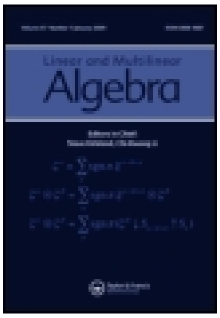

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A topological approach to left eigenvalues of quaternionic matrices

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A topological approach to left eigenvalues of quaternionic matrices

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It is known that a 2×2 quaternionic matrix has one, two or an infinite number of left eigenvalues, but the available algebraic proofs are difficult to generalize to higher orders. In this paper, a different point of view is adopted by computing the topological degree of a characteristic map associated to the matrix and discussing the rank of the differential. The same techniques are extended to 3×3 matrices, which are still lacking a complete classification.

Keywords: quaternionic matrices; left eigenvalues; characteristic map; topological degree

AMS Subject Classifications: 15A33; 15A18

1. Introduction

In 1985, Wood [1] proved that any $n \times n$ quaternionic matrix A has at least one *left eigenvalue*, that is, a quaternion $\lambda \in \mathbb{H}$ such that the matrix $A - \lambda\,\mathrm{Id}$ is singular. However, even for matrices of small size, the left spectrum is not fully understood yet (see Zhang's papers [2,3] for a survey). For instance, it was only in 2001 when Huang and So [4] proved that a 2×2 matrix may have one, two or an infinite number of left eigenvalues; a different proof was presented by the authors in [5]. While Wood used topological techniques, namely homotopy groups, the last two papers are of algebraic nature and seemingly difficult to generalize for $n > 2$.

In this article we try a different approach. The basic ideas will be those of characteristic map, linearization and topological degree. In the simplest case $n = 2$, we associate to each matrix A a polynomial μ_A whose roots are the left eigenvalues; computing the rank of its differential allows us to classify the different types of spectra.

In the second part of the paper, we extend those techniques to 3×3 matrices. This time the characteristic map may not be a polynomial but a rational map, usually with a point of discontinuity. This seems to require the use of a local version of degree, although a closer look allows us to reduce the problem to the global theory. In particular, this gives a new proof of the existence of left eigenvalues.

Unlike the 2×2 case, where the linear equations that appear correspond to the well-known Sylvester equation, the 3×3 situation is much more complex. Then for $n = 3$ a

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complete classification of spectra is still unknown. Nevertheless, our method allows to deal with specific examples and opens the way for a better understanding of the general case.

The paper is organized as follows. In Section 2, we recall some topological and algebraic preliminaries. Although our ideas are closely related to the theory of quasideterminants [6], we have preferred a development based on Study's determinant which parallels the commutative setting. Section 3 is devoted to a notion of *characteristic map* for the left eigenvalues of a quaternionic matrix A, that is, a map $\mu_A : \mathbb{H} \to \mathbb{H}$ such that $\mu_A(\lambda) = 0$ if and only if the matrix $A - \lambda\,\mathrm{Id}$ is not invertible. In Section 4, we give a complete classification of the left spectra of 2×2 matrices, depending on the rank of the characteristic map. In Section 5, we prove that any 3×3 matrix A has a characteristic map μ_A which is either a polynomial (when some of the entries outside the diagonal is null) or a rational function with a distinguished point π_A called its *pole*. When μ_A is continuous it has topological degree 3. However, π_A may be a point of discontinuity; in this case the matrix $B = A - \pi_A\,\mathrm{Id}$ turns out to be invertible and we prove that B and B^{-1} have diffeomorphic spectra and that B^{-1} admits a polynomial characteristic map. The last section offers several illustrative examples.

2. Preliminaries

2.1. *Topological degree*

The topological degree (or Brouwer degree) of a map can be defined by techniques either from algebraic topology [7] or from functional analysis [8–10]. We want to apply the following global result (cf. [11, p.101]):

Theorem 2.1 *Let M be a connected closed oriented manifold, let $\mu\colon M \to M$ be a differentiable map of degree k. Let $m \in M$ be a regular value such that the differential $\mu_{*\lambda}$ preserves the orientation for any λ in the fiber $\mu^{-1}(m)$. Then $\mu^{-1}(m)$ is a finite set with k points.*

Sometimes one has to deal with a local notion of degree. The main result is the following one [8, p.38].

Theorem 2.2 *Let Ω be a bounded open set in $\mathbb{R}^n$. Let $\mu\colon \overline{\Omega} \to \mathbb{R}^n$ be a map which is continuous on the closure $\overline{\Omega}$ and differentiable on Ω. Suppose that 0 is a regular value of μ and that $0 \notin \mu(\partial\Omega)$. Then*

$$\deg(\mu, \Omega, 0) = \sum_{\lambda \in \mu^{-1}(0)} \mathrm{sgn}[J_\mu(\lambda)]$$

where we denote by J_μ the Jacobian of μ.

A well-known consequence is that if $\deg(\mu, \Omega, 0) \neq 0$ then the equation $\mu(\lambda) = 0$ has at least one solution in Ω. In fact, for maps from the sphere into itself, the *existence* of solutions only depends on the *continuity*, by the following result [12, Ch. VIII, Ex. 2.5, p.191]:

Proposition 2.3 *Let $\mu\colon S^4 \to S^4$ be a continuous map whose degree is nonzero. Then μ is surjective.*

2.2. *Linearization*

We consider the space of quaternions $\mathbb{H}$ as a differentiable manifold diffeomorphic to $\mathbb{R}^4$. Then the differential $\mu_{*\lambda}\colon \mathbb{H} \to \mathbb{H}$ at the point $\lambda \in \mathbb{H}$ of the differentiable map $\mu\colon \mathbb{H} \to \mathbb{H}$ can be computed by means of the formula

$$\mu_{*\lambda}(X) = \frac{d}{dt}_{|t=0}\mu(\lambda + tX) = \lim_{t\to 0}\frac{1}{t}\left(\mu(\lambda + tX) - \mu(\lambda)\right).$$

LEMMA 2.4 (1) *Let $f, g\colon \mathbb{H} \to \mathbb{H}$ be two differentiable maps. Then the differential of the product is given by*

$$(f \cdot g)_{*\lambda}(X) = f_{*\lambda}(X) \cdot g(\lambda) + f(\lambda) \cdot g_{*\lambda}(X);$$

(2) *Assume that $f(\lambda) \neq 0$ for all $\lambda \in \mathbb{H}$. Let $1/f\colon \mathbb{H} \to \mathbb{H}$ be the map given by $(1/f)(\lambda) = f(\lambda)^{-1}$. Then*

$$(1/f)_{*\lambda}(X) = -f(\lambda)^{-1} f_{*\lambda}(X) f(\lambda)^{-1}.$$

2.3. *Sylvester equation*

Let $P, Q, R \in \mathbb{H}$ be three quaternions. We are interested (see formula (6)) in the rank of $\mathbb{R}$-linear maps $\Sigma\colon \mathbb{H} \to \mathbb{H}$ of the form $\Sigma(X) = PX + XQ$. The equation $\Sigma(X) = R$ has been widely studied, sometimes under the name of Sylvester equation [13].

