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Minimal foliations on Lie groups

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ABSTRACT

Let $F(G,H)$ be the foliation determined by a (not necessarily closed) Lie subgroup H of a Lie group G. We prove that there always exists a Riemannian metric on G for which the leaves of $F(G,H)$ are minimal submanifolds.

1. INTRODUCTION

Let $F(G,H)$ be the riemannian foliation determined by an analytic Lie subgroup H of a connected Lie group G. The aim of this paper is to prove the existence of some (bundle-like) riemannian metric on G for which the leaves of $F(G,H)$ become minimal submanifolds.

In [TY] examples are given of foliations $F(G,H)$ which are minimal but not totally geodesic, for left invariant metrics under rather restrictive hypotheses, particularly that H admits a bi-invariant metric.

We give a direct proof for $F(G,H)$ of Molino of Fédida's theorems about the structure of transversely complete foliations. This enables us to restrict the problem to abelian Lie foliations $F(K,H)$ with H a *dense* connected Lie subgroup of its closure K. In this case minimality holds for some (any, if H is unimodular) left invariant metric on K. For arbitrary foliations $F(G,H)$, the desired metric is constructed by glueing together the fibres of $G \to G/K$. Finally, we give an example where there is no left invariant minimizing metric.

2. PRELIMINARIES

2.1. Let (M,F) be a foliated manifold. The foliation F is said to be *Riemannian* if there exists a Riemannian metric g on M which is *bundle-like* for F, that is the functions $g(Y,Z)$ are constant along the leaves for all foliate, orthogonal to the leaves, vector fields Y,Z defined on an open set of M. Recall that *foliate* vector fields Y are defined by the condition $[Y, TF] \subset TF$.

We say that F is *minimal* for a given riemannian metric on M if the leaves are minimal submanifolds. This means that the Weingarten's maps $W^Y: TF \to TF$ have zero trace for all normal vectors Y. A particular case is that of *totally geodesic* foliations, where $W^Y = 0$, i.e. leaves are totally geodesic submanifolds.

2.2. Let G be a connected Lie group, H an analytic subgroup of G, that is H is a connected Lie group and there is an injective morphism of Lie groups from H into G. Let $F = F(G,H)$ be the foliation on G whose leaves are the left cosets gH, $g \in G$. Takagi and Yorozu proved the following:

THEOREM [TY]. *Let g be a left invariant riemannian metric on G. If g is $Ad(H)$-invariant then g is bundle-like for $F(G,H)$ and $F(G,H)$ is totally geodesic for g.*

We first observe that in this case the restriction g_K of g to the closure K of H is a bi-invariant metric on K. In fact, g is $Ad(K)$-invariant because H is dense in K and $Ad: G \to GL(\mathfrak{g})$ is a continuous map. As a consequence, both K and H are unimodular Lie groups.

Let $\mathfrak{g}$ (respectively $\mathfrak{h}$) be the Lie algebra of G (resp. H).

We shall use the following rank of indexes:

$$1 \le i, j, k, \ldots \le m; \qquad 1 \le \alpha, \beta, \chi, \ldots \le n,$$

if the dimension of the group is $m+n$ and the dimension of the subgroup is m.

PROPOSITION 1. *$F(G,H)$ is minimal for a left invariant metric g on G if and only if $\sum_{i=1}^{m} c_{i\alpha}^{i} = 0$, where $\{c_{AB}^{D}\}$ are the structure constants relative to some adapted to $\mathfrak{h}$ g-orthonormal basis of $\mathfrak{g}$.*

PROOF. The Weingarten's operator associated to a normal vector field Y defined in a neighborhood of $x \in G$ is given by

$$W_x^Y(X) := -pr_x(\nabla_X Y), \quad X \in T_x F(G,H)$$

where $pr_x: T_x G \to T_x F(G,H)$ is orthogonal projection.

By the very definition, $F(G,H)$ is minimal if and only if trace $W_x^Y = 0$ for all normal vectors Y, for all $x \in G$. When g is a left invariant metric on G, it suffices to prove the nullity of the trace at the identity of G. Let $\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ be any adapted orthonormal basis of $\mathfrak{g}$.

