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Manifolds of Maps in Riemannian Foliations*

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Abstract. Let (M', F') and (M, F) be two (compact or not) foliated manifolds, $C_F^\infty(M', M)$ the space of smooth maps which send leaves into leaves. In this paper we prove that $C_F^\infty(M', M)$ admits a structure of an infinite-dimensional manifold modeled on LF-spaces, provided that F is a Riemannian foliation or, more generally, when it admits an adapted local addition.

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Introduction

Manifolds of maps are a classical field of research. In [14], R. Palais proved that the space $C^\infty(M', M)$ of smooth maps between two compact manifolds is an infinite-dimensional manifold modeled on inductive limits of Hilbert spaces. The noncompact case was solved by P. Michor [9], who modeled $C^\infty(M', M)$ on LF-spaces, i.e. inductive limits of Fréchet spaces.

The group $\mathrm{Diff}(M)$ of diffeomorphisms of a compact manifold was also studied by J. Leslie [4] —who modeled it on Fréchet spaces— and H. Omori [12], [13] —modeled on inductive limits of Hilbert spaces. The non-compact case was studied by P. Michor [9].

In this paper, we consider two foliated manifolds (M, F), (M', F') and the space $C_F^\infty(M', M)$ of smooth maps which send leaves into leaves. Our aim is to endow $C_F^\infty(M', M)$ with a structure of manifold modeled on LF-spaces, by refining the results of Michor cited above. As a particular case, the group $\mathrm{Diff}(M, F)$ of foliation preserving diffeomorphisms of a Riemannian foliation will inherit a manifold structure modeled on the Lie algebra of foliate vector fields with compact support. Although the group $\mathrm{Diff}(M, F)$ was studied by J. Leslie in [5] and [6] when M is compact, we think that our approach (in the Riemannian case) is conceptually clearer.

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The contents of this paper are as follows: First, we choose a suitable locally path-connected topology in the space of smooth maps, namely the C^∞-topology of Whitney in the compact case and the $\mathcal{FD}$-topology of Michor in the noncompact case. See [7] or [10] for the definitions.

Second, we consider manifolds modeled by LF-spaces as a suitable framework which makes the use of Sobolev's completions unnecessary. We introduce the notion of a *foliate* vector field along a map $f \in C^\infty_F(M', M)$, in such a way that when $f = \mathrm{id}_M$ we recover the usual notion of foliate vector field on a foliated manifold [11]. Then we prove (1.2.2) that the space $\Gamma^F_c(f^*TM)$ of foliate vector fields along f with compact support is an LF-space. This will be the local model around f.

Third, we introduce the notion of a local addition $\mathcal{E}: S \subset TM \to M$ which is *adapted*, in the sense that it induces around any point a local addition in a small transversal to the foliation (the precise definition appears in 1.3.1). This allows us to give an explicit atlas for the manifold $C^\infty_F(M', M)$. Roughly speaking, the chart around f associates to each map h near f the foliate vector field X along f such that $\mathcal{E}(X_p) = h(p)$.

In the second part of the paper we show that when F is a Riemannian foliation [11] there always exists an adapted local addition $\mathcal{E}$. In a few words, $\mathcal{E}$ will be a mixture of the exponential map along the leaves and the restriction of the exponential map in the manifold to the orthogonal bundle of F. Since the leaves may not be totally geodesic submanifolds, we carefully state some results about geodesics on the leaves and the horizontal lift of geodesics in Riemannian submersions.

In this paper, all finite-dimensional manifolds M, M', ... are supposed to be Hausdorff, paracompact and connected. Compactness is *not* assumed.

1. The Manifold of Foliation Preserving Maps

1.1. TOPOLOGICAL PRELIMINARIES

We state here several basic definitions and results about topologies on spaces of smooth maps that we need in this paper. We refer the reader to [7] and [10] for further details.

Let M, M' be two smooth manifolds. For $0 \leqslant k \leqslant \infty$, let $J^k(M', M)$ be the space of k-jets of maps from M' to M.

DEFINITION 1. The *C^k-topology of Whitney* or *WO^k-topology* on $C^\infty(M', M)$ is given by the basis of all open sets

$$W^k(\Omega) = \{f \in C^\infty(M', M) \mid (j^k f)(M') \subseteq \Omega\}$$

where Ω is any open set in $J^k(M', M)$.

Let $h: M \to N$ be a smooth map. Then for any WO^k-topology, $0 \leqslant k \leqslant \infty$, the induced map $h_*: C^\infty(M', M) \to C^\infty(M', N)$, $h_*(f) = h \circ f$, is continuous. When h

is a *proper* map, the induced map $h^\sharp\colon C^\infty(N, M') \to C^\infty(M, M')$, $h^\sharp(f) = f \circ h$, is continuous. Notice that the restriction map to an open set may not be continuous.

DEFINITION 2. A basis of the $\mathcal{D}$-*topology* on $C^\infty(M', M)$ is given by the open sets

$$D(L, \Omega) = \{f \in C^\infty(M', M) \mid (j^\infty f)(L_n) \subseteq \Omega_n, \forall n\}$$

where $L = (L_n)$ is a locally finite countable family of closed sets in M' and $\Omega = (\Omega_n)$ is a family of open sets in $J^\infty(M', M)$.

If $k' \geqslant k$, the $WO^{k'}$-topology is finer than the WO^k-topology. The $\mathcal{D}$-topology is finer than the WO^∞-topology; both coincide when the manifold M' is compact.