LEMMA 2.5 (1) *Let $P = t + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then the matrix associated to the $\mathbb{R}$-linear map $X \mapsto PX$ with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is $L(P) = \Re(P)\,\mathrm{Id} + A(P)$, where $\Re(P)$ is the real part of P and*

$$A(P) = \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{bmatrix};$$

(2) *Analogously, if $Q = s + u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ then the matrix associated to the right translation $X \mapsto XQ$ is $R(Q) = \Re(Q)\,\mathrm{Id} + B(Q)$, where*

$$B(Q) = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & w & -v \\ v & -w & 0 & u \\ w & v & -u & 0 \end{bmatrix}.$$

Next Theorem is a reformulation of the results by Janovská and Opfer in [14], see also [15].

THEOREM 2.6 (1) *The rank of Σ is even, namely* 0, 2 *or* 4*;*

(2) rank $\Sigma < 4$ *if and only if P and $-Q$ are similar quaternions, i.e. they have the same norm and the same real part;*

(3) rank $\Sigma = 0$ *if and only if P is a real number and $Q = -P$.*

Proof The matrix associated to Σ is

$$J = \begin{bmatrix} t+s & -x-u & -y-v & -z-w \\ x+u & t+s & -z+w & y-v \\ y+v & z-w & t+s & -x+u \\ z+w & -y+v & x-u & t+s \end{bmatrix}.$$

Since

$$\det J = (t+s)^4 + 2(t+s)^2(x^2+y^2+z^2+u^2+v^2+w^2) + (x^2+y^2+z^2-u^2-v^2-w^2)^2 \geq 0, \quad (1)$$

the matrix J has rank 4 except when $t+s=0$ and $x^2+y^2+z^2=u^2+v^2+w^2$. In this case J is skew-symmetric, hence its rank is even. If rank $\Sigma = 0$ it follows that $x=y=z=0$ and $u=v=w=0$. □

2.4. *Resolution of arbitrary linear equations*

More generally, let us consider a linear equation of the form

$$P_1 X Q_1 + \cdots + P_n X Q_n = R, \quad \text{with } P_i, Q_i, R \in \mathbb{H}. \quad (2)$$

For each bilateral term $X \mapsto PXQ$, the matrices $L(P)$ and $R(Q)$ commute, because $P(XQ) = (PX)Q$. Then $A(P)$ and $B(Q)$ commute too. This implies that the quaternionic linear Equation (2) is equivalent to the real linear system $MX = R$, where $X, R \in \mathbb{R}^4$ and M is the 4×4 real matrix

$$M = \sum_i L(P_i)R(Q_i) = \sum_i a_i b_i \,\mathrm{Id} + \sum_i (a_i B_i + b_i A_i) + \sum_i A_i B_i$$

with $a_i = \Re(P_i)$, $b_i = \Re(Q_i)$, $A_i = A(Q_i)$ and $B_i = B(Q_i)$. Contrary to the case $n = 2$, when $n \geq 3$ the rank of M may be odd.

Example 2.7 The rank of the matrix associated to the bilateral linear equation $\mathbf{k}X + X(2-\mathbf{i}) - 2\mathbf{j}X\mathbf{j} = 0$ is 3.

2.5. *Determinants*

It is possible to generalize to the quaternion the norm $|\det|$ (that is, with real values) of the complex determinant. Papers [16, 17] are surveys of the general theory of quaternionic determinants. For the relationship between Study's determinant and quasideterminants see [6, p.76–85].

Definition 2.8 Let the quaternionic matrix $A \in \mathcal{M}_{n\times n}(\mathbb{H})$ be decomposed as $A = X + \mathbf{j}Y$ with $X, Y \in \mathcal{M}_{n\times n}(\mathbb{C})$. We shall call *Study's determinant* of A the non-negative real number

$$\mathrm{Sdet}(A) := (\det c(A))^{1/2},$$

where $c(A)$ is the complex matrix $c(A) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in \mathcal{M}_{2n\times 2n}(\mathbb{C})$.

Remark 1 Up to the exponent, this is the same determinant which appears in Theorem 8.1 of [2], as well as others considered in [16]. We have normalized the exponent to $1/2$ in order to ensure that $\mathrm{Sdet}(D) = |q_1 \ldots q_n|$ for diagonal matrices $D = \mathrm{diag}(q_1, \ldots q_n)$.

Proposition 2.9 ([17]) Sdet *is the only functional that verifies the properties:*

(1) $\mathrm{Sdet}(AB) = \mathrm{Sdet}(A) \cdot \mathrm{Sdet}(B)$*;*
(2) *If A is a complex matrix then* $\mathrm{Sdet}(A) = |\det(A)|$.

The following immediate consequences are very useful for computations.

COROLLARY 2.10 (1) $\mathrm{Sdet}(A) > 0$ *if and only if the matrix A is invertible;*

(2) *Let A and $B = PAP^{-1}$ be similar matrices, then* $\mathrm{Sdet}(A) = \mathrm{Sdet}(B)$*;*

(3) $\mathrm{Sdet}(A)$ *does not change when a (right) multiple of one column is added to another column;*

(4) $\mathrm{Sdet}(A)$ *does not change when a (left) multiple of one row is added to another row;*

(5) $\mathrm{Sdet}(A)$ *does not change when two columns (or two rows) are permuted.*

We shall need the following result too (we have not found it explicitly in the literature):

PROPOSITION 2.11 *For any matrix with two submatrices A, B of order m and n, respectively, it holds that* $\mathrm{Sdet}\begin{bmatrix} A & 0 \\ * & B \end{bmatrix} = \mathrm{Sdet}(A) \cdot \mathrm{Sdet}(B)$.

