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# TANGENTIAL LUSTERNIK–SCHNIRELMANN CATEGORY OF FOLIATIONS

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ABSTRACT

A new invariant of (integrable) homotopy type for foliations is introduced: the tangential category of a foliated manifold.

The classical Lusternik–Schnirelmann theory is generalized to foliations and the relations of the tangential category with other known invariants such as the fibrewise and the equivariant category are studied. Cohomological lower bounds are provided in terms of foliated cohomology.

If the foliation is a product, the tangential category coincides with the ordinary category of the leaves. In general it is just bounded below. Estimates are given of the tangential category for compact-Hausdorff foliations and suspensions. Examples show that the lower and upper bounds are realized.

## 1. *Introduction*

The Lusternik–Schnirelmann category of a topological space $X$ was introduced in the course of research on the calculus of variations in 1930 **[18]**. It is an invariant of homotopy type, defined as the least number of open subsets, contractible in $X$, required to cover $X$. For a survey of Lusternik–Schnirelmann category see **[15, 16]**.

There are many extensions of the original concept adapted to various contexts such as the *fibrewise category*, introduced by I. M. James and J. R. Morris **[17]** or the *equivariant category*, by E. Fadell **[10]**.

A good category for foliations should be an invariant of homotopy type, for some notion of homotopy compatible with the foliated structure.

We introduce the tangential Lusternik–Schnirelmann category for any foliated manifold $(M,\mathscr{F})$. An open subset $U$ of $M$ is said to be tangentially categorical if the inclusion map $(U,\mathscr{F}_U) \hookrightarrow (M,\mathscr{F})$ is integrably homotopic to a map which is constant on each leaf of $\mathscr{F}_U$. This definition yields a well-defined integer (possibly infinite) associated to $(M,\mathscr{F})$ which is an invariant of integrable homotopy. For a product foliation, the tangential category coincides with the category of the leaf. In general, the category of the leaves is just a lower bound, and the tangential category turns out to be a rather subtle invariant of the foliation.

We obtain cohomological lower bounds for the tangential category in terms of foliated (leafwise, tangential) cohomology. We explicitly define the natural notions of *relative* foliated cohomology and cup product, and we prove that $\mathrm{cat}_{\mathscr{F}}(M) \geqslant \mathrm{nil}\, \tilde{H}_{\mathscr{F}}(M)$, the latter being the index of nilpotency of the foliated cohomology in positive degrees.

We study the tangential category of compact-Hausdorff foliations; that is, foliations with all leaves compact and whose leaf space is Hausdorff. We also

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study the tangential category for suspensions. We give lower and upper bounds for the tangential category in both cases and as shown by examples these bounds depend upon the geometry of the foliation.

Finally we give the calculation of the tangential category of all foliations on the torus by explicit constructions of tangentially categorical coverings.

We also propose in [6] a transverse notion of category that complements the study of the categorical invariants of a foliation. See also [5, 14] for the compact-Hausdorff case.

## 2. *Tangential Lusternik–Schnirelmann category*

All manifolds and foliations are assumed to be of class $C^\infty$. If $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ are foliated manifolds, then $f : (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}')$ denotes a *foliated* map. That is, it is assumed that $f$ is smooth and sends leaves into leaves.

DEFINITION 1. An *integrable homotopy* between foliated maps $f, g$ is a homotopy $H : (M \times \mathbf{R}, \mathscr{F} \times \mathbf{R}) \longrightarrow (M', \mathscr{F}')$ such that $H_t = f$ for $t \leqslant 0$ and $H_t = g$ for $t \geqslant 1$. We denote this by $f \simeq_{\mathscr{F} \times \mathbf{R}} g$.

If $H : (M \times \mathbf{R}, \mathscr{F} \times \mathbf{R}) \longrightarrow (M', \mathscr{F}')$ is an integrable homotopy between foliated maps $f, g$, then for all $0 \leqslant t \leqslant 1$ we have a foliated map $H_t : (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}')$. Moreover, for each $x \in M$ the curve $t \longmapsto H_t(x)$ is a leafwise curve in $M'$. Thus, an integrable homotopy is exactly a homotopy for which all of the 'traces' are leafwise curves. As a consequence, it is easy to see that if $f \simeq_{\mathscr{F} \times \mathbf{R}} g$ then $f$ and $g$ induce the same map between the spaces of leaves.

The notions of (integrable) homotopy equivalence and (integrable) homotopy type of foliated manifolds are obvious.

Let $(M, \mathscr{F})$ be a foliated manifold, $U \subset M$ be an open set and $\mathscr{F}_U$ be the induced foliation. We say that $U$ is a *tangentially categorical* open set (in short, $\mathscr{F}$-categorical) if there exists a map $c : (U, \mathscr{F}_U) \longrightarrow (M, \mathscr{F})$, constant on each leaf of $\mathscr{F}_U$, such that $i_U \simeq_{\mathscr{F} \times \mathbf{R}} c$.

DEFINITION 2. The *tangential category* of $(M, \mathscr{F})$ is the least integer $k = \mathrm{cat}_{\mathscr{F}}(M)$ such that $M$ can be covered by $k$ $\mathscr{F}$-categorical open subsets. When such an integer does not exist, we put $\mathrm{cat}_{\mathscr{F}}(M) = \infty$.