With any of the topologies above, the space $C^\infty(M', M)$ fails to be locally path connected when M' is not compact. One must then add as new open sets the equivalence classes of the relation: $h \sim f$ if and only if they coincide outside a compact set. One obtains in this way a still finer topology, called the F$\mathcal{D}$-*topology* by Michor.

Let $(E, \tau, M, \mathbf{R}^n)$ be a C^∞ vector bundle over the manifold M, and $\Gamma(E)$ its set of C^∞ sections, which we consider as a subspace of $C^\infty(M, E)$ with the F$\mathcal{D}$-topology. The maximal topological vector space contained in $\Gamma(E)$ is the subspace $\Gamma_c(E)$ of sections with compact support [10]. It is an *LF-space*, that is to say a complete, locally path-connected vector space, which is an inductive limit of Fréchet spaces. When M is compact, the F$\mathcal{D}$-topology coincides with the C^∞-topology and $\Gamma(E)$ is a Fréchet space.

1.2. THE LOCAL MODEL

1.2.1. *The space of basic functions*

We begin by showing that the space of basic functions is a closed subspace in the space of all smooth functions.

Let F be a foliation on the manifold M. If (U, φ) is an adapted chart, with coordinates $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$, and L is the leaf passing through $p = \varphi^{-1}(x_0, y_0) \in U$, we call $P_U(p) = \varphi^{-1}(\mathbf{R}^m \times y_0)$ the plaque of p in U, that is the connected component of p in $L \cap U$.

An open set $V \subset U$ is said to be adapted if $\varphi(V)$ is an open cube in $\mathbf{R}^m \times \mathbf{R}^n$. When its closure $\overline{V}$ is contained in U, we can endow $\overline{V}$ with a structure of differentiable manifold with corners modeled on open sets of a quadrant of $\mathbf{R}^{m+n}$ [7]. Then the inclusion $\overline{V} \hookrightarrow M$ is a proper differentiable map. Moreover a map $\lambda\colon \overline{V} \to \mathbf{R}$ is smooth if and only if there exists an open set $\overline{V} \subset W \subseteq U$, and a smooth function $\tilde{\lambda}\colon W \to \mathbf{R}$, that extends λ. Let us define the smooth functions

$$\partial\lambda/\partial x_i = \partial\tilde{\lambda}/\partial x_i \mid_{\overline{V}}.$$

DEFINITION 3. A smooth function $\lambda\colon \overline{V} \to \mathbf{R}$ defined in the manifold with corners $\overline{V}$ is called *basic* iff $\partial\lambda/\partial x_i = 0$ for all $i = 1, \ldots, m$.

We remark that this definition does *not* mean that λ extends to a usual basic function on an open set, that is a smooth map which is constant along the leaves of the induced foliation. On the other hand, the restriction $C_b^\infty(\overline{V}) \to C_b^\infty(V)$ between the spaces of basic functions is well defined.

LEMMA 4. *$C_b^\infty(\overline{V})$ is a closed subspace of $C^\infty(\overline{V})$, with the WO^1-topology.*

Proof. The maps $F_i: (C^\infty(\overline{V}), WO^1) \to (C^\infty(\overline{V}), WO^0)$, $1 \leqslant i \leqslant m$, given by $F_i(\lambda) = \partial\lambda/\partial x_i$, are continuous at any $\lambda \in C^\infty(\overline{V})$. In fact, let us consider the atlas $\{(Z_t, \phi_t) \mid t = 1, \ldots, 2^{m+n}\}$ on $\overline{V}$ obtained just by copying via φ the usual differentiable structure of the closed cube. We can choose compact sets $K_t \subset Z_t$, whose interiors in $\overline{V}$ cover $\overline{V}$. Then a basic neighbourhood of $F_i(\lambda)$ in the WO^0-topology is [7]

$$W^0(\epsilon) = \{\mu \in C^\infty(\overline{V}) : |\mu(p) - F_i(\lambda)(p)| < \epsilon_t, \forall p \in K_t, \forall t\}$$

where $\epsilon = \{\epsilon_t\}$ is any family of positive real numbers. But

$$W^1(\epsilon) = \{\mu \in C^\infty(\overline{V}) : |\mu\phi_t^{-1}(z) - \lambda\phi_t^{-1}(z)| < \epsilon_t,$$
$$\|d(\mu\phi_t^{-1})_z - d(\lambda\phi_t^{-1})_z\| < \epsilon_t, \forall z \in \phi_t(K_t), \forall t\}$$

is a neighbourhood of λ in the WO^1-topology, such that $F_i(W^1(\epsilon)) \subseteq W^0(\epsilon)$.

Then $C_b^\infty(\overline{V}) = \bigcap_{i=1}^m F_i^{-1}(0)$ is a closed subspace. □

THEOREM 5. *Let (M, F) be a foliated manifold. The space $C_b^\infty(M)$ of basic functions is closed in $C^\infty(M)$ for the WO^1-topology, so for any other finer.*

Proof. Let $\{(U_i, \varphi_i) \mid i \in I\}$, $\{(V_j, \varphi_j) \mid j \in J\}$ be two adapted atlases in M, such that for any $j \in J$ there exists $i \in I$ with $\overline{V}_j \subset U_i$. Since the inclusions $\overline{V}_j \hookrightarrow M$ are smooth proper maps, the induced restriction maps $r_j: C^\infty(M) \to C^\infty(\overline{V}_j)$ are continuous for the WO^1-topology [7]. Then

$$C_b^\infty(M) = \bigcap_j r_j^{-1}(C_b^\infty(V_j)) = \bigcap_j r_j^{-1}(C_b^\infty(\overline{V}_j))$$

is closed by Lemma 4. □

1.2.2. *Foliate vector fields along a map*

From now on, (M, F) and (M', F') will be foliated manifolds, $f: M' \to M$ a smooth map sending leaves into leaves.

