Length of parallel curves and rotation index

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Abstract

We prove that the length difference between a closed periodic curve (possibly with self-intersections) and its parallel curve at a sufficiently small distance ε equals $2\pi\varepsilon\omega$, where ω is the rotation index. As an application, the rotation index of a curve can be estimated by means of Cauchy-Crofton's formula.

Key words: parallel curve, arclength, rotation index, Cauchy-Crofton's formula
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1. Introduction.

The aim of this note is to prove the following result. Let α be a closed periodic regular curve, possibly with self-intersections; let β be the parallel curve at distance $\varepsilon \geq 0$. Assume that ε is small enough to not exceed the radius of curvature of α when $\kappa > 0$ (κ is the signed curvature of α). Let ω be the rotation index of α.

Theorem 1.1. *The length difference $L(\alpha) - L(\beta)$ equals $2\pi\varepsilon\omega$.*

I believe that this result, although maybe not new, at least is not well known in differential geometry and I did not find it in this explicit form in the literature. It is related to Steiner's formula (Gray, 1990, p. 10). A particular case (when the curve α is simple, so $\omega = \pm 1$) appears in (DoCarmo, 1976, p. 47). The result in (Santaló, 2004, p. 8) for convex bodies is also related.

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Although elementary, it seems interesting because it actually finds the exact difference and shows that a relatively sophisticated invariant like the rotation index can be determined by a much simpler invariant, namely, the length of a curve. As an application, we show a simple way to estimate the rotation index by using techniques from integral geometry.

The converse is also a useful observation: computing the length by the rotation index, since this is a regular homotopy invariant (Montiel et al., 2005, p. 330). As a corollary, the difference of the length of a curve and its ε-parallel curve is a regular homotopy invariant. Possibly this could also be directly proved by a variational argument.

I have tried to do the paper as self-contained as possible.

2. Basic definitions and notations.

Let $\alpha(t)$ be a differentiable plane curve, defined on the interval $[a, b]$. The *length* of the curve is given by

$$L(\alpha) = \int_a^b |\alpha'(t)|\, dt. \tag{1}$$

Suppose that the curve is regular, which means that the speed $|\alpha'|$ never vanishes. Then the *arc-length parameter* $s(t)$, defined by $ds = |\alpha'|dt$ and $s(a) = 0$, serves to reparametrize the curve with unit speed.

The (signed) *curvature* of α is the function

$$\kappa = \det(\alpha', \alpha'')/|\alpha'|^3. \tag{2}$$

If the parameter is arc-length, the absolute value of the curvature is $|\kappa| = |\ddot{\alpha}|$, the module of the second derivative.

When $\kappa \neq 0$, the *unitary normal vector* $\overrightarrow{\mathbf{n}} = \ddot{\alpha}/|\ddot{\alpha}|$ is well defined. It is perpendicular to the tangent direction and it points inwards the curve. For an arbitrary parameter t, the vector α'' is not collinear to $\ddot{\alpha}$, but both are on the same side of the tangent line.

3. Parallel curves.

Let $\alpha(t)$ be an arbitrary regular parametrization. At each point $\alpha(t)$ we take a unitary vector $\overrightarrow{\mathbf{e}(t)}$ orthogonal to $\alpha'(t)$ and such that $\det(\alpha', \overrightarrow{\mathbf{e}}) > 0$. In other words, $\overrightarrow{\mathbf{e}}$ is obtained by rotating in the counter-clockwise sense the unitary tangent vector $\overrightarrow{\mathbf{t}} = \dot{\alpha} = \alpha'/|\alpha'|$ (in Alfred Gray's book Gray (1998), $\overrightarrow{\mathbf{e}}$ is denoted by $J\alpha'$).

Then $\overrightarrow{\mathbf{e}} = +\overrightarrow{\mathbf{n}}$ when $\kappa > 0$ (the curve turns left) and $\overrightarrow{\mathbf{e}} = -\overrightarrow{\mathbf{n}}$ when $\kappa < 0$ (the curve turns right).

Definition 3.1. We define the (left) *parallel* curve to α at distance $\varepsilon \geq 0$ as the curve $\beta = \alpha + \varepsilon\overrightarrow{\mathbf{e}}$.