PROPOSITION 3. $\mathrm{cat}_{\mathscr{F}}(M)$ *is invariant under integrable homotopies.*

*Proof.* Let $f : (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}')$ be an integrable homotopy equivalence with homotopic inverse $g : (M', \mathscr{F}') \longrightarrow (M, \mathscr{F})$. We shall prove that $\mathrm{cat}_{\mathscr{F}}(M) \geqslant \mathrm{cat}_{\mathscr{F}'}(M')$ (an analogous argument gives the inverse inequality).

Let $U \subset M$ be an $\mathscr{F}$-categorical open set. Then $V = g^{-1}(U)$ is $\mathscr{F}'$-categorical in $M'$. In fact, if $i_U \simeq_{\mathscr{F} \times \mathbf{R}} c$ for some map $c : (U, \mathscr{F}_U) \longrightarrow (M, \mathscr{F})$ which is constant on each leaf of $\mathscr{F}_U$, let $c' : (V, \mathscr{F}'_V) \longrightarrow (M', \mathscr{F}')$ be the map $c' = fcg\,|_V$. It is constant on each leaf $P'$ of $\mathscr{F}'_V$, because (due to connectedness) $g(P')$ must be contained in some leaf $L$, hence in some connected component $P$ of $L \cap U$, so $c'(P') \subset fc(P)$, where $c(P)$ is a point. Moreover, $c' = fcg\,|_V \simeq_{\mathscr{F} \times \mathbf{R}} fi_U g\,|_V = fgi_V \simeq_{\mathscr{F} \times \mathbf{R}} \mathrm{id}\, i_V = i_V$, and thus $c' \simeq_{\mathscr{F} \times \mathbf{R}} i_V$. □

When $\mathcal{F}$ is a foliation by a single leaf, an open subset is tangentially categorical if and only if it is categorical, so $\mathrm{cat}_{\mathcal{F}}(M) = \mathrm{cat}\, M$. For a foliation by points, we have $\mathrm{cat}_{\mathcal{F}}(M) = 1$.

A distinguished open set of a foliated chart is always categorical, so the tangential category is finite in compact manifolds.

Each leaf of a foliation supports two different topologies: the submanifold topology $\tau_L$ which has the plaques as a basis, and the relative topology $\tau \subset \tau_L$ induced by the ambient manifold. For our study of the tangential category of foliations, it is necessary to compare the category of a leaf with respect to these two topologies.

LEMMA 4. $\mathrm{cat}(L, \tau_L) \leqslant \mathrm{cat}(L, \tau)$.

*Proof.* Let $V$ be a categorical open set for the relative topology and $H : (V, \tau) \times \mathbf{R} \longrightarrow (L, \tau)$ be the contraction to a point of $L$. Since $\mathrm{id} : (V, \tau_L) \longrightarrow (V, \tau)$ is a continuous map, $H : (V, \tau_L) \times \mathbf{R} \longrightarrow (L, \tau)$ is continuous too. Then since $(L, \tau_L)$ is a weakly embedded submanifold of $M$ and $(V, \tau_L) \times \mathbf{R}$ is locally connected, the map $H : (V, \tau_L) \times \mathbf{R} \longrightarrow (L, \tau_L)$ is continuous. This means that $V$ is a categorical open set for $(L, \tau_L)$. □

In this paper, unless otherwise specified, we will assume that $\mathrm{cat}\, L = \mathrm{cat}(L, \tau_L)$, the category of the leaf as a submanifold of $M$.

PROPOSITION 5. *For any leaf $L \in \mathcal{F}$, $\mathrm{cat}\, L \leqslant \mathrm{cat}_{\mathcal{F}}(M)$.*

*Proof.* Let $U \subset M$ be an $\mathcal{F}$-categorical open set and $H : i_U \simeq_{\mathcal{F} \times \mathbf{R}} c$ be an integrable homotopy. Since $H$ is integrable and constant on the leaves of $\mathcal{F}_U$, the restriction of $H$ to any leaf $L$ is a map $(U \cap L) \times \mathbf{R} \longrightarrow L$ contracting each connected component of $U \cap L$ into a point. We can define a contraction on $U \cap L$ into a point $x_0 \in L$ by concatenation of the contraction on each connected component to a point $x \in L$ and the path from $x$ to $x_0$. This contraction is continuous for the relative topology and the result follows from Lemma 4. □

PROPOSITION 6. *If $\mathrm{cat}_{\mathcal{F}}(M) = 1$ then all leaves are closed and contractible.*

*Proof.* Since $M$ is $\mathcal{F}$-categorical, let $c : (M, \mathcal{F}) \longrightarrow (M, \mathcal{F})$ be the map such that $c(L) = p_L$, $p_L \in L$ for each leaf $L \in \mathcal{F}$. Then each set $M - L = c^{-1}(M - p_L)$ is open. Moreover, by Proposition 5 each leaf $L$ has category 1; hence it is contractible. □

COROLLARY 7. *Let $M$ be a compact manifold. Then $\mathrm{cat}_{\mathcal{F}}(M) = 1$ if and only if $\mathcal{F}$ is the foliation by points.*

The contractibility of every leaf is not sufficient to contract tangentially the manifold, as shown by the example of the linear foliation on the torus with irrational slope (see Section 6).