Let f^*TM be the pull-back of the tangent bundle $\tau: TM \to M$. Then a section of f^*TM is any smooth map $X: M' \to TM$ such that $X_p \in T_{f(p)}M$ for all $p \in M'$. We call it a *vector field along* f. This notion is of importance for applications in Lagrangian Mechanics, see for example [2] or [8].

DEFINITION 6. We say that $X \in \Gamma(f^*TM)$ is *foliate* if for any basic function $\lambda \in C_b^\infty(U)$ defined in the open set $U \subset M$ with $f^{-1}(U) \neq \emptyset$, the function $X\lambda: f^{-1}(U) \to \mathbf{R}$ given by $(X\lambda)(p) = (d\lambda)_{f(p)}(X_p)$ is basic too.

Remark: When $f = \mathrm{id}$, our definition is equivalent to that of foliate vector field on a foliated manifold that is an infinitesimal transformation of the foliation [11]. Also, let us remark that the map $f: M' \to M$ induces morphisms

$$\chi(M') \xrightarrow{f_*} \Gamma(f^*TM) \xleftarrow{f^\sharp} \chi(M)$$

given by $f_*(X) = (df)(X)$ and $f^\sharp(Y) = Y \circ f$, which in general are not isomorphisms. The interesting thing is that both maps send foliate vector fields into foliate vector fields along f, as the reader can easily verify.

Let $(U, (x, y))$ be an adapted chart in M. Then we have the local expression

$$X\,|_{f^{-1}(U)} = \sum_{i=1}^{m} a_i\left(\frac{\partial}{\partial x_i} \circ f\right) + \sum_{\alpha=1}^{n} b_\alpha\left(\frac{\partial}{\partial y_\alpha} \circ f\right) \tag{1}$$

of $X \in \Gamma(f^*TM)$, with coefficients in $C^\infty(f^{-1}(U))$. We have

PROPOSITION 7. *The vector field X along f is foliate if and only if for any pair of adapted open sets $U' \subset M'$, $U \subset M$ with $f(U') \subset U$, the transverse components $b_\alpha = Xy_\alpha$ in the local expression of $X\,|_{U'}$ in (1) are constant along the plaques.*

Now, if $V \subset U'$ is an adapted open set such that $\overline{V} \subset U'$, it is easy to see that any vector field X along f defined on $\overline{V}$ can be extended to an open set containing $\overline{V}$, just by extending the coordinate functions of its local expression. This allows us to slightly generalize our Definition 6 to the manifold with corners $\overline{V}$ by saying that X is *foliate* if $b_\alpha \in C_b^\infty(\overline{V})$, $\forall \alpha = 1, \ldots, m$, in the local expression (1). We remark that in general a foliate vector field along f on $\overline{V}$ does *not* extend to a foliate vector field along f in an open set, while its restriction to V *is* foliate.

LEMMA 8. *The space $\Gamma^F(\overline{V}, f^*TM)$ of foliate vector fields along f defined on $\overline{V}$ is closed in the space $\Gamma(\overline{V}, f^*TM)$ of all vector fields along f on $\overline{V}$, for the WO^1-topology.*

Proof. Since U' is a trivializing open set of the vector bundle $\tau_f: f^*TM \to M'$, we consider the component functions ψ_α, $1 \leqslant \alpha \leqslant n$, of the trivializing map $\psi: \tau_f^{-1}(U') \to U' \times \mathbf{R}^m \times \mathbf{R}^n$. Then the maps

$$(\psi_\alpha)_\sharp: \Gamma(\overline{V}, \tau_f^{-1}(U')) \to C^\infty(\overline{V})$$

are continuous for the WO^1-topology (see 1.1), and we have $(\psi_\alpha)_\sharp(X) = b_\alpha$ in the local expression (1) of $X \in \Gamma(\overline{V}, \tau_f^{-1}(U'))$.

By Lemma 4, $(\psi_\alpha)_\sharp^{-1}(C_b^\infty(\overline{V}))$ is closed in $\Gamma(\overline{V}, \tau_f^{-1}(U'))$ and

$$\Gamma^F(\overline{V}, \tau_f^{-1}(U')) = \bigcap_{\alpha=1}^{n} (\psi_\alpha)_\sharp^{-1}(C_b^\infty(\overline{V}))$$

is closed too. Since $\tau_f^{-1}(U')$ is a submanifold of f^*TM, the map $i_\sharp: \Gamma(\overline{V}, \tau_f^{-1}(U')) \to \Gamma(\overline{V}, f^*TM)$, induced by the inclusion, is a homeomorphism [7]. Thus $\Gamma^F(\overline{V}, f^*TM)$ is a closed subspace of $\Gamma(\overline{V}, f^*TM)$. □

THEOREM 9. *Let (M, F), (M', F') be foliated manifolds, $f: M' \to M$ a smooth map sending leaves into leaves. The space $\Gamma^F(f^*TM)$ of foliate vector fields along f is closed in the space $\Gamma(f^*TM)$ of all smooth vector fields along f, for the WO^1-topology, so for any other finer.*

The proof is analogous to that of Theorem 5.

