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Diffeological groups

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Abstract. We introduce the basic concepts of the theory of J. M. Souriau's diffeological spaces. As a particular example we study the space of leaves of a Lie foliation on a compact manifold and its group of diffeomorphisms.

1. Introduction

The notion of *diffeological space* was introduced by J. M. Souriau [19, 20, 21, 22] in the setting of geometric quantization theory, as an attempt of formalizing quantum mechanics. This generalization of the notion of differentiable manifold appeared yet essentially in [18] and [2]. As has been pointed out by some authors [13, 12], there is still a lack of commonly accepted knowledge of these spaces and their applications. Among the latter we can cite the symplectic integration problem for Poisson manifolds [1] (see also [3]); the space of all connections in a principal bundle [11]; the explicit method of construction for the homogeneous contact manifolds in [4]; the symplectic structure of the coadjoint orbits of a subgroup of diffeomorphisms [23] and Gelfand-Fuks cohomology [15].

In this talk we introduce the basic definitions, examples and properties of diffeological spaces, their coverings, tangent bundles, and diffeological groups. Other constructions have been developped by several authors, e.g. fibrations [10], differential forms and exterior bundles [14, 15], or homology groups [8].

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2. Diffeological spaces

Let M be a set. Any set map $U \subset \mathbb{R}^n \to M$ defined on an open set U of some $\mathbb{R}^n$, $n \geq 0$, will be called a *chart* (or plate, or generalized curve). A *diffeology* on M is any collection $\mathcal{P}$ of charts $\alpha \colon U_\alpha \subset \mathbb{R}^{n_\alpha} \to M$ verifying the following axioms:

1. Any constant map $c \colon U \subset \mathbb{R}^n \to M$ belongs to $\mathcal{P}$;

2. If $\alpha \in \mathcal{P}$ is defined on $U \subset \mathbb{R}^n$, and $h \colon V \subset \mathbb{R}^m \to U$ is a smooth map, then $\alpha \circ h \in \mathcal{P}$;

3. Let $\alpha \colon U \subset \mathbb{R}^n \to M$ be a set map such that each $t \in U$ has a neighbourhood U_t with $\alpha_{|U_t} \in \mathcal{P}$. Then $\alpha \in \mathcal{P}$.

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The pair $(M,\mathcal{P})$ will be called a *diffeological space*. Axiom 1 ensures that $\mathcal{P}\neq\emptyset$; Axiom 2 means that we can change coordinates inside $\mathcal{P}$; from Axiom 3 charts can be glued together. Often, we simply denote α_U a chart with domain U.

2.1. Generating sets.

When defining a diffeology, we can consider only a *generating set.* That is, given any family $\mathcal{G}$ of charts on M, we define the diffeology $\mathcal{P}=\langle\mathcal{G}\rangle$ formed by all maps α_U such that $\forall t\in U$, $\exists U_t$ with $\alpha_{|U_t}=\beta\circ h$, for some $\beta\in\mathcal{G}$ and some change of coordinates h.

For instance, the set charts with connected domains of a diffeology is a generating set for this diffeology.

2.2. Differentiable maps.

Let $(M,\mathcal{P})$, $(N,\mathcal{Q})$ be two diffeological spaces. A map $F\colon M\to N$ is said to be *differentiable* if $F\circ\alpha\in\mathcal{Q}$ for all $\alpha\in\mathcal{P}$. Clearly, it suffices to check this condition on a generating set of $\mathcal{P}$.

We denote $\mathcal{D}(M,N)$ the set of differentiable maps between $(M,\mathcal{P})$ and $(N,\mathcal{Q})$. The set of diffeomorphisms of the diffeological space $(M,\mathcal{P})$ –i.e. invertible differentiable maps f such that f^{-1} is also differentiable– is denoted by $\mathrm{Diff}(M)$.

2.3. Examples.

Example 2.1. Topological spaces
For any topological space X we take as charts all the *continuous* maps $\alpha\colon U\subset\mathbb{R}^n\to X$. Then, for two topological spaces X, Y, every continuous map $f\colon X\to Y$ is differentiable in the sense above, but in general, the set $\mathcal{D}(X,Y)$ is larger than the set $C^0(X,Y)$ of continuous maps. The two sets coincide when X,Y are topological manifolds.

Example 2.2. Finite dimensional manifolds
If M is a manifold, $\dim M=m<\infty$, the coordinate charts $\varphi^{-1}\colon\varphi(W)\subset\mathbb{R}^m\to W\subset M$ generate a diffeology, which consist of the usual smooth maps $\alpha\colon U\subset\mathbb{R}^n\to M$, $n\geq 0$.

For two manifolds M,N we have $\mathcal{D}(M,N)=C^\infty(M,N)$, the set of smooth maps in the usual sense.