2.6. *Jacobi identity*

Let C be a complex $n \times n$ matrix. Let $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_p\}$ be the two subsets of $\{1, \dots, n\}$ with the same size p. Let us denote by $C_{I,J}$ the submatrix formed by the rows with index in I and the columns with index in J. On the other side, let us denote by $C^{I,J}$ the complementary matrix obtained by suppressing the rows in I and the columns in J.

The following *Jacobi identity* is attributed to Kronecker in [18].

LEMMA 2.12 *Assume that the complex matrix C is invertible. Then*

$$\det (C^{-1})_{I,J} = (-1)^{I+J} \det C^{J,I} / \det C,$$

where $I + J$ means $i_1 + \cdots + i_p + j_1 + \cdots + j_p$.

A generalization to quasideterminants appears in [6, Theorem 1.5.4, p.74], see also Section 3.2. We shall use Study's determinant to establish an analogous result in the quaternionic setting.

PROPOSITION 2.13 *Let A be an invertible quaternionic matrix. Then*

$$\mathrm{Sdet}\, (A^{-1})_{I,J} = \mathrm{Sdet}\, A^{J,I} / \mathrm{Sdet}\, A.$$

Proof If $I = \{i_1, \dots, i_p\}$ we denote $I' = I + n = \{i_1 + n, \dots, i_p + n\}$; analogously $J' = J + n$. The result follows from Lemma 2.12 because

$$c\left((A^{-1})_{I,J}\right) = c(A^{-1})_{I \cup I', J \cup J'} = \left(c(A)^{-1}\right)_{I \cup I', J \cup J'}$$

and

$$c(A^{J,I}) = c(A)^{J \cup J', I \cup I'}.$$

□

3. Characteristic equation

The problem we are proposing here is to find a characteristic map for the left eigenvalues of a given matrix A, that is, to find a map $\mu_A \colon \mathbb{H} \to \mathbb{H}$ such that $\mu_A(\lambda) = 0$ if and only if λ is a left eigenvalue of A. Notice that the function $\mathrm{Sdet}(A - \lambda\, \mathrm{Id})$ is real-valued, so it is not of interest from the point of view of the topological degree, nor it is solvable in any obvious way.

3.1. *Left eigenvalues*

Let A be a $n \times n$ matrix with quaternionic coefficients.

Definition 3.1 The quaternion $\lambda \in \mathbb{H}$ is a *left eigenvalue* of A if the matrix $A - \lambda\,\mathrm{Id}$ is not invertible, or equivalently $\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = 0$.

Let $\sigma_l(A)$ be the left spectrum, i.e. the set of left eigenvalues, of the matrix A.

PROPOSITION 3.2 *The spectrum $\sigma_l(A)$ is compact.*

Proof The spectrum is a closed set because it is given by the equation $\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = 0$. It is bounded because $\lambda \in \sigma_l(A)$ if and only if there exists $v \in \mathbb{H}^n$, $v \neq 0$, such that $Av = \lambda v$; then

$$|\lambda| = \frac{|\lambda v|}{|v|} \leq \sup_{w \neq 0} \frac{|Aw|}{|w|} = |A|.$$

In fact, all the left eigenvalues of A are located in the union of n Geršgorin balls [3, Theorem 6, p.146]. □

PROPOSITION 3.3 *Let B be an invertible matrix. Then $\lambda \in \sigma_l(B)$ if and only if $\lambda^{-1} \in \sigma_l(B^{-1})$.*

Proof If $Bv = \lambda v$ then $B^{-1}(\lambda v) = B^{-1}Bv = v = \lambda^{-1}(\lambda v)$. □

3.2. *Background*

When the matrix A is hermitian, all left eigenvalues are real numbers so they coincide with the right eigenvalues [19]. Moreover, it is possible to define a true determinant for hermitian matrices [2,20], which allows to construct a characteristic polynomial $p(t) = \det(A - t\,\mathrm{Id})$ with real variable.

For the general case, $\mathrm{Sdet}(A)$ equals, up to an exponent, the determinant of AA^*. On the other hand, Zhang [2] pointed out that if the quaternionic matrix is decomposed as $A = X + \mathbf{j}Y$, with $X, Y \in \mathcal{M}_{n\times n}(\mathbb{C})$, then its left eigenvalues $\lambda = x + \mathbf{j}y$, with $x, y \in \mathbb{C}$, are the roots of the function $\sigma : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ given by

$$\sigma(x, y) = \det \begin{bmatrix} X - x\,\mathrm{Id} & -\overline{Y} + \overline{y}\,\mathrm{Id} \\ Y - y\,\mathrm{Id} & \overline{X} - \overline{x}\,\mathrm{Id} \end{bmatrix}. \quad (3)$$

This is equivalent to the equation $\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = 0$.

Another approach is due to Gelfand et al. [6]. These authors associated with each matrix $A \in \mathcal{M}_{n\times n}(\mathbb{H})$ the n^2 functions – that we shall call *quasicharacteristic functions* –, defined by

$$f_{ij}(\lambda) = |\lambda\,\mathrm{Id} - A|_{ij}, \quad 1 \leq i, j \leq n,$$

where $|\cdot|_{ij}$ is the (i, j)−quasideterminant. Let us denote by $A^{i,j}$ the submatrix of order $(n - 1)$ obtained by suppressing the row i and the column j in the matrix $A \in \mathcal{M}_{n\times n}(\mathbb{H})$. Then quasideterminants are defined inductively by the formula

$$|A|_{ij} = a_{ij} - \sum a_{iq}(|A^{i,j}|_{pq})^{-1} a_{pj},$$

where the sum is taken over the indices $p, q \in \{1, \ldots, n\}$ with $p \neq i, q \neq j$, such that the quasideterminant of lower order $|A^{i,j}|_{pq}$ is defined and it is non-null (see Proposition 1.5 of [21]).

When the matrix A is invertible, the entries of the inverse matrix A^{-1} are exactly $a^{ij} = |A|_{ji}^{-1}$. In the commutative case, this gives the well-known formula $a^{ij} = (-1)^{i+j} \det A^{j,i} / \det A$. For quaternionic matrices, the norm of the quasideterminant $|A|_{ij}$ of $A \in \mathcal{M}_{n\times n}(\mathbb{H})$ verifies

$$\left||A|_{ij}\right| \cdot \mathrm{Sdet}(A^{i,j}) = \mathrm{Sdet}(A). \tag{4}$$

This is a particular case of Jacobi identity for quaternions (Proposition 2.13).