Remark 3.2. It is unnecessary to consider the case $\varepsilon \leq 0$, as we can always reparametrize the curve α in the opposite direction.

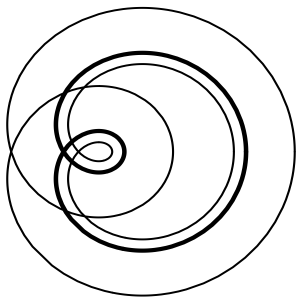

Figure 1: A curve (Pascal Snail) and two parallels

We now discuss the regularity of β. For that we have to take into account the *radius of curvature* $1/|\kappa|$ and the *evolute* of α, which is the geometric locus of the *centers of curvature* $\alpha + (1/|\kappa|)\overrightarrow{\mathbf{n}} = \alpha + (1/\kappa)\overrightarrow{\mathbf{e}}$.

By differentiating with respect to the arc-length parameter s of α, we obtain $\dot{\overrightarrow{\mathbf{e}}} = -\kappa\overrightarrow{\mathbf{t}}$, which is just a reformulation of the usual *Frénet formula* $\dot{\overrightarrow{\mathbf{n}}} = -|\kappa|\overrightarrow{\mathbf{t}}$ (Montiel et al., 2005; DoCarmo, 1976; Gray, 1998). Hence $d\beta/ds = (1 - \varepsilon\kappa)\dot{\alpha}$ and

$$|d\beta/ds| = |1 - \varepsilon\kappa|. \tag{3}$$

It follows that the parallel β has a singularity each time ε equals $1/\kappa$. This can only occur (as we are taking $\varepsilon \geq 0$) when $\kappa > 0$ and ε equals the radius of curvature, i.e. the parallel β touches the evolute of α at corresponding points (see figure 2).

Remark 3.3. The evolute itself has singularities at the places where the curvature attains a critical value; this is a consequence of the fact that the tangent vector to the evolute points in the *normal* direction to α.

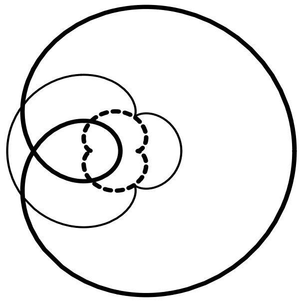

Figure 2: The same curve, its evolute (dashed) and one parallel with two singularities

By applying definition (1) to formula (3) we obtain the length of β.

Theorem 3.4. *The length of the left parallel curve β at distance $\varepsilon \geq 0$ to α is given by*

$$L(\beta) = \int_0^{L(\alpha)} |1 - \varepsilon\kappa(s)|\, ds.$$

In Corollary 3.6 we shall emphasize two particular cases of Theorem 3.4.

Definition 3.5. The *total curvature* of the curve α is the number

$$K = \int_0^{L(\alpha)} \kappa(s)\, ds = \int_a^b \kappa(t)|\alpha'(t)|\, dt.$$

Corollary 3.6. 1) *If $\kappa \leq 1/\varepsilon$ then $L(\beta) = L(\alpha) - \varepsilon K$;*
2) *If $\kappa \geq 0$ and $\varepsilon \geq 1/\kappa$ then $L(\beta) = \varepsilon K - L(\alpha)$.*

Example 3.7. Let $\alpha(t) = (R\cos t, R\sin t)$, $0 \leq t \leq \pi$, be a half-circle with a big radius $R > 0$. It has global curvature $K = \pi$. The parallel curve at distance $R + 1$ is a small half-circle of radius 1 wich goes backwards. Its length is $(R+1)\pi - \pi R = \pi$.

Remark 3.8. From (2) it follows that the curvature of the parallel curve $\beta = \alpha + \varepsilon\overrightarrow{\mathbf{e}}$ is given by $\kappa_\beta = \kappa/|1 - \varepsilon\kappa|$, see (Gray, 1998, p. 117). Then, when $\kappa < 1/\varepsilon$, β has the same evolute that α.

4. Closed curves.

Let $\alpha(t)$ be a regular curve defined in $[a, b]$. From now on we shall assume that our curve is *closed* and *periodic*, i.e. it satisfies $\alpha(a) = \alpha(b)$ and $\alpha'(a) = \alpha'(b)$.