Example 2.3. Fréchet manifolds [14]
Any smooth manifold modelled on a Fréchet space has a diffeological structure which determines in a unique manner the differentiable structure. Moreover, a diffeological morphism of Fréchet manifolds is a smooth map, that is $\mathcal{D}(M,N)=C^\infty(M,N)$.

Example 2.4. Vector spaces
Let V be a vector space. The *linear* diffeology on V is the diffeology generated by the set of *linear* maps $l\colon\mathbb{R}^n\to V$, $n\geq 0$. We remark that any *multilinear* map $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_k}\to V$ is differentiable too.

This is an example of a colimit diffeology (V is the direct limit of their finite dimensional subspaces). When $\dim V < \infty$, this diffeology coincides with the usual differentiable structure on V.

3. Basic constructions

3.1. Quotients.

One of the interesting aspects of diffeological spaces is that they include quotient spaces such as spaces of orbits of actions and other highly pathological objects as spaces of leaves of foliations, hence generalizing several known ad hoc categories [8]. In general, if $\sim$ is an equivalence relation on the diffeological space $(M,\mathcal{P})$, then we consider on the quotient set $M/\sim$ the diffeology generated by the maps $\pi \circ \alpha$, where $\alpha \in \mathcal{P}$ and π is the canonical projection.

For instance, on $[0,+\infty)$ we can define two different diffeological structures, respectively induced by the quotient maps $\pi\colon \mathbb{R} \to [0,+\infty)$, $\pi(x) = |x|$, or $\pi\colon \mathbb{R}^2 \to [0,+\infty)$, $\pi(v) = \|v\|$. They are not diffeomorphic [10].

We have the following universal property.

Proposition 3.1. *A map $F\colon M/\sim \to (N,\mathcal{Q})$ is differentiable if and only if $F \circ \pi$ is differentiable.*

For instance, let $(M,\mathcal{F}),(N,\mathcal{G})$ be two foliated manifolds and let $M \to N$ be a smooth map sending leaves into leaves. Then the induced map $M/\mathcal{F} \to N/\mathcal{G}$ between the spaces of leaves is differentiable in the diffeological sense.

3.2. Submanifolds.

Let $(M,\mathcal{P})$ be a diffeological space. On any subset $N \subset M$ we have an induced diffeology when we consider the charts $\alpha \in \mathcal{P}$, $\alpha\colon U \to M$ such that $\alpha(U) \subset N$.

Hence all submanifolds are *weakly embedded*, in the sense that a map $M' \to N \subset M$ is differentiable iff $M' \to M$ is differentiable.

3.3. Products.

Let $(M,\mathcal{P})$, $(N,\mathcal{Q})$ be diffeological spaces. We consider on $M \times N$ the diffeology generated by the maps $\alpha \times \beta\colon U \times V \to M \times N$, where $\alpha_U \in \mathcal{P}$, $\beta_V \in \mathcal{Q}$.

Clearly, the projections are differentiable, and a map $M' \to M \times N$ is differentiable iff its component functions are differentiable.

3.4. Colimits.

For any collection $(M_i,\mathcal{P}_i)$ of diffeological spaces we can consider the disjoint union $M = \coprod M_i$, with the diffeology generated by the maps $\lambda \circ \alpha$, where $\alpha \in \mathcal{P}_i$ and $\lambda_i\colon M_i \to M$ is the canonical inclusion.

More generally, let $\lambda_{ij}\colon M_i \to M_j$ be a directed system of diffeological spaces. We take on $M = \lim M_i = (\coprod M_i)/\sim$ the quotient of the diffeology above by the usual relation $\lambda_{ik}(x_i) = \lambda_{jk}(x_j)$.

3.5. Functional diffeology.

Let $\mathcal{D}(M,N)$ be the set of differentiable maps between two diffeological spaces. A generating set of the so-called *functional diffeology* on $\mathcal{D}(M,N)$ is formed by all maps $\alpha\colon U \subset \mathbb{R}^n \to \mathcal{D}(M,N)$ such that the map

$$\alpha^\wedge\colon U \times M \to U \times N$$

given by $\alpha^\wedge(t,x) = (t, \alpha(t)(x))$ is differentiable.

We have the folllowing nice property:

Proposition 3.2. (Exponential law) *The functional diffeological spaces*

$$\mathcal{D}(M, \mathcal{D}(N,P)) \cong \mathcal{D}(M \times N, P)$$

are diffeomorphic.

Proof. The diffeomorphism ϕ is given by $\phi(F)(x,y) = F(x)(y)$ with inverse map $\phi^{-1}(G)(x)(y) = G(x,y)$. From the definitions of the product and functional diffeologies, we have that for any chart α_U in $\mathcal{D}(M,\mathcal{D}(N,P))$, and charts β_V in M and γ_W in N, the map

$$(t,s,r) \in U \times V \times W \mapsto \alpha(t)(\beta(s))(\gamma(r)) \in P$$

is differentiable. Then $\phi \circ \alpha$ is differentiable, which shows that ϕ is differentiable.