Remark 2 From Equation (4) it follows that the roots of the quasicharacteristic functions are left eigenvalues. However, none of those functions gives the complete spectrum, as shown in the next Example. Also notice that the definition of *noncommutative left eigenvalue* considered in [6, Subsection 8.2, p.128] does not correspond to the notion we are discussing here.

Example 3.4 Let $A = \begin{bmatrix} \mathbf{i} & 0 & 0 \\ \mathbf{k} & \mathbf{j} & 0 \\ -3\mathbf{i} & 2\mathbf{k} & \mathbf{k} \end{bmatrix}$. Then $\sigma_l(A) = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The quasi-characteristic functions are

$$\begin{aligned}
f_{11}(\lambda) &= \lambda - \mathbf{i},\\
f_{12}(\lambda) &= -(\lambda - \mathbf{i})\mathbf{k}(\lambda - \mathbf{j}),\\
f_{13}(\lambda) &= -(\lambda - \mathbf{i}) \left(3\mathbf{i} - 2\mathbf{k}(\lambda - \mathbf{j})^{-1}\mathbf{k}\right)^{-1} (\lambda - \mathbf{k}),\\
f_{22}(\lambda) &= \lambda - \mathbf{j},\\
f_{23}(\lambda) &= -\frac{1}{2}(\lambda - \mathbf{j})\mathbf{k}(\lambda - \mathbf{k}),\\
f_{32}(\lambda) &= -2\mathbf{k} - (\lambda - \mathbf{k})\mathbf{k}(\lambda - \mathbf{j}),\\
f_{33}(\lambda) &= \lambda - \mathbf{k},
\end{aligned}$$

while $f_{21}(\lambda)$ and $f_{31}(\lambda)$ are not defined.

3.3. *Characteristic map*

We now introduce the notion of a characteristic map whose roots are the left eigenvalues, thus generalizing the usual characteristic polynomial in the real and complex settings. As we shall see, this notion fits naturally with the equation of order 2 given by Wood [1], as well as with the procedure proposed by So [22] in order to compute the left eigenvalues of 3×3 matrices.

Definition 3.5 The map $\mu : \mathbb{H} \to \mathbb{H}$ is a *characteristic map* of the matrix $A \in \mathcal{M}_{n\times n}(\mathbb{H})$ if, up to a constant, its norm verifies $|\mu(\lambda)| = \mathrm{Sdet}(A - \lambda\,\mathrm{Id})$ for all $\lambda \in \mathbb{H}$.

Example 3.6 Let $D = \mathrm{diag}(q_1, \ldots, q_n)$ be a diagonal matrix. Then $\mu(\lambda) = (q_1 - \lambda) \cdots (q_n - \lambda)$ is a characteristic map for D. The same result holds for triangular matrices.

Let us start with the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If A is a diagonal matrix, then $\sigma_l(A)$ reduces to the elements in the diagonal. Otherwise, we can always suppose that $b \neq 0$ (see Remark 4). Moreover, $\mathrm{Sdet}(A - \lambda\,\mathrm{Id})$ does not change after elementary transformations (Corollary 2.10), for instance

$$\mathrm{Sdet}\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \mathrm{Sdet}\begin{bmatrix} 0 & b \\ c-(d-\lambda)b^{-1}(a-\lambda) & d-\lambda \end{bmatrix}.$$

Then, as pointed out by Wood, computing the left spectrum is equivalent to finding the roots of a characteristic map like

$$\mu(\lambda) = c - (d-\lambda)b^{-1}(a-\lambda). \tag{5}$$

Remark 3 Huang [23] proposed another map when $c \neq 0$, namely $(\lambda - a)c^{-1}(\lambda - d) - b$. This polynomial is obtained by adding $(\lambda - a)c^{-1}$ by the second row to the first row. This expression is equivalent to $b - (a-\lambda)c^{-1}(d-\lambda)$, which is the one given by Wood at the end of [1] (there is a misprint in the original paper).

The left spectrum is not invariant by similarity. However, we shall use the following fact:

PROPOSITION 3.7 *If P is an invertible* real *matrix then* $\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = \mathrm{Sdet}(PAP^{-1} - \lambda\,\mathrm{Id})$. *Hence A and PAP^{-1} have the same characteristic maps.*

Remark 4 Let A be a matrix of order $n \geq 2$, let $P_{\alpha\beta}$ be the real matrix obtained by interchanging the rows α and β in the identity matrix I_n. Left (resp. right) multiplication by the matrix $P_{\alpha\beta}$ switches two rows (resp. columns) of A. Now let $i \neq j$ be two indices and let π be any permutation of $\{1, \ldots, n\}$ sending i to 1 and j to n. Then π can be written as a composition of transpositions, so by taking the product P of the corresponding matrices $P_{\alpha\beta}$, we obtain the matrix PAP^{-1} where the initial entry a_{ij} of A has moved to the place $(1, n)$.

4. Spectrum of matrices of order 2

In the next paragraphs, we shall classify the different possible spectra of 2×2 quaternionic matrices depending on the rank of the differential $\mu_{*\lambda}$ of a characteristic map.

4.1. *Preliminaries*

The characteristic map $\mu\colon \mathbb{H} \to \mathbb{H}$ given in (5) verifies that $\lim |\mu(\lambda)| = \infty$ when $|\lambda| \to \infty$. Then μ can be extended to a continuous (or even differentiable) map $\mu\colon S^4 \to S^4$ on the sphere $S^4 = \mathbb{H} \cup \{\infty\}$, the 1-point compactification of $\mathbb{R}^4$. A rigorous proof of the following result can be found in Eilenberg–Steenrod's book [24, p.304–310]:

PROPOSITION 4.1 *A polynomial map like μ and the power map λ^2 are homotopic, hence they have the same topological degree, which equals* 2.

From Lemma 2.4 it follows that the differential of μ is given by

$$\mu_{*\lambda}(X) = Xb^{-1}(a-\lambda) + (d-\lambda)b^{-1}X. \tag{6}$$

4.2. *Classification of left spectra*

Now we are in a position to reformulate the following result from Huang and So [4], see also [5,25]. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a quaternionic matrix with $b \neq 0$, and denote

$$a_0 = -b^{-1}c, \quad a_1 = b^{-1}(a-d), \quad \Delta = a_1^2 - 4a_0.$$

THEOREM 4.2 ([4]) *The matrix A has one, two or infinite left eigenvalues. The last case is equivalent to the following conditions: a_0, a_1 are real numbers such that $a_0 \neq 0$ and $\Delta < 0$.*

Remark 5 We shall call the infinite case *spherical*, because the spectrum $\sigma_l(A) = \{(1/2)(a + d + bq) : q^2 = \Delta\}$ is diffeomorphic to the sphere $S^2 \subset \mathbb{H}_0 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$.

