Symplectic matrices with predetermined left eigenvalues[★]

E. Macías-Virgós, M. J. Pereira-Sáez

Institute of Mathematics. Department of Geometry and Topology.
University of Santiago de Compostela. 15782- SPAIN

Abstract

We prove that given four arbitrary quaternion numbers of norm 1 there always exists a 2×2 symplectic matrix for which those numbers are left eigenvalues. The proof is constructive. An application to the LS category of Lie groups is given.

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1. Introduction

Left eigenvalues of quaternionic matrices are only partially understood. While their existence is guaranteed by a result from Wood [11], many usual properties of right eigenvalues are no longer valid in this context, see Zhang's paper [10] for a detailed account. In particular, a matrix may have infinite left eigenvalues (belonging to different similarity classes), as has been proved by Huang and So [5]. By using this result, the authors characterized in [7] the symplectic 2×2 matrices which have an infinite spectrum. In the present paper we prove that given four arbitrary quaternions of norm 1 there always exists a matrix in $Sp(2)$ for which those quaternions are left eigenvalues. The proof is constructive. This non-trivial result is of interest for the computation of the so-called LS category, as we explain in the last section of the paper.

2. Left eigenvalues of symplectic matrices

Let $Sp(2)$ be the Lie group of 2×2 symplectic matrices, that is quaternionic matrices such that $AA^* = I$, where $A^* = \bar{A}^t$ is the conjugate transpose.

By definition, a quaternion σ is a left eigenvalue of the matrix A if there exists a vector $v \in \mathbb{H}^2$, $v \neq 0$, such that $Av = \sigma v$; equivalently, $A - \sigma I$ is not invertible.

It is easy to prove that the left eigenvalues of a symplectic matrix must have norm 1.

Remark 2.1. $\mathbb{H}^2$ will be always considered as a *right* quaternionic vector space, endowed with the product $\langle v, w \rangle = v^* w$.

The following Theorem was proven by the authors in [7], using the results of Huang and So in [5].

Theorem 2.2. *The only 2×2 symplectic matrices with an infinite number of left eigenvalues are those of the form*

$$
L_q \circ R_\theta = \begin{bmatrix} q \cos \theta & -q \sin \theta \\ q \sin \theta & q \cos \theta \end{bmatrix}, \quad q \in \mathbb{H}, |q| = 1, \quad \theta \in \mathbb{R}, \sin \theta \neq 0.
$$

Any other symplectic matrix has one or two left eigenvalues.

The matrix $L_q \circ R_\theta$ above corresponds to the composition of a real rotation $R_\theta \neq \pm \mathrm{id}$ with a left translation L_q, $|q| = 1$. We need to characterize its eigenvalues.

Lemma 2.3. *Let the matrix $A = L_q \circ R_\theta$ be as in Theorem 2.2 and let $\sigma \in \mathbb{H}$ be a quaternion. The following conditions are equivalent:*

1. *σ is a left eigenvalue of A;*
2. *$\sigma = q(\cos \theta + \sin \theta \cdot \omega)$ with $\omega \in \langle i, j, k \rangle_{\mathbb{R}}$, $|\omega| = 1$;*
3. *$|\sigma| = 1$ and $\Re(\bar{q}\sigma) = \cos \theta$;*
4. *$\bar{q}\sigma$ is conjugate to $\cos \theta + i \sin \theta$.*

Proof. Part 2 can be checked by a direct computation from the definition or by using the results in [5]; part 3 follows because $\bar{q}\sigma - \cos \theta$ has not real part; finally part 4 is proved from the fact that two quaternions are conjugate if and only if they have the same norm and the same real part. $\qquad \square$

We also need the following elementary result.

Lemma 2.4. *Let $M \in \mathcal{M}(n+1, \mathbb{R})$ be a real matrix with rows $\sigma_1, \ldots, \sigma_{n+1}$ and let $w \in \mathbb{R}^{n+1}$, $w \neq 0$. Suppose that M has maximal rank $n+1$ and that its rows have euclidean norm 1. Then,*

$$|M \cdot w| < \sqrt{n+1}\,|w|.$$

Proof. Let the matrix $M = (m_{ij})$ and let $w = \sum_{j=1}^{n+1} w_j e_j$ where e_j are the vectors of the canonical basis. Then

$$
\begin{aligned}
|M \cdot w| &= \\
\left| \sum_{i=1}^{n+1} \left(\sum_{j=1}^{n+1} m_{ij} w_j \right) e_i \right| &= \\
\left[\sum_{i=1}^{n+1} \left(\sum_{j=1}^{n+1} w_j m_{ij} \right)^2 \right]^{1/2} &= \\
\left[\sum_{i=1}^{n+1} \langle w, \sigma_i \rangle^2 \right]^{1/2} &,
\end{aligned}
\tag{1}
$$

where $\langle , \rangle$ is the scalar product in $\mathbb{R}^{n+1}$. Moreover, for any $1 \leq i \leq n+1$ we have

$$\langle w, \sigma_i \rangle = |w|\,|\sigma_i| \cos \sphericalangle(w, \sigma_i) = |w| \cos \sphericalangle(w, \sigma_i) \leqslant |w|$$

and equality implies that w is a multiple of σ_i. Since the rows σ_i are $\mathbb{R}$-independent by hypothesis, the vector $w \neq 0$ can not be in the direction of the $n+1$ rows at the same time, so $\langle w, \sigma_i \rangle < |w|$ for some i. Then, from (1),

$$|M \cdot w| < \left((n+1)|w|^2 \right)^{1/2} = \sqrt{n+1}|w|.$$

$\square$

Next theorem is the main result of this paper.