COROLLARY 10. *The space $\Gamma_c^F(f^*TM)$ of foliate vector fields along f with compact support is a closed subspace of $\Gamma_c(f^*TM)$ with the $\mathcal{FD}$-topology. Then it is an LF-space.*

Let us remark that for f the identity of a foliated manifold M, we obtain in this way that the space $\chi_c(M, F)$ of foliate vector fields with compact support is an LF-space —a closed subspace of $\chi_c(M)$.

1.3. THE STRUCTURE OF MANIFOLD

1.3.1. *Local additions*

Let $\mathcal{E}: \mathcal{S} \subset TM \to M$ be a smooth map defined on an open neighbourhood of the zero section of the tangent bundle $\tau: TM \to M$. We say that $\mathcal{E}$ is a *local addition* [10] if

(i) $\mathcal{E}(0_p) = p$ for all $p \in M$;
(ii) $(\tau, \mathcal{E}): \mathcal{S} \subset TM \to M \times M$ is a diffeomorphism onto some open neighbourhood $\mathcal{D}$ of the diagonal.

In any Riemannian manifold the exponential map defines a local addition in a suitable open set $\mathcal{S} \subset TM$. More generally, we shall prove in Section 2 that for any Riemannian foliation F there exists a local addition which is adapted to F, in the sense that it induces around any point a local addition in a small transversal to the foliation. More precisely, if $(U, (x, y))$ is an adapted chart, let $\pi_U = (y_1, \dots, y_n)$ be the submersion from U onto the local transverse manifold $\overline{U} \subset \mathbf{R}^n$, which locally defines the foliation.

DEFINITION 11. A local addition $\mathcal{E}$ in a foliated manifold is said to be *adapted* if for any adapted chart $\pi_U: U \to \overline{U}$ there exist an open neighbourhood $\mathcal{S}_U \subset \mathcal{S}$ of the zero section in TU and a local addition $\overline{\mathcal{E}}_U: \overline{\mathcal{S}}_U = (\pi_U)_*(\mathcal{S}_U) \to \overline{U}$ such that $\mathcal{E}(\mathcal{S}_U) \subset U$ and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_U & \xrightarrow{\mathcal{E}} & U \\ (\pi_U)_* \downarrow & & \downarrow \pi_U \\ \overline{\mathcal{S}}_U & \xrightarrow{\overline{\mathcal{E}}_U} & \overline{U}. \end{array} \qquad (2)$$

1.3.2. *An explicit atlas*

In [9] and [10] a manifold structure is given to the space $C^\infty(M', M)$ by means of a (unadapted) local addition $\mathcal{E}$. More precisely, let $\mathcal{U}_f$ be the open neighbourhood of $f \in C^\infty(M', M)$ given by

$$\mathcal{U}_f = \{h \in C^\infty(M', M) \mid h \sim f \quad \text{and} \quad (f(p), h(p)) \in \mathcal{D}, \ \forall p \in M'\},$$

where $h \sim f$ means that f, h coincide outside some compact set (see 1.1).

Then we define

$$\varphi_f : \mathcal{U}_f \to \Gamma_c(f^*TM)$$

as $\varphi_f(h)(p) = X_p$, where X_p is the only vector in $\mathcal{S} \cap T_{f(p)}M$ such that $(\tau, \mathcal{E})(X_p) = (f(p), h(p))$. The vector field $X = \varphi_f(h)$ along f is smooth because $X = (\tau, \mathcal{E})^{-1} \circ (f, h)$.

Let (M, F), (M', F') be foliated manifolds. We shall see now that when we use an adapted local addition in M the manifold structure above induces another one on the subspace $C_F^\infty(M', M)$ of foliation preserving maps.

PROPOSITION 12. *Let $f \in C_F^\infty(M', M)$ and suppose that the local addition $\mathcal{E}$ on M is adapted to the foliation. Then a map $h \in \mathcal{U}_f$ sends leaves into leaves if and only if the vector field $X = \varphi_f(h)$ along f is foliate.*

Proof. We can suppose that $\mathcal{S}$ is the union of the $\mathcal{S}_U$ for some maximal atlas of adapted charts $\pi: U \to \overline{U}$ defining the foliation F, where $\mathcal{S}_U$ is given by Definition 11.

Let $p \in M'$ and $\pi: U \to \overline{U}$ an adapted chart in M such that $X_p \in \mathcal{S}_U$. Since the map $X: M' \to \mathcal{S}$ is smooth, there is an adapted coordinate open neighbourhood U' of p such that $f(U') \cup h(U') \subset U$.

If the map h sends leaves into leaves and $q \in U'$ is in the same plaque than p, then $\pi f(p) = \pi f(q)$ and $\pi h(p) = \pi h(q)$. For the local addition in the transverse manifold this means by the relation $\pi \circ \mathcal{E} = \overline{\mathcal{E}}_U \circ \pi_*$ in (2) that

$$(\tau, \overline{\mathcal{E}}_U)\pi_{*f(p)}X_p = (\pi f(p), \pi h(p)) = (\tau, \overline{\mathcal{E}}_U)\pi_{*f(q)}X_q,$$

hence $\pi_{*f(p)}X_p = \pi_{*f(q)}X_q$. Then the transverse components of X are constant along the plaques and we conclude that X is a foliate vector field along f (see 1.2.2).