## 3. *Fibrewise category*

We now compare the tangential category with the *fibrewise category* introduced by I. M. James and J. R. Morris in [17]. Recall that a fibrewise space $X$ over $B$

is a topological space $X$ together with a map $\pi : X \longrightarrow B$. An open set $U \subset X$ is said to be fibrewise categorical if there exists a *global* section $s : B \longrightarrow X$ such that the inclusion $i_U$ and the map $c = s\pi\,|_U$ are fibrewise homotopic (that is, integrably homotopic in the continuous sense). The fibrewise category $\operatorname{cat}_B X$ is the least number of fibrewise categorical open sets required to cover $X$. (If no such covering exists, the fibrewise category is said to be infinite.)

Consider a smooth version $\operatorname{cat}_B^\infty X$ of the fibrewise category, where we require that all objects and maps are smooth. If $(M, \mathscr{F})$ is a foliated manifold, and $M/\mathscr{F}$ denotes the leaf space with the quotient topology, then we can consider $M$ as a fibrewise space over $M/\mathscr{F}$. If there is no global section $s : M/\mathscr{F} \longrightarrow M$ then $\operatorname{cat}_{M/\mathscr{F}}^\infty M = \infty$. In any case, we have the following proposition.

PROPOSITION 8. $\operatorname{cat}_{\mathscr{F}}(M) \leqslant \operatorname{cat}_{M/\mathscr{F}}^\infty M$.

The requirement of the existence of a global section makes the fibrewise category infinite in most interesting foliations. A local version of the fibrewise category [**3**] provides a generalization that only requires local sections.

The existence of local sections implies that the fibres are closed and the leaf space is locally Hausdorff [**4**]. Thus, for all foliations containing either a non-closed leaf or a leaf with non-finite holonomy [**21**], the fibrewise category and local fibrewise category of $\mathscr{F}$ are infinite. On the other hand, the tangential category is always finite for a foliation on a compact manifold.

## 4. *Foliated cohomology*

We obtain lower bounds for the tangential category in terms of the foliated cohomology. We explicitly define the natural notions of *relative* foliated cohomology and cup product, and prove that $\operatorname{cat}_{\mathscr{F}}(M) \geqslant \operatorname{nil} \tilde{H}_{\mathscr{F}}(M)$, the latter being the index of nilpotency of the foliated cohomology in positive degrees. These results are directly analogous to the classical case.

Let us consider the following differential subcomplex $\Omega_0$ of the smooth forms $\Omega(M) : \omega \in \Omega_0^r$ if and only if $\omega \in \Omega^r(M)$ and $\omega(X_1, \ldots, X_r) = 0$ when $X_1, \ldots, X_r$ are tangent to the foliation.

We put $\Omega_0^0 = 0$. For $r > \dim \mathscr{F}$ we have $\Omega_0^r = \Omega^r(M)$.

It is straightforward to check that $\Omega_0$ is an ideal, closed under the differential. The factor space $\Omega_{\mathscr{F}}(M) = \Omega(M)/\Omega_0$ has a differential $d_F\bar{\omega} = \overline{d\omega}$ induced by the exterior differential. We call its cohomology $H_{\mathscr{F}}(M)$ the *foliated cohomology* [**8**, **12**, **13**] of the foliated manifold $(M, \mathscr{F})$. If the foliation is by points, the foliated cohomology is 0 in positive degrees.

Clearly, any foliated map induces a homomorphism in foliated cohomology. Moreover, two (integrably) homotopic maps induce the same homomorphism in foliated cohomology.

### 4.1. *Relative foliated cohomology*

Following the usual definition of the mapping cone [**1**], we associate to any foliated map $g : (M', \mathscr{F}') \longrightarrow (M, \mathscr{F})$ the complex

$$\Omega_{\mathscr{F}}^r(g) = \Omega_{\mathscr{F}}^r(M) \oplus \Omega_{\mathscr{F}}^{r-1}(M')$$

with differential

$$d_F(\bar{\omega}, \bar{\theta}) = (d_F\bar{\omega}, f^*\overline{\omega} - d_F\bar{\theta})$$

and we denote $H^r_{\mathcal{F}}(g)$ its cohomology groups.

From the short exact sequence

$$0 \longrightarrow \Omega^{r-1}_{\mathcal{F}}(M') \xrightarrow{\alpha} \Omega^r_{\mathcal{F}}(g) \xrightarrow{\beta} \Omega^r_{\mathcal{F}}(M) \longrightarrow 0 \tag{1}$$

with $\alpha(\eta) = (0, \eta)$ and $\beta(\omega, \theta) = \omega$ we have the long exact sequence in foliated cohomology

$$\ldots \longrightarrow H^{r-1}_{\mathcal{F}}(M') \xrightarrow{\alpha^*} H^r_{\mathcal{F}}(g) \xrightarrow{\beta^*} H^r_{\mathcal{F}}(M) \xrightarrow{g^*} H^r_{\mathcal{F}}(M') \longrightarrow \ldots \tag{2}$$

In particular, for the inclusion $i : (U, \mathcal{F}_U) \hookrightarrow (M, \mathcal{F})$ of an open set $U$ we call $H^r_{\mathcal{F}}(M, U) := H^r_{\mathcal{F}}(i)$ the *relative foliated cohomology* groups of the pair $(M, U)$, and we obtain from (2) the exact sequence

$$\ldots \longrightarrow H^{r-1}_{\mathcal{F}_U}(U) \longrightarrow H^r_{\mathcal{F}}(M, U) \longrightarrow H^r_{\mathcal{F}}(M) \xrightarrow{i^*} H^r_{\mathcal{F}_U}(U) \longrightarrow \ldots$$

PROPOSITION 9. $H_{\mathcal{F}}(M, M) = 0$.