Conversely, if ξ_Ω is a chart in $\mathcal{D}(M \times N, P)$, and β_V, γ_W are as above, then the map

$$(p,s,r) \in \Omega \times V \times W \mapsto \xi(p)(\beta(s),\gamma(r)) \in P$$

is differentiable, what means that $\phi^{-1} \circ \xi$ is a chart. Hence ϕ^{-1} is differentiable. ∎

4. Diffeological groups

A *diffeological group* is a diffeological space $(M,\mathcal{P})$ endowed with a group structure such that the division map $\delta\colon M \times M \to M$, $\delta(x,y) = xy^{-1}$, is differentiable.

From the definition of the subspace diffeology given in 3.2, it follows that any subgroup of a diffeological group is a diffeological group.

4.1. Examples.

According to Souriau, any group admits a diffeology which turns it into a diffeological group. Here we focus on the point that in many standard situations there is a natural diffeology compatible with the group structure.

Example 4.1. Lie groups
A (usual) Lie group is a diffeological group with respect to its manifold diffeology.

Example 4.2. The general linear group
$GL(\infty)$ (real or complex) is a diffeological group when endowed with the colimit diffeology defined by the directed system $GL(1) \to GL(2) \to GL(3) \cdots$. Consequently, the subgroups $O(\infty)$, $SO(\infty)$, $U(\infty)$ or $Sp(\infty)$ are diffeological groups.

Example 4.3. Groups of diffeomorphisms
Let M be a diffeological space. Then the group $\mathrm{Diff}(M)$ of diffeomorphisms inherits the diffeology induced by the functional diffeology. It is not hard to see that the composition map $\mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is differentiable, because for charts α_U, β_V in $\mathcal{D}(M,M)$ and γ_W in M, the map

$$(t,s,r) \in U \times V \times W \mapsto \alpha(t)(\beta(s)(\gamma(r)))$$

is differentiable, due to the differentiability of $\mathrm{id}_U \times (\beta^\wedge \circ (\mathrm{id}_V \times \gamma))$ and $\mathrm{id}_V \times \alpha^\wedge$.

The difficult point is whether it is true or not that the inversion map $I\colon \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is differentiable, which would require some kind of inverse function theorem. Alternatively, one could consider on $\mathrm{Diff}(M)$ the (smaller) diffeology defined by the charts α such that $\alpha^\wedge$ is a diffeomorphism.

Proposition 4.4. *Let M be a finite dimensional manifold. Then the group of diffeomorphisms* $\mathrm{Diff}(M)$ *is a diffeological group, with the diffeology induced from* $\mathcal{D}(M,M)$.

Proof. The inversion map is differentiable, because for any chart α_U in $\mathrm{Diff}(M)$, the maps $\alpha^\wedge$ and $(I\alpha)^\wedge$ are inverse, and the (usual) differential of $\alpha^\wedge$ is an isomorphism at any point. ■

Example 4.5. Quotients of the circle
Let $\mathcal{F}_\alpha$ be the foliation on the torus T^2 defined by lines of constant *irrational* slope α. The space of leaves with the quotient diffeology is the diffeological abelian group T_α obtained as the quotient $\mathbb{R}/\Gamma_\alpha$, where $\Gamma_\alpha \subset \mathbb{R}$ is the dense subgroup $\{m\alpha + n \colon m,n \in \mathbb{Z}\}$.

Proposition 4.6. [7] *The diffeological groups T_α, T_β are isomorphic if and only if there exist integer numbers $a,b,c,d \in \mathbb{Z}$ with $ad - bc = \pm 1$ such that $\alpha = (a\beta + b)/(c\beta + d)$.*

4.2. Lie foliations.

The foliation $\mathcal{F}_\alpha$ on the torus is defined by the closed 1-form $\omega = dy - \alpha dx$. The group Γ_α is the group of periods of ω. The following more general situation is well known (see for instance [16]). Let $\omega_1, \ldots, \omega_n$ be non-singular 1-forms on a compact manifold M, such that $d\omega_k = \sum c_{ij}^k \omega_i \wedge \omega_j$, where c_{ij}^k are the structure constants of some Lie algebra $\mathfrak{g}$. Then the foliation $\mathcal{F}$ (of codimension n) defined by the vector fields X on M such that $\omega_i(X) = 0$ for $i = 1, \ldots, n$ is called a G-Lie foliation, where G is the simply connected Lie group associated to $\mathfrak{g}$.

The space of leaves $M/\mathcal{F}$ of the foliation $\mathcal{F}$ is a quotient G/Γ, where Γ is a finitely generated (algebraic) subgroup of G. The leaves of $\mathcal{F}$ are dense in M if and only Γ is dense in G.

Lie foliations with dense leaves play a central role in the theory of riemannian foliations [17].