Theorem 2.5. *Let $\sigma_1, \ldots, \sigma_4$ be four quaternions with norm 1. Then there exists a matrix $A \in Sp(2)$ for which those quaternions are left eigenvalues.*

Proof. Accordingly to Theorem 2.2 the matrix must be of the form $A = L_q \circ R_\theta$, $|q| = 1$, $\sin \theta \neq 0$.

Now, part (3) of Lemma 2.3 means that, in order to find A, we have to fix a possible $\cos \theta \neq \pm 1$ –the exact value of $\cos \theta$ will be determined later–, and then to solve the system of linear equations

$$\Re(\bar{q}\sigma_m) = \cos \theta, \quad m = 1, \ldots, 4. \tag{2}$$

Moreover, the solution q must verify $|q| = 1$.

Let us write

$$q = t + xi + yj + zk, \quad t, x, y, z \in \mathbb{R},$$

and analogously

$$\sigma_m = t_m + x_m i + y_m j + z_m k, \quad m = 1, \ldots, 4.$$

Then system (2) can be written as

$$\begin{pmatrix} t_1 & x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ t_4 & x_4 & y_4 & z_4 \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \cos \theta \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

which we abbreviate as

$$M \cdot q = \cos \theta \cdot u. \tag{3}$$

If $0 < \operatorname{rank} M < 4$ we take $\cos \theta = 0$, so system (3) is homogeneous. The set of solutions is a non trivial vector space, hence it contains at least one solution q with norm $|q| = 1$. Then a matrix having $\sigma_1, \ldots, \sigma_4$ as eigenvalues would be, for instance, $A = L_q \circ R_{\pi/2}$.

On the other hand, if the matrix M has maximal rank 4, it is invertible and the unique solution of (3) is given by

$$q = \cos \theta \cdot M^{-1} \cdot u. \tag{4}$$

By Lemma 2.4 above we have

$$2 = |u| = |M \cdot M^{-1} \cdot u| < \sqrt{4} \, |M^{-1} \cdot u|,$$

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hence $|M^{-1}u| > 1$ and so we can choose θ such that

$$0 < |\cos\theta| = \frac{1}{|M^{-1}u|} < 1 \tag{5}$$

which implies

$$|q| = |\cos\theta||M^{-1}u| = 1.$$

With that angle θ and the solution q in (4) we have obtained a matrix $A = L_q \circ R_\theta$ having σ_m, $m = 1, \ldots, 4$ among its eigenvalues. $\qquad\square$

Example 2.6. The four quaternions $1, i, j, k$ give rise to the system

$$\mathrm{id} \cdot q = \cos\theta \cdot u,$$

with unitary solutions $q = \pm(1/2)u$ (we are using (4) and (5)). Then, the only two symplectic matrices having those four numbers as left eigenvalues are

$$\frac{1}{4}\begin{pmatrix} u & -\sqrt{3}u \\ \sqrt{3}u & u \end{pmatrix}$$

and

$$\frac{1}{4}\begin{pmatrix} u & \sqrt{3}u \\ -\sqrt{3}u & u \end{pmatrix},$$

where $u = 1 + i + j + k$.

3. Application to LS category

The Lusternik-Schnirelmann category of a topological space X, denoted by cat X, is the minimum number of open sets (minus one), contractible in X, which are needed to cover X. This homotopical invariant has been widely studied and has many applications which go from the calculus of variations to robotics, see [1, 2, 6]. The computation of the LS category of Lie groups and homogeneous spaces is a central problem in this field, where many questions are still unanswered. For instance, for the symplectic groups the only known results are cat $Sp(1) = 1$, cat $Sp(2) = 3$ and cat $Sp(3) = 5$ [3].

There is a standard technique which has been successfully applied in the complex setting, for instance to the unitary group $U(n)$ [9] and to the symmetric spaces $U(2n)/Sp(n)$ and $U(n)/O(n)$ [8]. It consists in considering, for a given complex number z with $|z| = 1$, the set $\Omega(z)$ of unitary matrices

A such that $A - zI$ is invertible. It turns out that this set is contractible. So, since a unitary $n \times n$ matrix can not have simultaneously $n + 1$ different eigenvalues, it is possible to cover the cited spaces by $n + 1$ contractible open sets $\Omega(z_1), \ldots, \Omega(z_{n+1})$, showing that they have category $\leq n$ (that n is also a lower bound can be proved with homological methods).

In the quaternionic setting, we must consider *left* eigenvalues. If $\sigma \in \mathbb{H}$, the open set

$$\Omega(\sigma) = \{A \in Sp(2) \colon A - \sigma I \text{ is invertible}\}$$

is contractible, for instance by means of the Cayley contraction

$$A_t = \frac{(1+t)A - (1-t)\sigma I}{(1+t)I - (1-t)\bar{\sigma}A}, \quad t \in [0,1]$$

(see [4] for a general discussion). Hence our main result in this paper (Theorem 2.5) implies that four contractible sets of the type $\Omega(\sigma)$ will never cover $Sp(2)$, despite the fact that $\operatorname{cat} Sp(2) = 3$.

Remark 3.1. We observe (cf. Lemma 2.3) that all the eigenvalues σ of the two matrices in Example 2.6 verify $\Re(\bar{q}\sigma) = \pm 1/2$. Then if we take $\sigma_5 = (i + j)/\sqrt{2}$, we have $\Re(\bar{q}\sigma_5) = \pm 1/\sqrt{2}$, hence the two matrices belong to $\Omega(\sigma_5)$. So five open sets associated to eigenvalues do suffice to cover the group.

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