*Proof.* $d_F(\bar{\omega}, \bar{\theta}) = 0$ means that $d\omega, \omega - d\theta \in \Omega_0$, so $d_F(\bar{\theta}, 0) = (\bar{\omega}, \bar{\theta})$. □

### 4.2. *Products in relative foliated cohomology*

Let $U, V \subset M$ be open sets. In order to define a product

$$H_{\mathcal{F}}(M, U) \otimes H_{\mathcal{F}}(M, V) \longrightarrow H_{\mathcal{F}}(M, U \cup V) \tag{3}$$

we take a smooth partition $\{f, g\}$ of unity subordinate to the covering $\{U, V\}$ of $U \cup V$. Then we define

$$\bullet : \Omega^p_{\mathcal{F}}(M, U) \times \Omega^q_{\mathcal{F}}(M, V) \longrightarrow \Omega^{p+q}_{\mathcal{F}}(M, U \cup V)$$

by

$$(\bar{\omega}, \bar{\theta}) \bullet (\bar{z}, \bar{t}) = (\overline{\omega \wedge z}, \bar{\eta}) \tag{4}$$

where $\eta \in \Omega^{p+q-1}(U \cup V)$ is given as follows.

(1) Let us consider in the intersection $U \cap V$ the forms $\theta\,|_{U\cap V}$, $t\,|_{U\cap V}$ and

$$(\omega, \theta) * (z, t) = ((-1)^p(\omega - d\theta) \wedge t - \theta \wedge (z - dt))\,|_{U\cap V}$$

which we can extend to $U$ by multiplying by $g$ because $\operatorname{supp} g \subset V$. Analogously we extend them to $V$ by multiplying by $f$.

(2) Then we put in (4)

$$\eta\,|_U = \theta \wedge z\,|_U + (-1)^p d(\theta \wedge gt) + g(\omega, \theta) * (z, t)$$

and

$$\eta\,|_V = (-1)^p\omega\,|_V \wedge t - (-1)^p d(f\theta \wedge t) - f(\omega, \theta) * (z, t).$$

The form $\eta$ is well defined because $f + g = 1$ in $U \cap V$, so $\eta\,|_U - \eta\,|_V$ equals

$$\theta \wedge z - (-1)^p\omega \wedge t + (-1)^p d(\theta \wedge t) + (-1)^p(\omega - d\theta) \wedge t - \theta \wedge (z - dt)$$

which is 0.

From the definition above it is not hard to prove that

$$(\bar{\omega}, \bar{\theta}) \bullet (\bar{z}, \bar{t}) = (-1)^{pq}(\bar{z}, \bar{t}) \bullet (\bar{\omega}, \bar{\theta}).$$

The product is well defined in cohomology and does not depend on the partition of unity [**4**].

### 4.3. *Category and nilpotency*

The ring $A$ is said to be nilpotent if there exists some integer $k > 0$ such that $a_1 \dots a_k = 0$ for any elements (not units) $a_1, \dots, a_k \in A$. The least integer $k = \operatorname{nil} A$ with this property is called the *index* of nilpotency of $A$.

For example, let $\tilde{H}_{\mathscr{F}}(M)$ be the foliated cohomology in degrees $r > 0$. Then $\operatorname{nil} \tilde{H}_{\mathscr{F}}(M) \leqslant p + 1$, $p = \dim \mathscr{F}$, since $H^r_{\mathscr{F}}(M) = 0$ for $r > p$.

LEMMA 10. *Let $U$ be an $\mathscr{F}$-categorical open set. Then the map $i^* : \tilde{H}_{\mathscr{F}}(M) \longrightarrow \tilde{H}_{\mathscr{F}_U}(U)$ induced in foliated cohomology is null in positive degrees.*

The proof is almost immediate from the fact that $(U, \mathscr{F}_U) \hookrightarrow (M, \mathscr{F})$ factors through a foliation by points up to integrable homotopy.

THEOREM 11. *For any foliated manifold,* $\operatorname{cat}_{\mathscr{F}}(M) \geqslant \operatorname{nil} \tilde{H}_{\mathscr{F}}(M)$.