Let us recall the notion of *rotation index* (also called *turning number*). For simplicity, we parametrize α by the arc-length $s \in [0, L(\alpha)]$, so the tangent vector $\overrightarrow{\mathbf{t}} = \dot{\alpha}$ has module 1. Write $\dot{\alpha} = (\cos\theta, \sin\theta)$. Then

$$\kappa = \det(\dot{\alpha}, \ddot{\alpha}) = d\theta/ds,$$

which proves that it is always possible to choose the angle θ in a differentiable way (unique for any preassigned value of $\theta(0)$). Namely

$$\theta(s) = \theta(0) + \int_0^s \kappa. \tag{4}$$

Clearly θ does not depend on the parametrization of α. Moreover, since the curve is periodic, the difference $\theta(b) - \theta(a)$ equals $2\pi\omega$, for some integer number ω.

Definition 4.1. The integer ω is called the *rotation index* of α. It measures how many times the curve turns with respect to a fixed direction (Gray, 1998, p. 159).

Example 4.2. The rotation index of the Pascal Snail in Figure 1 is $\omega = \pm 2$ depending on the sense of rotation.

The following result is immediate from (4).

Proposition 4.3. *The total curvature K of a closed periodic curve with rotation index ω equals $2\pi\omega$.*

Finally, if ε is not too large, directly from Corollary 3.6 we obtain Theorem 1.1.

In addition, we have the following consequence of the Hopf's theorem on turning tangents (Montiel et al., 2005, p. 333).

Corollary 4.4. *For a simple closed curve in the plane (i.e., one without self-intersections), the length of the ε-parallel curve minus the length of the original curve is always $\pm 2\pi\varepsilon$, for ε small enough.*

Remark 4.5. The following is a very well-known fact, which seems quite striking to non-mathematicians. Imagine that we surround the earth by the ecuator with a cable at ground level. If we next wanted the cable to stand a metre above ground level, how much extra cable would we need? The answer is: a little more than 6 metres. The reason is: $2\pi(R+1) - 2\pi R = 2\pi$. Of course this is a very particular case of our result.

5. Estimation of the rotation index.

An estimation of the rotation index of the closed curve α can be obtained by using Cauchy-Crofton's formula in order to estimate the lengths of α and the offset curve β, then applying Theorem 1.1.

More precisely, let us take a family of equidistant parallel lines $L_1, \ldots, L_p$ as in Figure 3.

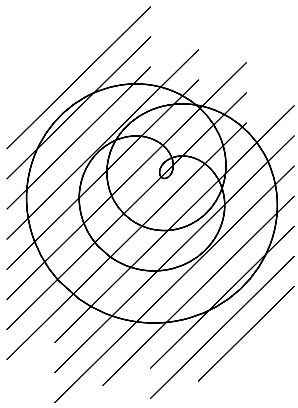

Figure 3: Estimating the rotation index

Assume that the curve α cuts L_i in n_i points. Let $n(\alpha) = n_1 + \cdots + n_p$ be the total number of intersections. Calling r the distance between L_i and L_{i+1} we get that $\tilde{l}(\alpha) = n(\alpha)\pi r/2$ is an approximation of the length of α (Santaló, 2004, p. 31). The accuracy of this estimator depends on r. Let β be the parallel curve to α at distance $\varepsilon > r$. Then

$$\tilde{\omega} = \frac{\tilde{l}(\alpha) - \tilde{l}(\beta)}{2\pi\varepsilon} = (n(\alpha) - n(\beta))\frac{r}{4\varepsilon}$$

is an estimation of the rotation index ω.

Example 5.1. In the example of Figure 3 we have taken the curve $\alpha(t)$ with polar coordinates $\rho(t) = 1+4\cos(t+2)$, $0 \leq \theta \leq 2\pi$, the offset β with $\varepsilon = 1.1$ and the distance $r = 0.45$. We count $n(\alpha) = 34$, $n(\beta) = 15$, so $\tilde{\omega} = 1.94318$, which is a very good estimation of $\omega = 2$. In practice one has to compute the average estimation after considering different directions.

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