Conversely, if $X = \varphi_f(h)$ is foliate, and $q \in U'$ is in the same plaque than p, we have $\pi_{*f(p)}X_p = \pi_{*f(q)}X_q$. Moreover $\pi f(p) = \pi f(q)$ because $f \in C_F^\infty(M', M)$. The same computation as above shows that $h(p)$ and $h(q)$ are in the same plaque of U'. Then h is a foliation preserving map. □

As a consequence, for each $f \in C_F^\infty(M', M)$, the map

$$\varphi_f : \mathcal{U}_f \cap C_F^\infty(M', M) \to \Gamma_c^F(f^*TM)$$

is a homeomorphism onto an open neighbourhood of the zero section in the space

$\Gamma_c^F(f^*TM)$ of foliate vector fields along f with compact support. In other words, we have proved

THEOREM 13. *Let (M, F), (M', F') be foliated manifolds, and suppose that F admits an adapted local addition. Then the space $C_F^\infty(M', M)$ of maps sending leaves into leaves admits a manifold structure, modeled on LF-spaces.*

We remark that this manifold structure does not depend on the adapted local addition ([10], page 91).

COROLLARY 14. *Let F be a foliation on the not necessarily compact manifold M. If F admits an adapted local additon, then the group $\mathrm{Diff}(M, F)$ of foliation preserving diffeomorphisms is a Lie group modeled on the Lie algebra $\chi_c(M, F)$.*

Proof. $\mathrm{Diff}(M, F)$ is an open subset of $C_F^\infty(M, M)$ because $\mathrm{Diff}(M)$ is open in $C^\infty(M, M)$. Then, if F admits an adapted local additon, $\mathrm{Diff}(M, F)$ inherits a structure of manifold from $\mathrm{Diff}(M)$, which is a Lie group [9]. Moreover, when $f \in \mathrm{Diff}(M, F)$, the space $\Gamma_c^F(f^*TM)$ is isomorphic to the Lie algebra $\chi_c(M, F)$ of foliate vector fields with compact support, which is a closed subspace of $\chi_c(M)$. □

2. The Adapted Local Addition in Riemannian Foliations

The foliation F is said to be *Riemannian* if there is some bundle-like metric in the ambient manifold, that is a Riemannian metric g such that the transverse part of g is constant along the leaves. Introduced by B. Reinhart [15], this is a natural framework for doing Riemannian geometry in foliated manifolds. We refer the reader to P. Molino's book [11] or Ph. Tondeur's survey in [16] for an account of the theory.

In this Section we shall prove that any Riemannian foliation admits an adapted local addition.

2.1. GEODESICS IN THE LEAVES

We begin by establishing some properties of the geodesics curves in a foliated manifold that we need in order to construct an adapted local addition.

Let F be any foliation (Riemannian or not) on the Riemannian manifold (M, g). We denote α_X the geodesic curve in M with initial conditions $p \in M$ and $X \in T_pM$. For each open set $W \subset M$, let $\Omega(W, r)$ be the open set in TM given by

$$\Omega(W, r) = \{X \in T_pM \mid p \in W, \ \|X\| < r\}.$$

Let (U, φ) be a coordinate chart in M and $p \in U$. There exist an open neighbourhood $V \subset U$ of p and a positive number $\epsilon > 0$ such that $\exp(\Omega(V, \epsilon)) \subset U$, that is for all $X \in \Omega(V, \epsilon)$ the geodesic α_X is defined on the interval $[0, 1]$ and contained in U.

In the same way, we can consider the induced metric g^L in the leaf L passing through p, the geodesic a_T in the leaf with initial conditions $p \in L$, $T \in T_pL$ and the open set $\Omega_F(W, r) = \Omega(W, r) \cap TF$ in the bundle tangent to F.

PROPOSITION 15. *Let* (U, φ) *an adapted chart in* (M, F). *Then for each point* $p \in U$ *there exist an adapted neighbourhood* $W \subset U$ *of* p *and* $r > 0$ *such that for all* $T \in \Omega_F(W, r)$, *the geodesic* a_T *in the leaf is defined in* $(-2, 2)$ *and contained in* U. *Moreover, the map* $(t, T) \in (-2, 2) \times \Omega_F(W, r) \mapsto a_T(t) \in U$ *is smooth.*

Proof. We define the smooth functions $\widetilde{\Gamma}^k_{ij}: \varphi(U) \to \mathbf{R}$, $1 \leqslant i, j, k \leqslant m$, as

$$\widetilde{\Gamma}^k_{ij}(x, y) = \frac{1}{2}\sum_{l=1}^{m} g^{lk}(x, y)\left(\frac{\partial g_{jl}}{\partial x_i}(x, y) + \frac{\partial g_{il}}{\partial x_j}(x, y) - \frac{\partial g_{ij}}{\partial x_l}(x, y)\right),$$

where $(g^{lk}) = (g_{lk})^{-1}$. If $\varphi(p) = (x_0, y_0)$, then $g^L_{ij}(x) = g_{ij}(x, y_0)$, and the Christoffel symbols for the metric in the leaf L of p are $\Gamma^k_{ij}(x) = \widetilde{\Gamma}^k_{ij}(x, y_0)$.