*Proof.* Let $\{U_1, \dots, U_k\}$ be a covering of $M$ by tangentially categorical open sets. For each $U = U_j$ we consider the long exact sequence of the pair $(M, U)$, that is

$$\dots \longrightarrow H^r_{\mathscr{F}}(M, U) \xrightarrow{\beta^*} H^r_{\mathscr{F}}(M) \xrightarrow{i^*} H^r_{\mathscr{F}_U}(U) \longrightarrow \dots$$

where $\beta^*$ is onto for $r > 0$ due to Lemma 10. Let $x_1, \dots, x_k$ be arbitrary elements of $\tilde{H}_{\mathscr{F}}(M)$. For each $x_j$, there exists $z_j \in H_{\mathscr{F}}(M, U_j)$ such that $\beta^*(z_j) = x_j$. Moreover, from the definition of $\beta$ in (1) it is obvious that if $x \in \Omega_{\mathscr{F}}(M, U)$ and $y \in \Omega_{\mathscr{F}}(M, V)$ then

$$\beta_{U \cup V}(x \bullet y) = \beta_U(x) \wedge \beta_V(y).$$

Thus the product $x_1 \dots x_k = \beta^*(z_1 \bullet \dots \bullet z_k) = 0$ because $z_1 \bullet \dots \bullet z_k \in H_{\mathscr{F}}(M, M) = 0$. This means that $\operatorname{nil} \tilde{H}_{\mathscr{F}}(M) \leqslant k$. □

## 5. *Computations*

In this section we study further properties of the tangential category for some classes of examples. One theme is the comparison of the tangential category of a foliation with the category of each of its leaves. We describe lower and upper bounds for compact-Hausdorff foliations and suspensions.

### 5.1. *Twisted products*

The simplest foliations are products. The tangential category of a product foliation $F \times B \longrightarrow B$ is just the category of the fibre, or $\operatorname{cat}_{\mathscr{F}}(F \times B) = \operatorname{cat} F$.

Twisted product foliations generalize product foliations, by introducing a twisting via the action of a finite group $G$. The tangential category for twisted products is related to the *equivariant category* introduced by E. Fadell [**10**] in 1985 and much studied over the years [2, 20]. Thus twisted products are a good class of examples in which to compare the tangential category with previously defined invariants.

We recall Fadell's definition in this context. An invariant subset $U$ of a $G$-space $X$ is said to be $G$-categorical if there exists an equivariant homotopy $H : U \times I \longrightarrow X$ of the inclusion such that $H_1 : U \longrightarrow X$ has an image in a single orbit. The *equivariant category* is the least number of $G$-categorical open sets required to cover $X$. If $G$ acts freely on $M$ then the equivariant category $\operatorname{cat} X/G$ coincides with the ordinary category of the orbit space. When the action is not necessarily free it is just bounded below by the category of the orbit space, $\operatorname{cat} X/G \leqslant \operatorname{cat}_G X$.

To construct a *twisted product*, we assume there is given a free action of a finite group $G$ on the compact manifold $F$, and a possibly non-free action of $G$ on $B$. Then $G$ acts on the product $F \times B$ by $(g,(x,b)) \mapsto (xg^{-1}, gb)$. Let $M = F \times_G B$ be the quotient space of such an action and $q : F \times B \longrightarrow F \times_G B$ be the quotient map. Then $M$ is a manifold foliated by compact leaves of the form $q(F \times b)$. A point $\bar{b} \in B/G$ is *regular* when its isotropy group $G_b$ is trivial. Observe that the set of regular points is a dense connected open subset of $B/G$. The fibres $F = q(F \times b)$ with $\bar{b}$ a regular point are regular leaves of the foliation.

PROPOSITION 12. $\operatorname{cat} F \leqslant \operatorname{cat}_{\mathscr{F}}(F \times_G B) \leqslant \operatorname{cat}_G F = \operatorname{cat} F/G$.

*Proof.* Since $F$ is a regular leaf, by Proposition 5 we have $\operatorname{cat} F \leqslant \operatorname{cat}_{\mathscr{F}}(F \times_G B)$. Let $U \subset F$ be a $G$-categorical open set for $F$ and $h : U \times \mathbf{R} \longrightarrow F$ be an equivariant homotopy of the inclusion such that $h_1(U)$ is contained in a single orbit.

Define $H : (U \times_G B) \times \mathbf{R} \longrightarrow F \times_G B$ by $H_t([x,b]) = [h_t(x), b]$. Clearly, $H$ is well defined and $H_0$ is the inclusion. The leaves of $\mathscr{F}|_{U \times_G B}$ are the connected components of $[x, b_0]$, with $x \in U$. Then $H_t$ is a foliated map and $H_1([x, b_0]) = [x_0, b_0]$ with $x_0$ in the orbit of $h_1(x)$, so $H_1$ is constant on the leaves of $\mathscr{F}|_{U \times_G B}$. □

The leaves in the twisted product are copies of the original fibre $F$ factored by subgroups of $G$: for all $L \in \mathscr{F}$, $L = F/H$ where $H$ is a subgroup of $G$. Thus if there is a leaf $L_0 \in \mathscr{F}$ which is a quotient of the fibre $F$ by the whole group $G$, the tangential category is the category of this fibre: $\operatorname{cat}_{\mathscr{F}}(F \times_G B) = \operatorname{cat} F/G$. In other words, the tangential category coincides with the ordinary category of the leaf $L_0$ of maximal category if $L_0 = F/G$.