Let $\lambda \in \mathbb{H}$ be an eigenvalue of A, that is, $\mu(\lambda) = 0$ for the map μ in (5). In the next Propositions we shall apply Theorem 2.6 to the differential $\Sigma = \mu_{*\lambda}$ computed in (6). Accordingly to the notation of Sylvester equation in Section 2.3, we denote

$$P = (d - \lambda)b^{-1}, \quad Q = b^{-1}(a - \lambda).$$

First, we study the two non-generic cases.

PROPOSITION 4.3 *If* rank $\mu_{*\lambda} = 0$*, then a_0, a_1 are real numbers and $\Delta = 0$. Moreover λ equals $(a + d)/2$ and this is the only left eigenvalue of the matrix.*

Proof We know from Theorem 2.6 that $P = t \in \mathbb{R}$ and $Q = -t$, then $a_1 = -2t$ and $2\lambda = a + d$. From $\mu(\lambda) = 0$ it follows that $a_0 = +t^2$, then $\Delta = 0$. Now it is easy to check (using for instance Theorem 2.3 in [4]) that $\lambda = a + tb$ is the only left eigenvalue of A. □

LEMMA 4.4 *Let A, B be two similar quaternions that do not commute. Then the equation $\lambda^2 - (A + B)\lambda + AB = 0$ has the unique solution $\lambda = B$.*

Proof If $\lambda \neq B$ is a solution, it follows from $(\lambda - B)\lambda = A(\lambda - B)$ that λ and A are similar, then $\Re(\lambda) = \Re(A) = \Re(B)$ and $|\lambda| = |A| = |B|$. By substituting in the equation we see that the real parts and norms disappear, so we can suppose that A, B, λ are pure imaginary quaternions with norm 1. Hence $\lambda^2 = -1 = B^2$ so the equation reduces to $(A + B)\lambda = (A + B)B$, which implies $\lambda = B$, a contradiction.

Alternatively, the uniqueness of λ can be proved by using Theorem 2.3, case 4.1, of the solution of quadratic equations in [26]. □

PROPOSITION 4.5 *If* rank $\mu_{*\lambda} = 2$ *two things may happen:*

(1) *either the spectrum is spherical and all the eigenvalues have rank* 2*;*
(2) *or the matrix has just one eigenvalue.*

Proof By using the diffeomorphism $a + b\sigma_l(A') = \sigma_l(A)$, we can substitute A by the so-called 'companion matrix' $A' = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$. Since the rank is 2, we have from Theorem 2.6 that $P = t + \alpha$ and $Q = -t + \beta$ with $\alpha, \beta \in \mathbb{H}_0 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$, $|\alpha| = |\beta| \neq 0$. Then

$a_1 = -2t + \beta - \alpha$. The first possibility is that $\beta = \alpha$, then $a_1 = -2t$. It follows from $\mu(\lambda) = 0$ that $a_0 = t^2 + |\beta|^2 \neq 0$ and $\Delta = -4|\beta|^2 < 0$. Then we have the spherical case. In particular $\lambda = (-a_1 - 2\beta)/2$. The other eigenvalues have the form $(-a_1 + q)/2$ with $q^2 = -4|\beta|^2$, then the differential of μ verifies $P = t - q/2$ and $Q = -t - q/2$, and so they have rank 2 too. The second possibility is that $\beta \neq \alpha$. Then $a_1 = -2t + \beta - \alpha$, $a_0 = (t + \alpha)(t - \beta)$, and Lemma 4.4 shows that the only eigenvalue is $\lambda = t - \beta$. □

Now we consider the generic case.

Proposition 4.6 *If* rank $\mu_{*\lambda} = 4$ *then the matrix has two different eigenvalues.*

Proof Since the differential has maximal rank at the eigenvalue λ, the matrix A cannot correspond to Propositions 4.3 or 4.5, hence, all its eigenvalues are of rank 4. Then by the inverse function theorem the fiber $\mu^{-1}(0)$ is discrete (in fact compact) and its cardinal equals (Theorem 2.1) the degree of the map μ, which is 2 by Proposition 4.1. Notice that the Jacobian is nonnegative by formula (1). □

Remark 6 In [27], Janovská and Opfer show that for quaternionic polynomials there are several types of zeros accordingly to the rank of some real 4×4 matrix, but their procedure does not seem to have an immediate geometrical meaning.

5. Characteristic maps of 3 × 3 matrices

The only known results about the left spectrum of 3×3 matrices are due to So [22], who did a case by case study, depending on some relationships among the entries of the matrix. He showed that when $n = 3$ left eigenvalues could be found by solving quaternionic polynomials of degree not greater than 3. In general, there is not any known method for solving the resulting equations.

In the following paragraphs, we shall develop a method for matrices of order 3 which is analogous to that of Section 4, that is, we shall find a map μ_A such that $|\mu_A(\lambda)| = \mathrm{Sdet}(A - \lambda\,\mathrm{Id})$. This time, however, the characteristic map μ_A will be in most cases a rational function instead of a polynomial (the latter occurs when the matrix A has some null entry outside the diagonal).

Let us consider the quaternionic matrix $A = \begin{bmatrix} a & b & c \\ f & g & h \\ p & q & r \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{H})$.

5.1. *Polynomial case*

We start studying the simplest situation, when there exists some zero entry outside the diagonal.