Moreover, since a curve $a: [0, 1] \to U$ is contained in the plaque $P_U(p)$ if and only if $\varphi(a(t)) = (x(t), y_0)$, we have that the geodesic equation in the leaves is

$$\frac{d^2x_k}{dt^2} + \sum_{i,j=1}^{m} \widetilde{\Gamma}^k_{ij}(x, y)\frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \qquad k = 1, \ldots, m = \dim F,$$

$$\frac{dy_\alpha}{dt} = 0, \qquad \alpha = 1, \ldots, n = \operatorname{codim} F,$$

where $\varphi(a(t)) = (x(t), y(t)) \in \mathbf{R}^{m+n}$.

The remainder of the proof follows the classical argument (see for example [1]). □

Thus we can define on the open set $\Omega_F(W, r) \subset TF$ the exponential map along the leaf $\exp^F(T) = a_T(1) \in U$, and we have $\exp^F(\Omega_F(W, r)) \subset U$.

COROLLARY 16. *Let* U *be an adapted chart. Then, for each* $p \in U$ *there exist* $r > 0$ *and adapted open neighbourhoods* $W \subset V \subset U$ *of* p *such that* $\exp^F(\Omega_F(W, r)) \subset V$ *and* $\exp(\Omega(V, r)) \subset U$.

2.2. RIEMANNIAN SUBMERSIONS

Let F be a Riemannian foliation on the manifold M. We shall use the following property of bundle-like metrics. For a given covering of M by adapted charts $\pi_U: U \to \overline{U}$, there is a Riemannian metric on each transverse manifold $\overline{U}$ such that π_U is a Riemannian submersion, and the transverse change of coordinates $\gamma_{VU}: \pi_U(U \cap V) \to \pi_V(U \cap V)$ with $\pi_V = \gamma_{VU} \circ \pi_U$ are isometries.

It is well known that if a geodesic cuts horizontally one leaf, then it cuts horizontally all the leaves encountering it [15]. Also, since $\pi: U \to \overline{U}$ is a Riemannian submersion, the projection of a horizontal geodesic in U is a geodesic in $\overline{U}$ and that the geodesics in the base can be locally lifted to horizontal geodesics [3].

PROPOSITION 17. *Let F be a foliated manifold endowed with a bundle-like metric g. Let (U, φ) be an adapted chart. Then for each $p \in U$, there exist an adapted open neighbourhood $W \subset U$ of p and $\epsilon > 0$ such that for all $q, \widetilde{q} \in W$ there is an unique horizontal geodesic $\alpha: [0, 1] \to U$ of length $\|\alpha'(0)\| < \epsilon$ joining q and the plaque of $\widetilde{q}$, that is $\alpha(0) = q$, $\alpha(1) \in P_U(\widetilde{q})$.*

Proof. Let us choose $V \subset U$ adapted open neighbourhood of p and $\delta > 0$ such that $\exp(\Omega(V, \delta)) \subset U$. Let $\overline{\alpha}: [0, 1] \to \overline{U}$ be a geodesic with $\overline{\alpha}(0) \in \pi(V)$ and $\|\overline{\alpha}'(0)\| < \delta$. Then for each $q \in \pi^{-1}(\overline{\alpha}(0)) \cap V$, we consider the horizontal lift $H \in T_qU$ of the tangent vector $\overline{\alpha}'(0)$. Since $H \in \Omega(V, \delta)$, the geodesic α_H is defined in [0, 1] and contained in U. Uniqueness imposes $\pi \circ \alpha_H = \overline{\alpha}$.

Let us take a neighbourhood $B \subset \overline{U}$ of $\pi(p)$, and $\epsilon > 0$ such that any two points $\pi(q), \pi(\widetilde{q}) \in B$ can be joined by an unique geodesic $\overline{\alpha}: [0, 1] \to \overline{U}$ of length $\|\overline{\alpha}'(0)\| < \epsilon$ [1]. We can suppose $\epsilon \leqslant \delta$. Then $W = \pi^{-1}(B) \cap V$ satisfies the desired property. □

2.3. THE ADAPTED LOCAL ADDITION

From now on, F will be a Riemannian foliation on the manifold M and g a bundle-like metric for F. In the following paragraphs, we prove that F admits an adapted local addition.

2.3.1. *The map* $\mathcal{E}: \Omega(W, r) \to U$

Let $\pi: U \to \overline{U}$ be a Riemannian submersion locally defining F. From Corollary 16 we know that U can be covered by adapted open sets W such that for some $r > 0$ and some adapted open set V both depending on W one has $\exp^F(\Omega_F(W, r)) \subset V$ and $\exp(\Omega(V, r)) \subset U$. For each adapted open set W, we are going to define a smooth map $\mathcal{E}: \Omega(W, r) \to U$ (see Figure 1).

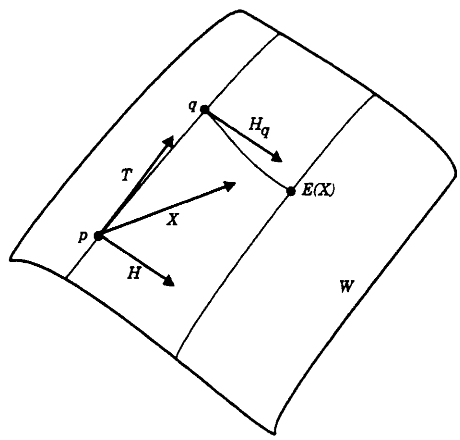

Figure 1. The map $\mathcal{E}$.