Proposition 4.7. *Let $M/\mathcal{F} = G/\Gamma$ be the space of leaves of a Lie foliation with dense leaves on a compact manifold. Then* $\mathrm{Diff}(M/\mathcal{F})$ *is a diffeological group (with the diffeology induced by $\mathcal{D}(M/\mathcal{F}, M/\mathcal{F})$).*

Proof. Let $\alpha\colon U \to \mathrm{Diff}(M/\mathcal{F}) \subset \mathcal{D}(M/\mathcal{F}, M/\mathcal{F})$ be a chart, i.e. the map $\alpha^\wedge\colon U \times G/\Gamma \to U \times G/\Gamma$, $\alpha^\wedge(t,[g]) = (t, \alpha(t)([g]))$, is differentiable. In order to show that the inversion map I is differentiable, we only have to check that $\alpha^\wedge$ is a diffeomorphism, because $(\alpha^\wedge)^{-1} = (I\alpha)^\wedge$.

We can suppose without loss of generality that the open set U is simply connected. Then, since G is the universal covering of G/Γ (see Section 5), the map $\alpha^\wedge$ can be lifted to some differentiable map $F\colon U \times G \to U \times G$ with $F(t,g) = (t, \xi(t,g))$ such that $[\xi(t,g)] = \alpha(t)([g])$.

Now, let us fix $t \in U$. Since $\alpha(t)^{-1}$ is a diffeomorphism of G/Γ we can lift it to some diffeomorphism λ of the universal covering G. But the composition $\lambda \circ \xi(t,\cdot)$ takes values in Γ, hence it is constant because Γ is a countable group and G is a connected Lie group. Then $\lambda \circ \xi(t,\cdot) = L_\gamma$ for some $\gamma \in \Gamma$, which shows that $\xi(t,\cdot)$ is an injective map with injective differential. Analogously, $\xi(t,\cdot) \circ \lambda = L_\mu$, then $\xi(t,\cdot)$ is a diffeomorphism.

Finally, it is not hard to check that F is a diffeomorphism, and that F^{-1} is a lifting of $(\alpha^\wedge)^{-1}$, hence $(I\alpha)^\wedge$ is differentiable. ∎

We now show how to compute the group above.

Let $\mathrm{Aut}_\Gamma(G) \subset \mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$ be the group of automorphisms of the Lie group G that preserve Γ, and consider the semi-direct product $\mathrm{Aut}_\Gamma(G) \ltimes G$. It contains a normal subgroup isomorphic to Γ, where we identify $\gamma \in \Gamma$ to the pair (i_γ, γ^{-1}) (here i_γ = inner automorphism).

Proposition 4.8. $\mathrm{Diff}(G/\Gamma)$ *is isomorphic (as diffeological groups) to the quotient* $(\mathrm{Aut}_\Gamma(G) \ltimes G)/\Gamma$.

Proof. Let φ be a lifting of a diffeomorphism of G/Γ. Then φ is a diffeomorphism of G and for any $\gamma \in \Gamma$ there exists $\xi(\gamma) \in \Gamma$ such that $\varphi(g\gamma) = \varphi(g)\xi(\gamma)$ for all $g \in G$. Let us define $\xi = \xi_\varphi = L^{-1}_{\varphi(e)} \circ \varphi$. Since ξ is an automorphism of Γ, and Γ is a dense subgroup, we have $\xi \in \mathrm{Aut}(G)$. Finally, two diffeomorphisms φ, ψ of G induce the same diffeomorphism of G/Γ iff $\xi_\psi = L_\mu^{-1}\xi_\varphi R_\mu$ for some $\mu \in \Gamma$. ∎

The result above generalizes that of P. Donato and P. Iglesias in [7], where they computed the group $\mathrm{Diff}(T_\alpha)$ of a linear foliation on the torus. Explicitly they obtain $T_\alpha \times \mathbb{Z} \times \mathbb{Z}_2$ when α is a *quadratic* irrational number, and $T_\alpha \times \mathbb{Z}_2$ otherwise.

5. Coverings and fundamental group

5.1. Homotopic maps.

Let $(M, \mathcal{P})$ be a diffeological space. We shall consider the equivalence relation on M generated by the following relation: two points are related iff they are in the image of some connected chart. In other words, we say that two points $x, y \in M$ are in the same *path connected component* if there is a collection $\alpha_1, \dots, \alpha_k$ of charts with connected domains $U_1, \dots, U_k$ and there are points $s_1, t_1 \in U_1, \dots, s_k, t_k \in U_k$ such that $x = \alpha_1(s_1)$, $\alpha_1(t_1) = \alpha_2(s_2), \dots, \alpha_{k-1}(t_{k-1}) = \alpha_k(s_k)$, and $\alpha_k(t_k) = y$.

Let us denote $\pi_0(M)$ the quotient diffeological space.

Proposition 5.1. *The diffeological space $\pi_0(M)$ is discrete (that is, every differentiable chart must be locally constant).*

Analogously, let $(M, \mathcal{P})$, $(N, \mathcal{Q})$ be diffeological spaces. Two maps $F, G \in \mathcal{D}(M, N)$ are said to be *homotopic* if they are in the same connected component of the functional diffeological space $\mathcal{D}(M, N)$.