First, suppose that the matrix has the zero entry $c = 0$, that is,

$$\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = \mathrm{Sdet} \begin{bmatrix} a-\lambda & b & 0 \\ f & g-\lambda & h \\ p & q & r-\lambda \end{bmatrix}.$$

There are three possibilities:

(1) if $b, h = 0$, we have a triangular matrix, so we can take

$$\mu(\lambda) = (r-\lambda)(g-\lambda)(a-\lambda); \tag{7}$$

(2) if $b = 0$ but $h \neq 0$, then Proposition 2.11 allows us to reduce to the 2×2 case and we obtain

$$\mu(\lambda) = \Big(q - (r-\lambda)h^{-1}(g-\lambda)\Big)(a-\lambda); \tag{8}$$

(3) finally, if $b \neq 0$, we can proceed as follows. We create a zero in the first row by substracting to the first column C_1 the multiple $C_2 b^{-1}(a-\lambda)$ of the second column:

$$\mathrm{Sdet}(A - \lambda\,\mathrm{Id}) = \mathrm{Sdet}\begin{bmatrix} 0 & b & 0 \\ f-(g-\lambda)b^{-1}(a-\lambda) & g-\lambda & h \\ p - qb^{-1}(a-\lambda) & q & r-\lambda \end{bmatrix},$$

then we permute the two last columns in order to reduce to the 2×2 case. In this way we can take as a characteristic map the polynomial of degree 3

$$\mu(\lambda) = p - qb^{-1}(a-\lambda) - (r-\lambda)h^{-1}\Big(f - (g-\lambda)b^{-1}(a-\lambda)\Big). \tag{9}$$

THEOREM 5.1 *If the matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{H})$ has some zero entry outside the diagonal, then A admits a polynomial characteristic map.*

Proof Let the entry be $a_{ij} = 0$, with $i \neq j$. Then according to Remark 4 there is a real invertible matrix P such that the transformation PAP^{-1} does not change the characteristic maps and gives a matrix with $a_{13} = 0$. □

5.2. *Rational case*

In the more general situation, when $c \neq 0$, we can compute the Study's determinant of the matrix A by creating zeroes in the first row. Then

$$\mathrm{Sdet}(A) = \mathrm{Sdet}\begin{bmatrix} 0 & 0 & c \\ f - hc^{-1}a & g - hc^{-1}b & h \\ p - rc^{-1}a & q - rc^{-1}b & r \end{bmatrix}.$$

From Proposition 2.11 and the results for 2×2 matrices it follows:

PROPOSITION 5.2 *If $c \neq 0$, then* $\mathrm{Sdet}(A)$ *is given:*

(1) *when $g - hc^{-1}b \neq 0$, by*

$$|c| \cdot |g - hc^{-1}b| \cdot |p - rc^{-1}a - (q - rc^{-1}b)(g - hc^{-1}b)^{-1}(f - hc^{-1}a)|;$$

(2) *otherwise, by*

$$|c| \cdot |q - rc^{-1}b| \cdot |f - hc^{-1}a|.$$

Definition 5.3 We call the point $\pi_A = g - hc^{-1}b$ a *pole* of the matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{H})$.

Notice that π_A is the quasideterminant $|A^{3,1}|_{21}$ (see page 7).

By applying Proposition 5.2 to the matrix $A-\lambda\,\mathrm{Id}$ we obtain the following characteristic map for A (we omit the term $|c|$).

PROPOSITION 5.4 *Let A be a matrix of order 3 such that $c \neq 0$. A characteristic map can be defined as follows:*

(1) *if $\pi_A = g - hc^{-1}b$ is the pole of A, then*

$$\mu(\pi_A) = \left(q - (r - \pi_A)c^{-1}b\right)\left(f - hc^{-1}(a - \pi_A)\right);$$

(2) *for $\lambda \neq \pi_A$ we define*

$$\mu(\lambda) = (\pi_A - \lambda)\Big[p - (r - \lambda)c^{-1}(a - \lambda) - \left(q - (r - \lambda)c^{-1}b\right)(\pi_A - \lambda)^{-1}\left(f - hc^{-1}(a - \lambda)\right)\Big]. \tag{10}$$

Remark 7 The map in (10) is exactly the same formula given by So in [22, p.563], even if our method is completely different. This is why we chose to compute determinants starting from the right bottom corner.

5.3. *Continuity at the pole*

Up to now we have defined maps which verify $|\mu(\lambda)| = \mathrm{Sdet}(A - \lambda\,\mathrm{Id})$ in norm. Since Sdet is a continuous map we have $\lim_{\lambda \to \pi_A} |\mu(\lambda)| = |\mu(\pi_A)|$. However, the following example shows that μ may not be continuous at the pole π_A.

Example 5.5 Let $A = \begin{bmatrix} 0 & \mathbf{i} & 1 \\ 3\mathbf{i} - \mathbf{k} & 0 & 1 \\ \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & 0 \end{bmatrix}$. The pole $\pi_A = -\mathbf{i}$ is not a left eigenvalue; in fact

$$\mu(\pi_A) = (-1 + \mathbf{j} + \mathbf{k} + 1)(3\mathbf{i} - \mathbf{k} - \mathbf{i}) = 1 - \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

We observe that the directional limits

$$\lim_{\varepsilon \to 0} \mu(-\mathbf{i} + \varepsilon q) = -q(\mathbf{j} + \mathbf{k})q^{-1}(2\mathbf{i} - \mathbf{k})$$

depend on $q \in \mathbb{H}$, hence $\lim_{\lambda \to \pi_A} \mu(\lambda)$ does not exist.

THEOREM 5.6 *The characteristic rational map μ_A is continuous if and only if the pole π_A is a left eigenvalue of A.*

Proof Assume that π_A is a left eigenvalue. Let $(q_n)_n$ be a sequence converging to π_A. Then $|\mu(q_n)| = \mathrm{Sdet}(A - q_n\,\mathrm{Id})$ converges to $\mathrm{Sdet}(A - \pi_A\,\mathrm{Id}) = 0$, that is, $\mu(q_n) \to 0 = \mu(\pi_A)$.

Now we shall prove the converse. The map μ defined in Proposition 5.4 is of the form

$$\mu(\lambda) = (\pi_A - \lambda)\Big[p(\lambda) - q(\lambda)(\pi_A - \lambda)^{-1} f(\lambda)\Big], \quad \lambda \neq \pi_A, \tag{11}$$

while $\mu(\pi_A) = q(\pi_A) f(\pi_A)$. Let us assume that $\lim_{\lambda\to\pi_A} \mu(\lambda)$ exists and equals $\mu(\pi_A)$. We must check that $\mu(\pi_A) = 0$. If $f(\pi_A) = 0$ we have finished. Otherwise we deduce from (11) that

$$\lim_{\lambda\to\pi_A} (\lambda - \pi_A) q(\lambda)(\lambda - \pi_A)^{-1} = -q(\pi_A). \tag{12}$$

From Lemma 5.7 it follows that the limit on the left side equals $q(\pi_A)$, hence $q(\pi_A) = 0$, which ends the proof. □

LEMMA 5.7 *Let $Q = Q(\lambda)$ be a continuous map and suppose that there exists the limit $l_0 = \lim_{\lambda\to 0} \lambda Q(\lambda)\lambda^{-1}$. Then $l_0 = Q(0)$.*

Proof Take a sequence of real numbers $(\varepsilon_n)_n \to 0$. Then

$$l_0 = \lim \varepsilon_n Q(\varepsilon_n)\varepsilon_n^{-1} = \lim Q(\varepsilon_n) = Q(0).$$

□

Notice that the differentiability of μ at the pole π_A is not ensured.