We have

$$\text{trace } W^{Y_\alpha} = \sum_{i=1}^{m} g(W^{Y_\alpha}(X_i), X_i) = -\sum_{i=1}^{m} g(\nabla_{X_i} Y_\alpha, X_i)$$

$$= -\sum_{i=1}^{m} g([X_i, Y_\alpha], X_i) = -\sum_{i=1}^{m} c_{i\alpha}^{i}. \qquad \square$$

3. THE STRUCTURE OF $F(G,H)$

The closure of H in G is a connected closed Lie subgroup K of G, and $\pi: G \to G/K$ is a locally trivial (principal) bundle whose fibres are the closures of the leaves of $F(G,H)$. Let us call π the *basic fibration* of $F(G,H)$ and $W = G/K$ the *basic manifold*. The restriction of $F(G,H)$ to each fibre of π defines a foliation isomorphic to $F(K,H)$.

We make use of the following result from Mal'cev:

THEOREM [Ma]. *Let H be a dense connected Lie subgroup of a Lie group K. Then H is normal in K and the quotient $\mathfrak{k}/\mathfrak{h}$ of their Lie algebras is an abelian Lie algebra.*

Thus $F(K,H)$ is an *abelian Lie foliation with dense leaves* as defined by Fédida [F]. As a matter of fact $F(G,H)$ is a transversely complete foliation in the Molino's sense [Mo]. We now give for this particular case a version of the Fédida's structure theorem.

Let $\tilde{K}$ be the universal covering of K. Let $f: \tilde{K} \to \mathbb{R}^n$, $n = \operatorname{codim} F(K,H)$, be the morphism of simply connected Lie groups associated to the morphism $\mathfrak{k} \to \mathfrak{k}/\mathfrak{h}$ of Lie algebras. Then the closed subgroup $\tilde{H} = \ker f$ of $\tilde{K}$ is just the connected Lie subgroup of $\tilde{K}$ with Lie algebra $\mathfrak{h}$. Moreover $\tilde{H}$ is the universal covering of H.

This means that the lifting of $F(K,H)$ to $\tilde{K}$ is the foliation $F(\tilde{K},\tilde{H})$, which is simple for the fibration $f: \tilde{K} \to \mathbb{R}^n$.

We remark that this construction makes it possible to easily obtain a bundle-like metric for $F(G,H)$, starting with any metric g_G (that one can suppose it is left invariant) on G, the usual metric on $\mathbb{R}^n$, and any metric g_W on the basic manifold. Takagi and Yorozu's hypotheses imply that g_W can be taken as induced by g_G.

4. MINIMALITY OF $F(K,H)$

We first study minimality of the foliation $F(K,H)$. That is, H is a *dense* connected Lie subgroup of a Lie group K. When K is compact, the foliation is minimal by a result of Haefliger on abelian Lie foliations [Ha]; the theorem of Takagi and Yorozu cited above also applies because there is a bi-invariant metric on K. Our next result includes the non-unimodular case.

THEOREM 2. *Let H be a dense connected Lie subgroup of a Lie group K. Then there exists a left invariant riemannian metric on K such that the foliation $F(K,H)$ is minimal.*

We shall divide the proof into several propositions.

PROPOSITION 3. *If H is not unimodular, then $F(K,H)$ is minimal for some left invariant metric.*

PROOF. Let $\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\}$ be any basis of the Lie algebra $\mathfrak{k}$ of K, adapted to the Lie subalgebra $\mathfrak{h}$ of H. Let us consider a change of basis

$$X_i' = X_i, \quad i = 1, \ldots, m.$$

$$Y_\alpha' = \sum_{j=1}^{m} a_\alpha^j X_j + Y_\alpha, \quad \alpha = 1, \ldots, n.$$

Let $\{c_{AB}^D\}$ be the structure constants relative to the first basis. Since $\mathfrak{k}/\mathfrak{h}$ is an abelian Lie algebra, the structure constants $\{c_{AB}'^D\}$ of the new adapted basis verify

$$c_{ij}'^k = c_{ij}^k;$$

$$c_{i\alpha}'^k = \sum_j a_\alpha^j c_{ij}^k + c_{i\alpha}^k;$$

$$c_{\alpha\beta}'^k = c_{\alpha\beta}^k + \sum_{i,j} a_\alpha^i a_\beta^j c_{ij}^k + \sum_i a_\alpha^i c_{i\beta}^k - \sum_j a_\beta^j c_{j\alpha}^k.$$

Let g be a left invariant metric so that $\{X_1', \ldots, X_m', Y_1', \ldots, Y_n'\}$ is an orthonormal basis of $\mathfrak{k}$. Then $F(K,H)$ is minimal for g if and only if (Prop. 1)

$$0 = \sum_{i=1}^{m} c_{i\alpha}'^i = \sum_{k=1}^{m} a_\alpha^k \Big(\sum_{j=1}^{m} c_{jk}^j\Big) + \sum_{j=1}^{m} c_{j\alpha}^j \quad \text{for all } \alpha,$$

that is

$$\text{(1)} \qquad C(a_\alpha^1 \cdots a_\alpha^m)^t = -\sum_{j=1}^{m} c_{j\alpha}^j, \quad \alpha = 1, \ldots, n,$$

with $C = (\sum_{j=1}^m c_{j1}^j \cdots \sum_{j=1}^m c_{jm}^j)$.

