Grassmannian $G_q^n = \mathcal{O}_q$ is the orbit of the matrix $\mathrm{diag}\,(-I_q, +I_{n-q})$.

Then, according to Theorem 3·1,

$$\text{cat } Sp(n) + 1 \leqslant \sum_{q=0}^{n} \left(\text{cat}_G\, G_q^n + 1\right),$$

which is exactly formula (1·1).

In the general case of an arbitrary function h_X on the Lie group G the components of the critical set are the products

$$\Sigma[n_0, i_1, \ldots, i_k] = O(n_0) \times G_{i_1}^{n_1} \times \cdots \times G_{i_k}^{n_k}, \quad 0 \leqslant i_q \leqslant n_q.$$

By applying Theorem 3·1 and formula (3·1) we obtain

THEOREM 3·3.

$$\text{cat } G + 1 \leqslant \sum (\text{cat}_G\, \Sigma[n_0, i_1, \ldots, i_k] + 1).$$

Example 3. Let $G = U(2)$ and h_X as in Example 1 with $X = I$. Then

$$\text{cat } U(2) + 1 \leqslant 2(\text{cat}_G(*) + 1) + (\text{cat}_G\, G_1^2 + 1) = 3$$

because $\text{cat}_{U(2)}\, G_1^2 = 0$. Indeed, although cat $S^2 = 1$, it happens that G_1^2, which is the orbit by the adjoint action of the matrix diag $(1, -1)$, is contained in the open set

$$\Omega_G(\mathbf{i}) = \{A \in G : \det(A - \mathbf{i}I) \neq 0\},$$

which is contractible [**4**]. (It is known that cat $U(n) = n$ [**12**].)

Example 4. Let $G = Sp(n)$ and take $n_0 = n - 1$. Then

$$\text{cat } Sp(n) \leqslant 2\,\text{cat}_{Sp(n)}\, Sp(n-1) + 1.$$

Note that $\Sigma(h_X) = Sp(n-1) \times \Sigma(1)$ and that the two copies of $Sp(n-1)$ lie in two different critical levels $h_X^{-1}(\pm t_1)$.

In order to improve Theorem 3·3 we should study the critical levels of an arbitrary height function h_X. The relative category is invariant by diffeomorphisms, then, according to Theorem 2·1 we can suppose that $X = \text{diag}(0, \ldots, t_1, \ldots, t_k)$. Each component $\Sigma[n_0, i_1, \ldots, i_k]$ of the critical set is formed by the matrices

$$A = \begin{pmatrix} A_0 & & & \\ & A_1 & & \\ & & \cdots & \\ & & & A_k \end{pmatrix}$$

such that $A_q \in O(n_q)$, $0 \leqslant q \leqslant k$, and each A_q, $q \geqslant 1$, verifies $A_q^2 = I$ and diagonalizes to the real matrix

$$\begin{pmatrix} -I_{i_q} & \\ & +I_{n_q - i_q} \end{pmatrix}.$$

Then

$$h_X(A) = \Re\,\text{Tr}\,(XA) = t_1(n_1 - 2i_1) + \cdots + t_k(n_k - 2i_k).$$

In particular, when h_X is a Morse function (that is, $n_0 = 0$, $n_1 = \ldots = n_k = 1$) we have that the possible critical values are

$$t_1\varepsilon_1 + \cdots + t_k\varepsilon_k, \quad \varepsilon_q = \pm 1.$$

COROLLARY 3·4.

$$\operatorname{cat} Sp(n) \leqslant \binom{n+1}{2}.$$

Proof. Let us consider the Morse function h_X with $X = \operatorname{diag}(1, 2, \ldots, n)$. In each critical level there is a matrix of the form $A = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$. Then we observe that:

(i) the maximum value is $C = 1 + \cdots + n = \binom{n+1}{2}$;

(ii) if c is a critical value then $-c$ is a critical value too;

(iii) changing $+1$ by -1 we observe that the critical values are either

$$-C, -C+2, \ldots, -2, 0, 2, \ldots, C$$

when C is even or

$$-C, -C+2, \ldots, -3, -1, 1, 3, \ldots, C$$

when C is odd. In both cases the number of critical levels is $\binom{n+1}{2} + 1$.

Then the result follows from Corollary 3·2.

Remark. An explicit categorical covering of $G = Sp(2)$ is formed by the four open sets $\Omega_G(\pm I)$, $\Omega(\pm P)$, where $P = \operatorname{diag}(-1, 1)$ and $\Omega_G(A) = \{X \in G : A + X \text{ is invertible}\}$.

Remark. Theorem 3·1 is proved in [**10**] for several generalizations of LS category, including the so-called *ball category* bcat (coverings by smoothly embedded balls [**13**]). Taking into account that for the function given in Corollary 3·4 the critical points of each critical level are contained in the ball $\Omega_G(\mathbf{i}I)$ we actually obtain $\operatorname{bcat} Sp(n) \leqslant \binom{n+1}{2}$.

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