It is an open question whether it is always possible or not to find a polynomial, or at least a continuous characteristic map for a given matrix A.

5.4. *Extension to the infinite point*

Each of the characteristic maps we have introduced up to now can be extended to the sphere $S^4 = \mathbb{H} \cup \{\infty\}$.

PROPOSITION 5.8 *The polynomial maps μ defined in Subsection 5.1 verify that $\lim_{|\lambda|\to\infty} |\mu(\lambda)| = \infty$.*

Proof When $b = 0$, the result follows from formulae (7) and (8); when $b \neq 0$, the expression of μ is that of formula (9), so

$$\frac{|\mu(\lambda)|}{|\lambda|^2} \geq \frac{|(r-\lambda)h^{-1}(g-\lambda)b^{-1}(a-\lambda)|}{|\lambda|^2} - \frac{|-p+qb^{-1}(a-\lambda)+(r-\lambda)h^{-1}f|}{|\lambda|^2}.$$

□

PROPOSITION 5.9 *The rational map μ defined in the $c \neq 0$ case by formula (10) can be extended to $S^4 = \mathbb{H} \cup \{\infty\}$ (maybe with a discontinuity at the pole π_A).*

Proof We have

$$\mu(\lambda) = (\pi_A - \lambda) p_2(\lambda) - (\pi_A - \lambda) q_1(\lambda)(\pi_A - \lambda)^{-1} f_1(\lambda)$$

with polynomials

$$\begin{aligned} p_2(\lambda) &= p - (r-\lambda)c^{-1}(a-\lambda), \\ q_1(\lambda) &= q - (r-\lambda)c^{-1}b, \\ f_1(\lambda) &= f - hc^{-1}(a-\lambda). \end{aligned} \tag{13}$$

Then

$$|\mu(\lambda)| \geq |(\pi_A - \lambda)p_2(\lambda)| - |q_1(\lambda)f_1(\lambda)|$$

and

$$\lim_{|\lambda|\to\infty} \frac{|\mu(\lambda)|}{|\lambda|^3} \geq \lim_{|\lambda|\to\infty} \frac{|(\pi_A - \lambda)p_2(\lambda)|}{|\lambda|^3} \geq \lim_{|\lambda|\to\infty} |c^{-1}| \frac{|r-\lambda|}{|\lambda|} \frac{|a-\lambda|}{|\lambda|} = |c^{-1}|.$$

□

5.5. *Discontinuous case*

Let us now assume that the characteristic map μ_A defined in Proposition 5.4 is not continuous at the pole π_A, or equivalently that π_A is not a left eigenvalue of A (Theorem 5.6). Then the matrix $B = A - \pi_A \operatorname{Id}$ is invertible and its pole is $\pi_B = 0$. Moreover, $\sigma_l(A) = \sigma_l(B) + \pi_A$. On the other hand, from Proposition 3.3, we know that the spectra of B and B^{-1} are diffeomorphic, because $\sigma_l(B^{-1}) = \sigma_l(B)^{-1}$.

THEOREM 5.10 *Let A be a matrix such that the pole π_A is not a left eigenvalue. Let $B = A - \pi_A \operatorname{Id}$. Then the matrix B^{-1} has a polynomial characteristic map.*

Proof According to formula (4) the norm of the entry (1, 3) of the matrix B^{-1} equals

$$\operatorname{Sdet}(B^{3,1})/\operatorname{Sdet}(B) = |\pi_B|/\operatorname{Sdet}(B) = 0,$$

then Theorem 5.1 applies. □

Here is an alternative proof of Theorem 5.10. Let $\mu_B(\lambda) = -\lambda R(\lambda)$, with

$$R(\lambda) = p(\lambda) + q(\lambda)\lambda^{-1}f(\lambda), \quad \lambda \neq 0,$$

be the characteristic map given in Proposition 5.4 (we assume $\pi_B = 0$.) Then it is immediate that $\lambda R(\lambda^{-1})\lambda$ is a polynomial in λ (of degree 3 and independent term $-c^{-1}$) and we only have to apply the following result.

PROPOSITION 5.11 *Let μ_B be a characteristic map of the invertible matrix $B = A - \pi_A \operatorname{Id}$, with a discontinuity at the pole $\pi_B = 0$. Then*

$$\mu_{B^{-1}}(\lambda) = \operatorname{Sdet}(B)^{-1}\lambda^2\mu_B(\lambda^{-1})\lambda, \quad \lambda \neq 0,$$

is a characteristic map for B^{-1}.

Proof From Proposition 2.9 we have

$$\operatorname{Sdet}(\lambda^{-1}\operatorname{Id}) \cdot \operatorname{Sdet}(B^{-1} - \lambda \operatorname{Id}) \cdot \operatorname{Sdet}(B) = \operatorname{Sdet}(\lambda^{-1}\operatorname{Id} - B)$$

then

$$|\lambda^{-3}| \cdot \operatorname{Sdet}(B^{-1} - \lambda \operatorname{Id}) \cdot \operatorname{Sdet}(B) = |\mu_B(\lambda^{-1})|.$$

□

Remark 8 The idea that a rational map like $R(\lambda)$ can be converted into a polynomial $\lambda^{-1}R(\lambda)\lambda^{-1}$ with variable λ^{-1} is due to So (see [22, Lemma 3.5, p.558]).

6. Topological study of the 3 × 3 case

We shall consider separately the polynomial, rational continuous and discontinuous cases considered in Section 5.

6.1. *Polynomial case*

Let us start with matrices having some null entry outside the diagonal (as we have seen this case can be reduced to the case $c = 0$). We know that the characteristic map is a polynomial of degree 3 which can be extended in a continuous way to the sphere S^4. Then since there is a unique term of higher degree 3, the map μ_A is homotopic to λ^3, so it has topological degree 3.