5.2. Fundamental group.

Let us fix a base point x_0 in the diffeological space M. We consider the loop space $\Omega(M, x_0)$ of base point preserving differentiable maps $(S^1, 1) \to (M, x_0)$, where S^1 is the circle (with its usual structure of manifold). Then $\Omega(M, x_0)$ is a diffeological space (as a submanifold of $\mathcal{D}(S^1, M)$), with base point the constant map x_0.

The *fundamental group* $\pi_1(M, x_0)$ is then defined as $\pi_0\Omega(M, x_0)$.

5.3. Universal covering.

Let $I = [0, 1]$ be the closed unit interval with the diffeological structure induced by the real line. By considering the space $P(M, x_0)$ of paths starting at x_0 as a submanifold of $\mathcal{D}([0, 1], M)$, one can construct the *universal covering* of any diffeological space $(M, \mathcal{P})$.

More precisely [10], a map $\pi\colon \tilde{M} \to M$ is a *covering* if it is a fibration with discrete fibres, that is, there exists a discrete diffeological space Γ (see above) such that, for any *global* chart $\alpha\colon \mathbb{R}^n \to M$ the pull-back is trivial, $\alpha^* = \mathbb{R}^n \times \Gamma$.

Proposition 5.2. *Any (connected) diffeological space M has a unique (up to diffeomorphism) simply connected covering $\tilde{M}$. It is universal in the sense that any other covering is a quotient of $\tilde{M}$.*

For instance, the universal covering of T_α (see Example 4.5) is the real line $\mathbb{R}$. More generally, the universal covering of the space of leaves G/Γ of a Lie foliation (see Example 4.2) is the simply connected Lie group G. In this case we show how to compute the fundamental group, generalizing [7].

Proposition 5.3. $\pi_1(G/\Gamma, [e]) = \Gamma$.

Proof. The loop space $\Omega(G/\Gamma, [e])$ is a quotient of the space of paths α in G such that $\alpha(0), \alpha(1) \in \Gamma$. We assign to α the element $\alpha(1)^{-1}\alpha(0) \in \Gamma$. Moreover two such paths define the same loop iff they differ by a translation R_γ by an element of $\gamma \in \Gamma$ (because Γ is a countable group). Hence we can suppose that $\alpha(0) = e$. Finally, if two paths starting at e have the same end point, they are in the same connected component because the Lie group G is simply connected. Hence $\pi_0\Omega(G/\Gamma, [e]) = \Gamma$. ∎

Example 5.4. Groups of diffeomorphisms [5]
Let $G = \mathrm{Diff}_0(\mathbb{R}^n)$ be the group of diffeomorphisms of $\mathbb{R}^n$ fixing the origin. Then its universal covering is the subspace $\tilde{G}$ of $G \times GL^+(n,\mathbb{R}) \times \mathbb{R}^n$ of all elements (f, A, v) such that $f'(0) = Av$.

As for finite dimensional Lie groups, the coverings of diffeological groups are diffeological groups. The universal covering (in the diffeological sense) of a topological space equals the usual one when this latter exists (by unicity). Moreover, diffeological coverings have lift properties analogous to the usual ones. We have implicitely used this lifting property in the proof of Proposition 4.8, which can be generalized as follows.

Proposition 5.5. *Let M be a diffeological space whose universal covering $\tilde{M}$ is a manifold. Then* $\mathrm{Diff}(M)$ *is a diffeological group (for the functional diffeology).*

6. Tangent bundle

6.1. Tangent space.

The following notion of tangent space for arbitrary diffeological spaces was defined by G. Hector in [8].

Let $x \in M$ be a point in the diffeological space $(M, \mathcal{P})$. We shall consider charts $\alpha \colon U \subset \mathbb{R}^m \to M$ which are *centered at* x. That means that U is a neighbourhood of the origin $0 \in \mathbb{R}^m$ and $\alpha(0) = x$. If $u \in T_0U = \mathbb{R}^m$, we shall denote $u \cdot \alpha$ the pair (u, α). Let $h \colon V \subset \mathbb{R}^n \to U$ be a change of coordinates, with $0 \in V$ and $h(0) = 0$. Then for any $v \in T_0V = \mathbb{R}^n$ we stablish the identification

$$v \cdot (\alpha \circ h) \equiv (dh)_0(v) \cdot \alpha. \tag{1}$$

The *tangent space* T_xM is the real vector space of all finite linear combinations $u_1 \cdot \alpha_1 + \cdots + u_k \cdot \alpha_k$, submitted to the identification above.