## 5.2. *Compact Hausdorff foliations*

A compact-Hausdorff foliation is a foliation with all leaves compact and its leaf space is Hausdorff. The structure of compact-Hausdorff foliations has been studied by various authors [7, 9]. The basic structure theorem for compact-Hausdorff foliations, due to D. B. A. Epstein, states that for any leaf $L \in \mathscr{F}$ there exists a finite subgroup $G$ of the orthogonal group $O(n)$, $n = \operatorname{codim} \mathscr{F}$; a homomorphism $h : \pi_1(L) \longrightarrow G$ and a leaf-preserving diffeomorphism of $\tilde{L} \times_G D$ onto a neighborhood of $L$, where $D$ is an open ball in $\mathbf{R}^n$ and $\tilde{L}$ is the covering space associated to the kernel of $h$. We call this neighborhood a *standard* neighborhood of $L$.

We can estimate the tangential category of a compact-Hausdorff foliation by using the above results on twisted products.

PROPOSITION 13. *Let $\mathscr{F}$ be a compact-Hausdorff foliation of a compact manifold $M$ and $L \in \mathscr{F}$ be a leaf with standard neighborhood $U$. Then* $\operatorname{cat}_{\mathscr{F}}(U, \mathscr{F}_U) = \operatorname{cat} L$.

*Proof.* The central leaf $L$ of $U$ is the quotient of its covering space $\tilde{L}$ by its holonomy group $G$, so $\operatorname{cat} L \leqslant \operatorname{cat}_{\mathscr{F}}(U, \mathscr{F}_U) \leqslant \operatorname{cat} \tilde{L}/G = \operatorname{cat} L$ by Propositions 12 and 5. □

Let $E \subset M$ be the set of exceptional leaves (leaves with non-trivial holonomy).

COROLLARY 14. *If $E$ is discrete,* $\operatorname{cat}_{\mathscr{F}}(E, \mathscr{F}_E) = \max\{\operatorname{cat} L | L \in \mathscr{F}\}$.

*Proof.* Let $E = \bigcup_{i \in I} L_i$ be the exceptional set. For each $L_i \subset E$, take $D_i \subset D$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$, where $U_i = \tilde{L} \times_{G_i} D_i$.

Let $U = \bigcup_{i \in I} U_i$, so $\operatorname{cat}_{\mathscr{F}}(U, \mathscr{F}_U) = \operatorname{cat}_{\mathscr{F}}(E, \mathscr{F}_E)$. Consider $\{W_{i_1}, \ldots, W_{i_r}\}$ an $\mathscr{F}$-categorical covering for $U_i$ and put $W_j = \bigcup_{i \in I} W_{i_j}$ (let $W_{i_j} = \emptyset$ if $j > \operatorname{cat} L_i$). Put $m = \max\{\operatorname{cat} L_i \mid i \in I\}$, so $\{W_1, \ldots, W_m\}$ is a tangentially categorical covering for $(U, \mathscr{F}_U)$.

The regular leaves of the foliation are covering spaces of the exceptional leaves, so their categories are no greater than $\max\{\operatorname{cat} L_i \mid i \in I\} = \max\{\operatorname{cat} L \mid L \in \mathscr{F}\}$. Since $E$ is saturated, this maximum is also a lower bound for its tangential category by Proposition 5. □

COROLLARY 15. *If $n_S$ is the number of standard neighborhoods required to cover $M$ then* $\operatorname{cat}_{\mathscr{F}}(M) \leqslant n_S \max\{\operatorname{cat} L | L \in \mathscr{F}\}$.

### 5.3. *Suspensions*

Let $\phi : \pi_1(N) \longrightarrow \operatorname{Diff}(T)$ be a representation of the fundamental group of the manifold $N$ as a group of diffeomorphisms of the manifold $T$. Let $p : \tilde{N} \longrightarrow N$ be the universal covering of $N$, and $E_\phi = \tilde{N} \times_\phi T$ be the quotient manifold of the action of $\pi_1(N)$ on $\tilde{N} \times T$. The quotient map $q : \tilde{N} \times T \longrightarrow E_\phi$ is a covering map and the projection $\Pi : E_\phi \longrightarrow N$ defined by $\Pi[\tilde{x}, z] = p(\tilde{x})$ is a locally trivial bundle with typical fibre $T$. The projection $\tilde{N} \times T \longrightarrow T$ induces a foliation $\mathscr{F}$ on $E_\phi$ which is transverse to the fibres of $\Pi$. It is called the *suspension* of $\phi$.

PROPOSITION 16. *Let $(E_\phi, \mathscr{F})$ be the suspension of $\phi : \pi_1(N) \longrightarrow$* $\operatorname{Diff}(T)$. *Then* $\operatorname{cat} \tilde{N} \leqslant \operatorname{cat}_{\mathscr{F}}(E_\phi) \leqslant \operatorname{cat} N$.

*Proof.* As $\tilde{N}$ is a covering of every leaf of the suspension, $\operatorname{cat} \tilde{N} \leqslant \operatorname{cat} L$, for all $L \in \mathscr{F}$ and by Proposition 5, $\operatorname{cat} \tilde{N} \leqslant \operatorname{cat}_{\mathscr{F}}(E_\phi)$.