Since H is not unimodular, there exists i such that $\sum_{j=1}^m c_{ji}^j \neq 0$. Then rank $C = 1$ and system (1) can be solved for all α. □

We remark that each solution of system (1) in the preceding proof gives a different distribution complementary to the foliation. Then, there is an infinity of left invariant minimizing metrics with different orthogonal distributions.

For H an unimodular group, we shall need the following:

PROPOSITION 4. *Let H be a dense connected Lie subgroup of a Lie group K. If H is unimodular then K is unimodular.*

PROOF. We must prove that $\det Ad_K(y) = 1$ for all $y \in K$.

As we have seen at §3, $F(K,H)$ lifts up to the simple foliation $F(\tilde{K}, \tilde{H})$. If $\pi : \tilde{G} \to G$ is a covering, then $Ad_{\tilde{G}}(x) = Ad_G(\pi(x))$ for all $x \in \tilde{G}$. This shows that $\tilde{H}$ is unimodular.

Now, $\tilde{H}$ is a normal closed subgroup of $\tilde{K}$, then $\det Ad_{\tilde{K}}(h) = \det Ad_{\tilde{H}}(h)$ for all $h \in \tilde{H}$ because $Ad_{\tilde{K}/\tilde{H}}(h) = Id$. In fact, in our case $Ad_{\mathfrak{k}/\mathfrak{h}}$ is trivial because $\mathfrak{k}/\mathfrak{h}$ is abelian.

Thus $\det Ad_K(h) = \det Ad_H(h) = 1$ for all $h \in H$.

Finally, let $y \in K$. Since H is dense in K there exists $\{h_n\} \subset K$ with $y = \lim h_n$, then $\det Ad_K(y) = \lim \det Ad_K(h_n) = \lim \det Ad_H(h_n) = 1$. □

PROPOSITION 5. *Let H be a connected dense Lie subgroup of a Lie group K. If H is unimodular then $F(K,H)$ is minimal for any left invariant metric on K.*

PROOF. Let g be any left invariant metric, $\{X_1,\dots,X_m,Y_1,\dots,Y_n\}$ an adapted to $\mathfrak{h}$ orthonormal basis of $\mathfrak{k}$ with structure constants $\{c_{AB}^D\}$. Since K and H are unimodular and $\mathfrak{k}/\mathfrak{h}$ is abelian, we have

$$0=\text{trace } ad_{\mathfrak{K}}(Y_\alpha)=\sum_{j=1}^{m} c_{\alpha j}^{j}$$

which is equivalent to minimality by Prop. 1. □

5. MINIMALITY OF $F(G,H)$

We now study minimality of the foliation $F(G,H)$, with G a connected Lie group and H an arbitrary analytic subgroup.

THEOREM 6. *Let (M,F) be a foliated manifold such that the closures of the leaves are the fibres of a locally trivial bundle $p: M\to B$, with fibre N and structure group K.*

Let g_N be a riemannian metric on N such that the induced foliation on the fibres is minimal. If K acts by isometries on the fibres, then there exists a riemannian metric on M such that the foliation F is minimal.

PROOF. Our proof is inspired by an argument of Ghys [G].

Let $\{U_i\}$ be a covering of B by coordinate open sets. Let us denote $\theta_i: p^{-1}(U_i)\to U_i\times N$ the trivialising maps, $\phi_i=pr\circ\theta_i$ with $pr: U_i\times N\to N$ the projection.

On $U_i\times N$ we consider the metric $g_i=\langle\ \rangle_i\times g_N$, where $\langle\ \rangle_i$ is the pull-back $\psi_i^*\langle\ \rangle$ of the usual inner product in $\mathbb{R}^n$ by a local chart (U_i,ψ_i) on B. Let $\{f_i\}$ be a partition of unity subordinated to $\{U_i\}$.

We define on M the metric $g=\sum (f_i\circ p)\theta_i^* g_i$.