A more formal definition is the following one (see the original paper [8] for details). Let E_x be the direct sum $\oplus_\alpha E_\alpha$, where $\alpha \colon U \subset \mathbb{R}^m \to M$ are the charts centered at the point $x \in M$, and $E_\alpha = T_0U$. Let us denote $j_\alpha \colon E_\alpha \to E_x$ the natural inclusion. Then T_xM is the quotient of E_x and the subspace generated by the elements

$$j_\beta(X) - j_\alpha \circ (dh)_0(X), \quad X \in E_\beta,\ \beta = \alpha \circ h.$$

Due to (1) it is clear that any tangent vector $u \cdot \alpha$ is equivalent to $\partial_t \cdot \gamma$, where γ is a 1-chart. Also remark that, just as in finite dimensional manifolds, for 1-plates γ_1, γ_2, relation (1) means that there exists a (higher dimensional) chart α and coordinate changes h_1, h_2 such that $h_1'(0) = h_2'(0)$ and $\alpha h_i = \gamma_i, i = 1, 2$.

6.2. Examples.

Example 6.1. For a topological space X we have $T_pX = 0$ for all $p \in X$.

Example 6.2. Let $A \subset \mathbb{R}^2$ the set $OX \cup OY$ union of the two coordinate axis with the induced diffeology. Then T_pA has dimension 1 excepting at the origin, where T_0A has dimension 2.

Proposition 6.3. *Let $G = \mathrm{Diff}(M)$ the group of diffeomorphisms of a finite dimensional compact manifold M. Then $T_{\mathrm{id}}(G) = \chi(M)$, the space of vector fields on M.*

Proof. Let $\alpha\colon U \to \mathrm{Diff}(M)$ be a 1-chart defined on the interval $U = (-\epsilon, +\epsilon) \subset \mathbb{R}$. We assign to it the vector field $X_p = (d/dt)_{t=0}\alpha(t)(p)$. Details can be seen in [8]. ∎

6.3. Tangent map.

If $F \in \mathcal{D}(M, N)$ is a differentiable map, then a linear map $(TF)_p\colon T_pM \to T_{F(p)}N$ is well defined by

$$(TF)_p(v \cdot \alpha) = v(F \circ \alpha)$$

The chain rule is obvious.

The following result will be useful later (see Corollary 6.5).

Proposition 6.4. *Let G be a diffeological group. Then the tangent map at (e, e) of the product $\pi\colon G \times G \to G$ is the sum in T_eG.*

Proof. Let $(u \cdot \alpha_U, v \cdot \beta_V) \in T_{(e,e)}(G \times G) = T_eG \times T_eG$, and let us consider the chart $\gamma\colon U \times V \to G$ given by $\gamma(s,t) = \alpha(s)\beta(t)$. For the coordinate changes $h_U\colon U \to U \times V$, $h_U(s) = (s, 0)$ and $h_V\colon V \to U \times V$, $h_V(t) = (0, t)$, we have

$$\begin{aligned} u \cdot \alpha + v \cdot \beta &= (u, 0) \cdot \gamma + (0, v) \cdot \gamma \\ &= (u, v) \cdot \gamma \\ &= (u, v) \cdot (\pi \circ (\alpha \times \beta)) \\ &= (T\pi)_{(e,e)}(u\alpha, v\beta). \end{aligned}$$

∎

Corollary 6.5. *Let G a diffeological group, T_eG the tangent space at $e \in G$. Then any vector in T_eG can be written as $u \cdot \alpha$ for some chart α in G centered at e.*

Proof. Let $v = u_1 \cdot \alpha_1 + \cdots + u_k \cdot \alpha_k \in T_eG$ be a tangent vector. From Proposition 6.4 we can define a product chart $\alpha\colon U_1 \times \cdots \times U_k \to G$ such that $v = u \cdot \alpha$, where $u = (u_1, \ldots, u_k) \in T_0(U_1 \times \cdots \times U_k)$.

Now it is clear that given any linear chart $l\colon \mathbb{R}^m \to T_eG$ and taking a basis, we can construct a chart α in G such that l factors trough $T\alpha$. ∎

Remark that this property is not true for general diffeological spaces (think of example 6.2).

6.4. Tangent bundle.

Let $TM = \bigcup T_pM$. Instead of the colimit of the linear structures, we define on it another diffeology as follows. Let $\alpha\colon U \subset \mathbb{R}^n \to M$ be a chart on $(M, \mathcal{P})$. We consider the map $T\alpha\colon TU = U \times \mathbb{R}^n \to TM$ given as

$$(T\alpha)(t, v) = v \cdot (\alpha \circ L_t) \in T_{\alpha(t)}M \tag{2}$$

where L_t is the the translation by t (this is necessary for having a chart centered at $\alpha(t)$). Remark that with our notation, $(t, v) \in T_tU$ should be written $\partial_s \cdot (t + sv)$.

The diffeology generated by these shifts will be denoted $T\mathcal{P}$.