Let $U \subset N$ be a smooth categorical open set, and $H : U \times \mathbf{R} \longrightarrow N$ be a homotopy $i_U \simeq x_0$. We shall prove that $\Pi^{-1}(U)$ is $\mathscr{F}$-categorical in $E_\phi$. First we define a map $\tilde{H} : p^{-1}(U) \times \mathbf{R} \longrightarrow \tilde{N}$ such that

$$\tilde{H}(\alpha(\tilde{x}), t) = \alpha \tilde{H}(\tilde{x}, t) \qquad \forall\, \alpha \in \pi_1(N) \tag{5}$$

and

$$\tilde{H}(\tilde{x}, 1) \in p^{-1}(x_0) \tag{6}$$

by considering the path $H_t(\tilde{x}) = H(p(\tilde{x}), t)$ in $N$, which connects $p(\tilde{x})$ to $x_0$, and lifting it to a unique path $\tilde{H}(\tilde{x}, t)$ in $\tilde{N}$ such that $\tilde{H}_0(\tilde{x}) = \tilde{x}$. The smoothness of $\tilde{H}$ is proved in the standard way.

Now, since $\Pi^{-1}(U) = q(p^{-1}(U) \times T)$, we define $H^\phi : \Pi^{-1}(U) \times \mathbf{R} \longrightarrow E_\phi$ by the formula

$$H^\phi([\tilde{x}, z], t) = [\tilde{H}(\tilde{x}, t), z]$$

which is well defined by (5). Clearly $H_0^\phi[\tilde{x}, z] = [\tilde{H}_0(\tilde{x}), z] = [\tilde{x}, z]$ is the inclusion. Moreover $H^\phi$ is a foliated homotopy, since the leaf of $\mathscr{F}$ passing through $[\tilde{x}, z]$ is $\pi_\phi(\tilde{N} \times \{z\})$, and $H_t^\phi[\tilde{y}, z] = [\tilde{H}_t(\tilde{y}), z]$ does not change $z$ for any $t$.

It only remains to show that the map $H_1^\phi$ is constant along the leaves of the induced foliation $\mathscr{F}_U$ on $\Pi^{-1}(U) \subset E_\phi$.

Let $[\tilde{x}, z] \in \Pi^{-1}(U)$. Then its leaf is a connected component $L_U$ of $L \cap \Pi^{-1}(U)$, where $L = \pi_\phi(\tilde{N} \times \{z\})$. In this way, if $[\tilde{y}, z] \in L_U$ there is a continuous path $\gamma(s) = [\tilde{x}_s, z] \in L_U$ joining $[\tilde{x}, z]$ to $[\tilde{y}, z]$. It is not hard to prove that $\tilde{N} \times \{z\}$ is a covering of $L$ with $\ker\phi$ as group of transformations, so we can lift $\gamma(s)$ to some path $\tilde{\gamma}(s) = (\tilde{x}_s, z)$ in $p^{-1}(U)$ with $\tilde{\gamma}(0) = (\tilde{x}, z)$. Then by (6), the continuous map $\tilde{H}_1\tilde{\gamma}(s)$ takes values in the discrete fibre $p^{-1}(x_0)$, hence it is constant. Thus $H_1^\phi[\tilde{y}, z] = H_1^\phi[\tilde{x}, z]$ as claimed. □

## 6. *Foliations on the torus $T^2$*

In this section we show that any foliation of dimension 1 on the torus has tangential category 2. On the other hand, when $\dim\mathscr{F} = 0$ we have $\mathrm{cat}_{\mathscr{F}}(M) = 1$ by definition, and when $\dim\mathscr{F} = 2$ we have $\mathrm{cat}_{\mathscr{F}}(T^2) = \mathrm{cat}(T^2) = 3$, the Lusternik–Schnirelmann category of the torus. Thus, the tangential category for foliations of the torus is an indiscriminate invariant.

Assume that $\mathscr{F}$ is a 1-dimensional foliation of $T^2$. Since the leaves of $\mathscr{F}$ are $S^1$ or $\mathbf{R}$, there is always some leaf which is either not closed or not contractible; hence $\mathrm{cat}_{\mathscr{F}}(T^2) > 1$ by Proposition 6, so we must just prove that $\mathrm{cat}_{\mathscr{F}}(T^2) \leqslant 2$.

We begin by recalling [11] that any foliation $\mathscr{F}$ on $T^2$ is obtained by glueing together a finite number of suspensions and a finite number of Reeb components.

### 6.1. *Suspensions*

Let $f \in \mathrm{Diff}(S^1)$ be such that $T^2 = \mathbf{R} \times_f S^1$. We denote $p : \mathbf{R} \times S^1 \longrightarrow T^2$, the quotient projection with $p(s, x) = [s, x]$. Let $T_i = p(\{s_i\} \times S^1)$, $i = 1, 2$, be two complete transversal submanifolds. Then the open sets $U_i = T^2 - T_i$, $i = 1, 2$, are tangentially categorical. Alternatively one can apply Proposition 16.

### 6.2. *The Reeb foliation on $T^2$*

We are indebted to Professor P. Schweitzer for suggesting the ideas in this example.

Let us consider the action on $H = \{(x, y) \in \mathbf{R}^2 - \{0\} \mid y \geqslant 0\}$ generated by the homothety $\varphi(v) = \frac{1}{4}v$. Since the second projection $pr_2 : H \longrightarrow \mathbf{R}$ is an equivariant submersion, the horizontal foliation induces a foliation on the quotient space $S^1 \times I$. The leaves of this *Reeb component* are the images of the straight lines $\mathbf{R} \times \{y\}$, $y > 0$, and the boundary $S^1 \times \{t\}$, $t = 0, 1$. By identifying the latter, we obtain the so-called Reeb foliation on $T^2$.