We shall verify that F is minimal for g. Let $x\in M$, Y_x a vector normal to F at x, $V_x\in T_xF$ a vector tangent to F. Then

$$\begin{aligned}
2g_x(\nabla_V Y,V) &= Y_x g(V,V)+2g_x(V,[V,Y])\\
&=\sum_i Y_x(f_i\circ p)(\theta_i^*g_i)_x(V,V)+\sum_i (f_i\circ p)(x)Y_x(\theta_i^*g_i(V,V))\\
&\quad+2\sum_i f_i(p(x))(\theta_i^*g_i)_x([V,Y],V)\\
&=\sum_i Y_x(f_i\circ p)\, g_{i_{\theta_i(x)}}(\theta_{i*_x}V,\theta_{i*_x}V)\\
&\quad+\sum_i f_i(p(x))\{Y_x(\theta_i^*g_i(V,V))+2(\theta_i^*g_i)_x([V,Y],V)\}\\
&=\sum_i Y_x(f_i\circ p)\, g_{N_{\phi_i(x)}}(\phi_{i*_x}V,\phi_{i*_x}V)\\
&\quad+\sum_i f_i(p(x))2(\theta_i^*g_i)_x(\nabla_V^i Y,V)
\end{aligned}$$

where ∇^i is the Levi-Civitá connection associated to $\theta_i^* g_i$. But $\theta_{i*}\nabla^i_V Y = \nabla^{\langle\,\rangle_i}_{p_* V} p_* Y + \nabla^N_{\phi_{i*}V}\phi_{i*}Y$, then

$$(\theta_i^* g_i)_x(\nabla^i_V Y, V) = g_{i_{\theta_i(x)}}(\nabla^{\langle\,\rangle_i}_{p_{*_x}V} p_{*_x} Y + \nabla^N_{\phi_{i*_x}V}\phi_{i*_x}Y, \theta_{i*_x}V)$$
$$= g_{N_{\phi_i(x)}}(\nabla^N_{\phi_{i*_x}V}\phi_{i*_x}Y, \phi_{i*_x}V).$$

Finally, we have

$$2g_x(\nabla_V Y, V) = \sum_i Y_x(f_i \circ p)\, g_{N_{\phi_i(x)}}(\phi_{i*_x}V, \phi_{i*_x}V)$$
$$+ 2\sum_i f_i(p(x))\, g_{N_{\phi_i(x)}}(\nabla^N_{\phi_{i*_x}V}\phi_{i*_x}Y, \phi_{i*_x}V)$$

where ∇^N denotes the Levi-Civitá connection of g_N.

Let $\{\psi_{ij} : U_i \cap U_j \to K\}$ be the transition functions of the bundle, $\phi_i(x) = \psi_{ij}(p(x))\phi_j(x)$. So $\phi_i(x)$ and $\phi_j(x)$ differ in a translation on the fibre $p^{-1}(p(x))$ by an element of K. But K acts on the fibres by isometries, then

$$g_{N_{\phi_i(x)}}(\phi_{i*_x}V, \phi_{i*_x}V) = g_{N_{\phi_j(x)}}(\phi_{j*_x}V, \phi_{j*_x}V).$$

Thus

$$g_x(\nabla_V Y, V) = \sum_i f_i(p(x)) g_{N_{\phi_i(x)}}(\nabla^N_{\phi_{i*_x}V}\phi_{i*_x}Y, \phi_{i*_x}V)$$

because $Y_x(\sum_i (f_i \circ p)) = Y_x(1) = 0$.

Let $\{V_1, \ldots, V_m\}$ be an orthonormal basis of $T_x F$, then trace $W_x^Y = -\sum_{j=1}^m g_x(\nabla_{V_j} Y, V_j) = 0$ because (N, F) is minimal for g_N. □

In our case, $B = G/K$, the structure group K acts by left translations, and the metric minimizing $F(K, H)$ is left invariant. We have then proved:

THEOREM 7. *Let G be a connected Lie group, H a connected, not necessarily closed, Lie subgroup of G. Then the foliation $F(G, H)$ is minimal for some riemannian metric on G.*

The following example was pointed out to us by A. Reventós, showing that in general it is not possible to find *left invariant* minimizing metrics.

Let $\mathfrak{g} = \langle x, y, z\rangle$ be a Lie algebra given by $[x, y] = 0$, $[x, z] = x$, $[y, z] \in \langle x, y\rangle$ arbitrary. Consider $\mathfrak{h} = \langle x\rangle$. Then for an adapted basis $\langle \lambda x, Y, Z\rangle$, Proposition 1 would imply $[\lambda x, Y] = [\lambda x, Z] = 0$, hence $Y, Z \in \langle x, y\rangle$.

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