Proposition 6.6. *The original diffeology $\mathcal{P}$ on M is the quotient diffeology of $(TM, T\mathcal{P})$ for the projection $\pi\colon TM \to M$. The induced diffeology on each fiber T_pM is the linear diffeology*

The proof is an exercise.

More generally, let $F\colon M \to N$ be a differentiable map between diffeological spaces. The *shift* $TF\colon TM \to TN$ is defined by

$$(TF)(v \cdot \alpha) = v \cdot (F \circ \alpha) \tag{3}$$

Proposition 6.7. *The shift map $T\colon \mathcal{D}(M, N) \to \mathcal{D}(TM, TN)$ given by (3) is well defined and differentiable for the functional diffeologies.*

Proof. If $F \in \mathcal{D}(M, N)$ then $TF\colon TM \to TN$ is differentiable because for a chart $T\beta\colon U \times \mathbb{R}^n \to TM$ in the generating set we have $TF \circ T\beta = T(F \circ \beta)$, which is a chart in TN. Hence T is well defined.

Let us now verify that T is a differentiable map by taking any chart α_U in $\mathcal{D}(M, N)$, $U \subset \mathbb{R}^m$, and any chart $T\beta\colon V \times \mathbb{R}^n \to TM$ in the generating set of $T\mathcal{P}$. Then the map $U \times TV = U \times V \times \mathbb{R}^n \to TN$ given by

$$(t, s, v) \mapsto (T\alpha)^\wedge(t, T\beta(s, v)) = v \cdot (\alpha(t) \circ \beta \circ L_s)$$

is differentiable, because it is the composition of the inclusion (zero coordinate in $\mathbb{R}^m$) $U \times V \times \mathbb{R}^n \subset T(U \times V) = (U \times V) \times (\mathbb{R}^m \times \mathbb{R}^n)$ and the map $T(\alpha(t)\beta(s))(0, v)$, where $(t, \alpha(t)\beta(s)) = \alpha^\wedge(t, \beta(s))$. Here we are using the identification $v \cdot (\alpha(t)\beta L_s) = (0, v) \cdot (\alpha(t)\beta(s))$ in the definition of a tangent vector. ■

Proposition 6.8. *The tangent bundle of any diffeological group G is trivial, $TG \sim G \times T_eG$ (diffeomorphism).*

Proof. First we prove that the bijective map $F\colon G \times T_eG \to TG$ given by $F(x, X) = (TL_x)_e(X)$ is differentiable, where T_eG is endowed with the linear structure. Let $\alpha\colon U \subset \mathbb{R}^m \to G$ be a chart, and $l\colon \mathbb{R}^k \to T_eG$ a linear map. Then

$$F \circ (\alpha \times l)(s, w) = (TL_{\alpha(s)})_e(l(w)).$$

Since this map is linear in w, we can suppose without loss of generality that $k = 1$. Let $l(1) = v \cdot \beta \in T_eG$, with $\beta\colon V \subset \mathbb{R}^n \to G$ a chart, and $v \in T_0U$. Then

$$F \circ (\alpha \times l)(s, \lambda) = (\lambda v) \cdot (L_{\alpha(s)}\beta). \tag{4}$$

If we consider the chart $H\colon U \times V \to G$ with $H(s,t) = \alpha(s)\beta(t)$, and the map $h\colon U \times \mathbb{R} \to TU \times TV$ given by $h(s,\lambda) = ((s,0),(0,\lambda v))$, we obtain

$$(TH \circ h)(s,\lambda) = (0,\lambda v) \cdot (H \circ L_{(s,0)}). \tag{5}$$

But the latter tangent vector is equivalent to $(\lambda v) \cdot (L_{\alpha(s)}\beta)$ in (4), because the map $h_s\colon V \to L_s^{-1}(U) \times V$, $h_s(t) = (0,t)$ satisfies $H \circ L_{(s,0)} \circ h_s = L_{\alpha(s)} \circ \beta$ and $(dh_s)_0(\lambda v) = (0, \lambda v)$. Then equation (5) means that $F \circ (\alpha \times l) = TH \circ h$, hence it is a chart for TG.

We prove now that the inverse $F^{-1}\colon TG \to G \times T_eG$ is differentiable too. Let $T\alpha\colon TU \to TG$ a chart in the generator set, that is the shift of some chart $\alpha\colon U \subset \mathbb{R}^m \to G$. We must show that $F^{-1} \circ T\alpha$ is a chart in the product, where

$$(F^{-1} \circ T\alpha)(s,u) = F^{-1}(u \cdot (\alpha \circ L_s)) = \left(\alpha(s), u \cdot (L^{-1}_{\alpha(s)} \circ \alpha \circ L_s)\right), \tag{6}$$

as can be verified just from the definition of tangent map. In order to prove that the second coordinate function is differentiable, we define a new chart $H\colon U \times U \to G$ given by $H(s,t) = \alpha(s)^{-1}\alpha(t)$, and take the map $h\colon TU = U \times \mathbb{R}^m \to TU \times TU = T(U \times U)$ with $h(s,u) = ((s,s),(0,u))$.