Let $X \in \Omega(W, r)$ be a tangent vector in a point $p \in W$ with $\|X\| < r$. If we decompose it in its tangent and normal parts, $X = T + H$, it follows from Corollary 16 that the point $q = \exp_p^F(T)$ belongs to $V \bigcap P_U(p)$.

Now, let $H_q \in T_qM$ be the only horizontal vector such that $\pi_{*q}H_q = \pi_{*p}H$. Then $H_q \in \Omega(V, r)$, so the corresponding horizontal geodesic in the manifold allows us to define $\mathcal{E}(X) = \exp_q(H_q) \in U$.

The map $\mathcal{E}: \Omega(W, r) \to U$ is smooth because the horizontal lift $l: \pi^* T\overline{U} \to TU$ is a smooth map.

In the next paragraphs we will prove that $\mathcal{E}$ serves to define an adapted local addition. Let $\mathcal{E}_p$ be the restriction of $\mathcal{E}$ to $\Omega(W, r) \cap T_pM = B(0_p, r)$ for $p \in W$.

PROPOSITION 18. *For each $p \in W$, there exists $\epsilon_p > 0$ such that $\mathcal{E}_p$ maps $B(0_p, \epsilon_p) \subset T_pM$ diffeomorphically onto an open neighbourhood of p.*

Proof. We will show that in some neighbourhood of p the map $\mathcal{E}_p$ admits a right inverse map $\mathcal{L}_p$. Then the result follows from the inverse function theorem.

The construction of $\mathcal{L}_p(q) = T(q) + H(q)$ is as follows (see Figure 2).

First, without loss of generality we can suppose that the plaque $P_W(p)$ is a g^L-normal neighbourhood of p, that is to say the exponential map $\exp^F$ along the leaf is a diffeomorphism from $P_W(p)$ onto $\{T \in T_pF \mid \|T\| < \delta_T\}$ for some $\delta_T > 0$.

Second, we can choose $\widetilde{W} \subset W$ and $\delta_H > 0$ as in Proposition 17.

Now, if $q \in \widetilde{W}$, let $H(q) = \alpha'(0)$ where $\alpha: [0, 1] \to W$ is the only horizontal geodesic such that $\alpha(0) = p$, $\alpha(1) \in P_W(q)$ and $\|\alpha'(0)\| < \delta_H$.

On the other hand, let $\beta: [0, 1] \to W$ be the only horizontal geodesic such that $\beta(0) = q$, $\beta(1) \in P_W(p)$ and $\|\beta'(0)\| < \delta_H$. Then we define $T(q) = a'(0) \in T_pM$ where $a: [0, 1] \to P_W(p)$ is the geodesic in the leaf joining p and $\beta(1)$ with $\|a'(0)\| < \delta_T$.

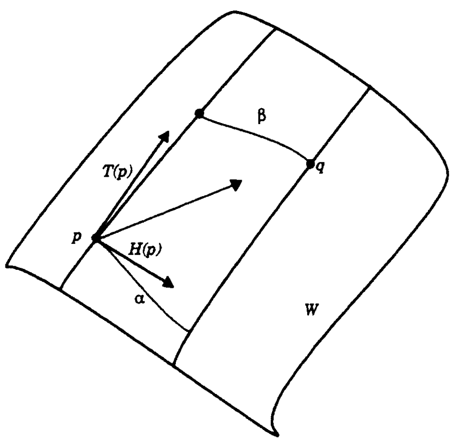

Figure 2. The map $\mathcal{L}_p$

Notice that we can suppose $\delta_T, \delta_H < r/2$, so we have $\mathcal{L}_p(\widetilde{W}) \subset B(0_p, r)$. From the definition of $\mathcal{E}$, we know that $\mathcal{E}_p(T(q) + H(q))$ is the endpoint $\exp_z(H_z)$ of the horizontal geodesic with initial conditions $z = \exp_p^F(T(q)) = \alpha(1) = \beta(1)$ and H_z the only horizontal vector such that $\pi_{*z}H_z = \pi_{*p}H(q) = \pi_{*p}\alpha'(0)$. Since the projection of the horizontal geodesic β^{-1} onto the transverse manifold $\overline{U}$ coincides with that of the horizontal geodesic α joining p and the plaque of q, we have that $H_z = (\beta^{-1})'(0)$, hence $\mathcal{E}_p(T(q) + H(q)) = \exp_z(H_z) = q$.

Finally, it is not difficult to prove that $\mathcal{L}_p$ is a smooth map. □

2.3.2. *The map* $(\tau, \mathcal{E})\colon \Omega(W, r) \to U \times U$

Recall that the map $\mathcal{E}$ has been defined on a class of open sets W covering the adapted chart $\pi\colon U \to \overline{U}$. Let $\tau\colon TM \to M$ be the tangent bundle.

PROPOSITION 19. *We can suppose that the map* $(\tau, \mathcal{E})\colon \Omega(W, r) \to U \times U$ *is a diffeomorphism onto an open subset of* $U \times U$.

Proof. In fact we will prove that for each $p \in W$ there exists some open neighbourhood $\widetilde{W} \subset W$ and some $\epsilon < r$ such that $(\tau, \mathcal{E})$ is a diffeomorphism between $\Omega(\widetilde{W}, \epsilon)$ and some open neighbourhood of $(p, p) \in U \times U$.