In order to compute its tangential category, let us consider $H$ the fundamental domain of the action above,

$$D = \{(x, y) \in H \mid 1 \leqslant \sqrt{x^2 + y^2} \leqslant 4\}$$

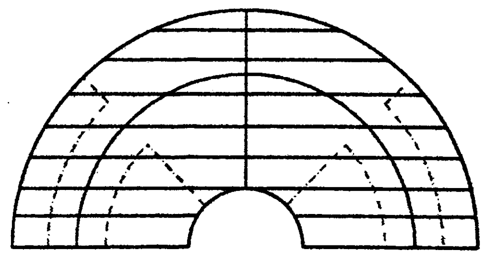

FIGURE 1. *The set* $M \cup P$.

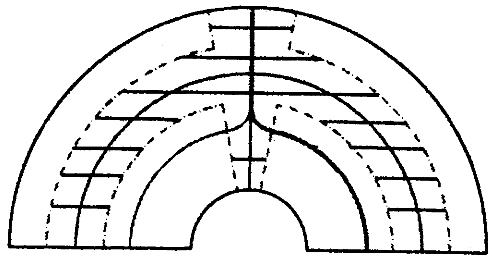

FIGURE 2. *The smooth curve* $(x(y), y)$.

and seek for a covering $\{U_1, U_2\}$ of $D$ by open sets which are tangentially categorical for the horizontal foliation and invariant by the action. Their projections $\{V_1, V_2\}$ on the torus will be a covering of $T^2$ by tangentially categorical open sets.

First, let $M = \{r_\theta \in D \mid \theta = \pi/2\}$ (meridian) and $P = \{r_\theta \in D \mid r = r_0\}$ (parallel) for some $1 < r_0 < 4$, where $r_\theta = (r, \theta)$ are polar coordinates in $D$ (see Figure 1). Then $U_1 = D - \{M \cup P\}$ is an $\mathscr{F}$-categorical open set since it is a distinguished open set of the foliated chart. Second, an open neighborhood of $M \cup P$ is given by $U_2 = A \cup B$, where

$$A = \left\{r_\theta \mid \theta \in \left(\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\right)\right\}$$

$$B = \{r_\theta \mid r \in (r_0 - \delta, r_0 + \delta)\}.$$

and $\epsilon, \delta > 0$ are arbitrarily small.

We now prove that $U_2$ is tangentially categorical.

Let $y_0 = (r_0 - \delta)\cos\epsilon$ be the second cartesian coordinate of the point with polar coordinates $r = r_0 - \delta$, $\theta = \pi/2 - \epsilon$. Then we define $c : U_2 \longrightarrow D$ as

$$c(x, y) = \begin{cases} (0, y) & \text{if } (x, y) \in A \text{ or } y \geqslant y_0 \\ (\pm x(y), y) & \text{if } (x, y) \in B \text{ and } 0 \leqslant y \leqslant y_0 \end{cases}$$

where $(x(y), y)$ is any smooth curve as in Figure 2. The map $c$ is smooth, constant along the plaques, and compatible with the action. Moreover, it is integrably homotopic to the inclusion by the homotopy

$$G : U_2 \times \mathbf{R} \longrightarrow D, \qquad ([(x, y)], t) \mapsto [(1 - t)(x, y) + tc(x, y)].$$

## 6.3. *The general case*

Suspensions are characterized by the existence of a closed complete transversal submanifold. When $\mathscr{F}$ is not a suspension, it can be obtained from some foliation

$\tilde{\mathscr{F}}$ on the cylinder $S^1 \times I$, which is tangential to the boundary, by identifying the two components of the boundary by an orientation-preserving homeomorphism of $S^1$.

THEOREM 17. *Let $\mathscr{F}$ be a foliation of dimension 1 on $T^2$. Then* $\mathrm{cat}_{\mathscr{F}}(T^2) = 2$.

*Proof.* It only remains to prove that $\mathrm{cat}_{\mathscr{F}}(T^2) \leqslant 2$ when $\mathscr{F}$ contains a Reeb component. Suppose $\mathscr{F}$ is obtained by glueing $m$ suspensions and $n$ Reeb components. Let $T_j$, $j = 1, \ldots, n$, be parallels on each suspension, and $M_i$, $P_i$, $i = 1, \ldots, m$, be the distinguished meridians and parallels on each Reeb component. We take

$$P = \left(\bigcup_{i=1}^{n} T_i\right) \cup \left(\bigcup_{j=1}^{m} P_j\right)$$

(a parallel in $T^2$) and the set

$$C = P \cup \left(\bigcup_{i=1}^{n} M_i\right)$$

whose complementary $V_1 = T^2 - C$ is tangentially categorical because it is a disjoint union of foliated open sets.

Finally, we construct an $\mathscr{F}$-categorical neighborhood $V_2$ of the set $C$ as before. Then $\{V_1, V_2\}$ is an $\mathscr{F}$-categorical covering of $T^2$. □

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