We have

$$(TH \circ h)(s,u) = (0,u) \cdot (H \circ L_{(s,s)}). \tag{7}$$

But the latter vector is in fact equivalent to the second coordinate in (6), because the map $h_s\colon L_s^{-1}(U) \to L_s^{-1}(U) \times L_s^{-1}(U)$ given by $h_s(t) = (0,t)$ verifies $H \circ L_{(s,s)} \circ h_s = L^{-1}_{\alpha(s)} \circ \alpha L_s$ and $(dh_s)_0(u) = (0,u)$.

Hence equation (7) means that the second coordinate function in (6) equals $TH \circ h$, a differentiable map for $T_eG \subset TG$. ∎

6.5. Vector fields.

We define vectors fields on the diffeological space M as differentiable sections of the tangent bundle $\pi\colon TM \to M$.

Even if a tangent vector at a point can have a linear expression $u_1 \cdot \alpha_1 + \cdots + u_k \cdot \alpha_k$ with several summands, vector fields are locally given by a single expression $u \cdot \alpha$.

More precisely, let X be a differentiable vector field and α_U a chart in M. Since $X\alpha\colon U \to TM$ is a chart in TM, it is locally of the form $X\alpha = T\beta \circ h$, for some chart β_V in M and some change of coordinates $h\colon V \to TU = U \times \mathbb{R}^n$, let us say $h(s) = (t(s), v(s))$. Then

$$X_{\alpha(s)} = (T\beta)(t(s), v(s)) = v(s) \cdot (\beta \circ L_s)$$

That means for instance, that on the submanifold $A \subset \mathbb{R}^2$ of Example 6.2 above, any smooth vector field has a singularity at the origin.

7. The Lie algebra of a diffeological group

Let G be a diffeological group. Instead of taking invariant vector fields, that we cannot integrate due to the non bounded dimensions of the charts in the diffeology, we shall take the vector space $\mathfrak{g} = T_eG$ and try to define directly a bracket product there, in analogy with the usual adjoint representation.

Let $u \cdot \alpha, v \cdot \beta \in T_eG$. For each $s \in U$ we have the inner automorphism $i_{\alpha(s)}$, hence we can compute the tangent map $(Ti_{\alpha(s)})_e \colon T_eG \to T_eG$ and its value

$$(Ti_{\alpha(s)})_e(v \cdot \beta) = v \cdot (i_{\alpha(s)} \circ \beta) \in T_eG.$$

Then we have the curve $\xi \colon s \in U \mapsto v \cdot (i_{\alpha(s)} \circ \beta) \in \mathfrak{g} = T_eG$ and we consider its tangent map $T_0\xi \colon T_0U \to T_0(T_eG)$ and the value $T_0\xi(0,u)$, which will be by definition the bracket product of $u \cdot \alpha$ and $v \cdot \beta$.

The reader may check that this definition is independent of the expression of the tangent vectors $u \cdot \alpha, v \cdot \beta \in T_eG$. But the crucial point is to prove that the map ξ above is differentiable.

Lemma 7.1. *The map $\xi \colon U \to T_eG$ given by $\xi(s) = v \cdot (i_{\alpha(s)} \circ \beta)$ is differentiable.*

Proof. Let us consider the map $F \colon U \times V \to G$ given by $F(s,t) = \alpha(s)\beta(t)\alpha(s)^{-1}$, and the map $h \colon U \to TU \times TV = U \times \mathbb{R}^m \times V \times \mathbb{R}^n$ given by $h(s) = ((s,0),(0,v))$. Then

$$(TF \circ h)(s) = (0,v) \cdot (F \circ L_{(s,0)}). \tag{8}$$

But this vector is in fact equivalent to $v \cdot (i_{\alpha(s)} \circ \beta)$, because we have the map $h_s \colon V \to L_s^{-1}(U) \times V$ given by $h_s(t) = (0,t)$, which verifies $(dh_s)_0(v) = (0,v)$ and $F \circ L_{(s,0)} \circ h_s(t) = F(s,t) = i_{\alpha(s)} \circ \beta(t)$.

Then equation (8) means that $\xi = TF \circ h$, a differentiable map for the diffeology of TG. ■

Proposition 7.2. *For the diffeomorphism group of a compact finite dimensional manifold M the product defined above is a Lie bracket. Moreover, the isomorphism $T_{\mathrm{id}}\,\mathrm{Diff}(M) = \chi(M)$ of Proposition 6.3 is an isomorphism of Lie algebras.*

Proof. Seeing $s \mapsto \alpha(s)$ and $t \mapsto \beta(t)$ as (neccessarily complete) flows of (time dependent) vector fields, the preceding construction of the bracket product clearly reduces to the usual Lie bracket of vector fields. This establishes the isomorphism, and, a posteriori, antisymmetry and Jacobi identity of the bracket product. ■

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