First, since W is a trivializing open set of the tangent bundle, we can write

$$(\tau, \mathcal{E})_{*0_p}\colon T_pW \times \mathbf{R}^{m+n} \to T_pU \times T_pU$$

which is given by $(\tau, \mathcal{E})_{*0_p}(X, v) = (X, X + (\mathcal{E}_p)_{*0_p}(v))$. But it follows from Proposition 18 that the map

$$(\mathcal{E}_p)_{*0_p}\colon \mathbf{R}^{m+n} \cong T_{0_p}(T_pW) \xrightarrow{\sim} T_pU$$

is an isomorphism, so the map $(\tau, \mathcal{E})_{*0_p}$ is an isomorphism $T_{0_p}\Omega(W, r) \xrightarrow{\sim} T_{(p,p)}(U \times U)$. Then there exists an open neighbourhood $\widetilde{\Omega}$ of 0_p in $\Omega(W, r)$ where $(\tau, \mathcal{E})$ is a diffeomorphism onto an open neighbourhood of (p, p).

Finally, a standard argument as in [1], shows that $\widetilde{\Omega}$ contains some $\Omega(\widetilde{W}, \epsilon)$ as stated. □

For the transverse manifold $\overline{U}$ we denote $\overline{\tau}\colon T\overline{U} \to \overline{U}$ the tangent bundle and $\overline{\mathcal{E}}$ the exponential map.

PROPOSITION 20. *We can suppose that the map* $(\overline{\tau}, \overline{\mathcal{E}})$ *is a diffeomorphism from* $\Omega(\pi(W), r)$ *onto an open subset of* $\overline{U} \times \overline{U}$. *Moreover, the following diagram is commutative*

$$\begin{array}{ccc} \Omega(W, r) & \xrightarrow{\mathcal{E}} & U \\ \pi_* \downarrow & & \downarrow \pi \\ \Omega(\pi(W), r) & \xrightarrow{\overline{\mathcal{E}}} & \overline{U}. \end{array}$$

Proof. It suffices to restrict the domain of definition in order to have $\Omega(\pi(W), r) \subseteq \overline{\mathcal{S}}$ with $\overline{\mathcal{E}}: \overline{\mathcal{S}} \to \overline{U}$ a local addition. The commutativity follows directly from our definition of $\mathcal{E}$. □

2.3.3. *The map* $\mathcal{E}: \mathcal{S} \subset TM \to M$

In the preceding subsections we have considered open subsets W which cover a given adapted coordinate open set U. Also, for each W we have defined a map $\mathcal{E}_W: \Omega(W, r_W) \to U \times U$ whose properties were established in Propositions 19 and 20. Let us take

$$\mathcal{S}_U = \bigcup_W \Omega(W, r_W)$$

which is an open neighbourhood of the zero section in TU. Since the maps $\mathcal{E}_W$ have been defined by means of the maps $\exp^F$, exp, τ and π, which only depend on U, it is clear that we have a well defined map $\mathcal{E}_U: \mathcal{S}_U \subset TU \to U$. Moreover, since $\pi: U \to \overline{U}$ is a Riemannian submersion, it follows that $\Omega(\pi(W), r_W) = \pi_*(\Omega(W, r_W))$. Then, from Proposition 20, the exponential map $\overline{\mathcal{E}}_U: \overline{\mathcal{S}}_U = \pi_*(\mathcal{S}_U) \to \overline{U}$ is a local addition in the transverse manifold $\overline{U}$, and that the diagram (2) in 1.3.1 is commutative.

PROPOSITION 21. *Let* $\mathcal{S}$ *be the union of the* $\mathcal{S}_U$ *for some maximal atlas of adapted charts* $\pi_U: U \to \overline{U}$ *defining the Riemannian foliation* F. *The combined map* $\mathcal{E}: \mathcal{S} \to M$ *defined by* $\mathcal{E}(X) = \mathcal{E}_U(X)$ *if* $X \in \mathcal{S}_U$ *is a smooth adapted local addition.*

Proof. Since each map $(\tau_U, \mathcal{E}_U)$ is an injective local diffeomorphism, we must only check that $\mathcal{E}$ is well defined, that is $\mathcal{E}_U(X) = \mathcal{E}_V(X)$ when $X \in \mathcal{S}_U \cap \mathcal{S}_V$.

If $X = T + H \in T_pM$, let $q = \exp_p^F(T) \in U \cap V$. By definition, $\mathcal{E}_U(X) = \exp_q(H')$ and $\mathcal{E}_V(X) = \exp_q(H'')$, where H', H'' are the only horizontal vectors in T_qM such that, respectively, $(\pi_U)_{*q}H' = (\pi_U)_{*p}H$ and $(\pi_V)_{*q}H'' = (\pi_V)_{*p}H$.

Let γ_{VU} be the isometry such that $\pi_U = \gamma_{VU} \circ \pi_V$. Since p and q are in the same plaque, we have

$$(\pi_U)_{*q}H'' = (\gamma_{VU})_*(\pi_V)_{*q}H'' = (\gamma_{VU})_*(\pi_V)_{*p}H = (\pi_U)_{*p}H.$$

Then $H' = H''$ and $\mathcal{E}_U(X) = \mathcal{E}_V(X)$. □

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