PROPOSITION 6.1 *Let λ be a left eigenvalue of the matrix A with $c = 0$. Then the differential of the polynomial characteristic map μ_A in Subsection 5.1 is given by*

(1) *if $b, h = 0$ then*

$$\mu_{*\lambda}(X) = -X(g-\lambda)(a-\lambda) - (r-\lambda)(g-\lambda)X - (r-\lambda)X(a-\lambda);$$

(2) *if $b = 0, h \neq 0$ then*

$$\begin{aligned}\mu_{*\lambda}(X) =& Xh^{-1}(g-\lambda)(a-\lambda) - \Big(q - (r-\lambda)h^{-1}(g-\lambda)\Big)X \\ &+ (r-\lambda)h^{-1}X(a-\lambda);\end{aligned}$$

(3) *otherwise,*

$$\begin{aligned}\mu_{*\lambda}(X) =& \Big(qb^{-1} - (r-\lambda)h^{-1}(g-\lambda)b^{-1}\Big)X \\ &+ Xh^{-1}\Big(f - (g-\lambda)b^{-1}(a-\lambda)\Big) - (r-\lambda)h^{-1}Xb^{-1}(a-\lambda).\end{aligned}$$

The proof is a direct application of the derivation rules given in Lemma 2.4.

The expressions obtained are of the form $PX + XQ + RXS = 0$, whose rank can be computed with the method given in Section 2.4.

Example 6.2 Let $A = \begin{bmatrix} a & 0 & 0 \\ f & g & 0 \\ p & q & r \end{bmatrix}$ be a triangular matrix. The differential $\mu_{*\lambda}(X)$ of the characteristic map

$$\mu(\lambda) = (r-\lambda)(g-\lambda)(a-\lambda),$$

at the eigenvalues $\lambda = a, g, r$ is given, respectively, by $(a-r)(g-a)X$, $(g-r)X(a-g)$ and $X(r-g)(a-r)$. Hence, unlike the case $n = 2$, the rank depends on the multiplicity of each eigenvalue, and can be either 0 or 4.

6.2. *Rational case*

When none of the entries outside the diagonal is zero, the characteristic map is a rational function, with a distinguished point π_A.

Let us first suppose that the pole π_A is a left eigenvalue. We know from Theorem 5.6 and (10) that μ_A is a continuous map on S^4 of the form

$$(\pi_A - \lambda)\left[p(\lambda) - q(\lambda)(\pi_A - \lambda)^{-1} f(\lambda)\right].$$

By examining formulae (13) it is clear that $p(\lambda)$ is homotopic to $-\lambda^2$ by the homotopy

$$tp - (tr - \lambda)(1 - t + tc^{-1})(ta - \lambda), \quad t \in [0, 1].$$

Analogously $q(\lambda) \sim \lambda$ and $f(\lambda) \sim \lambda$, so $\mu_A(\lambda)$ is homotopic to

$$(\pi_A - \lambda)\left[-\lambda^2 - \lambda(\pi_A - \lambda)^{-1}\lambda\right]$$

(notice that this map is continuous at $\lambda = 0$), which in turn is homotopic to $\lambda^3 - \lambda^2$ by the homotopy

$$(t\pi_A - \lambda)\left[-\lambda^2 - \lambda(t\pi_A - \lambda)^{-1}\lambda\right].$$

Finally, the homotopy $\lambda^2(\lambda - t)$ shows that the map μ_A is homotopic to λ^3. All these homotopies can be extended to the infinity.

Hence we have proved

PROPOSITION 6.3 *When the rational characteristic map μ_A is continuous it has topological degree* 3.

On the other hand, suppose that π_A is not a left eigenvalue. Then the polynomial case applies to $(A - \pi_A \operatorname{Id})^{-1}$ by Theorem 5.10. So we do not have to use the local theory of degree, whose main difficulty is the need of considering homotopies which are *admissible* with respect to the domain Ω of definition, see [8, p.28].

COROLLARY 6.4 *Any* 3×3 *quaternionic matrix has at least one left eigenvalue.*

Proof In all cases the eigenvalues (or its inverses) can be computed as the roots of a continuous map μ of degree 3; then, $\mu^{-1}(0)$ is not void (see Section 2). □

6.3. *Final remarks*

In order to simplify the computation of the rank, by taking $B = A - \pi_A \operatorname{Id}$, we can always assume that the pole is $\pi_B = 0$.

PROPOSITION 6.5 *For a* 3×3 *matrix with* $c \neq 0$ *the differential of the characteristic map* μ *given in formula (10) at the point* $\lambda \neq \pi_B = 0$ *is*

$$\begin{aligned}
&\mu_{*\lambda}(X) = \\
&X\left[-p + (r - \lambda)c^{-1}(a - \lambda) + (q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}(f - hc^{-1}(a - \lambda))\right] \\
&\quad + (-\lambda)Xc^{-1}(a - \lambda) - (-\lambda)Xc^{-1}b(-\lambda)^{-1}(f - hc^{-1}(a - \lambda)) \\
&\quad - (-\lambda)(q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}X(-\lambda)^{-1}(f - hc^{-1}(a - \lambda)) \\
&\quad + \left[(-\lambda)(r - \lambda)c^{-1} - (-\lambda)(q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}hc^{-1}\right]X.
\end{aligned}$$

When μ_A is continuous but not differentiable at the pole π_A the rank at π_A could be computed by taking a different characteristic map $\mu_{PAP^{-1}}$. In particular, by moving around the entries outside the diagonal of a given matrix (see Remark 4 of Section 3.3) one can obtain up to six different characteristic maps, each one with a different pole.

It is an open question whether there exists a matrix A verifying that all the poles of the matrices PAP^{-1}, with P real, are eigenvalues. If such an example does not exist then the non-polynomial case would not be necessary.

7. Examples

Here are some miscellaneous examples.

Example 7.1 Discontinuous map. Let A be the matrix given in Example 5.5. Then

$$B = A - \pi_A \,\mathrm{Id} = \begin{bmatrix} \mathbf{i} & \mathbf{i} & 1 \\ 3\mathbf{i} - \mathbf{k} & \mathbf{i} & 1 \\ \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & \mathbf{i} \end{bmatrix}$$

is an invertible matrix with pole $\pi_B = 0$. By computing the quasideterminants we obtain the inverse matrix

$$B^{-1} = \frac{1}{10} \begin{bmatrix} 4\mathbf{i} - 2\mathbf{k} & -4\mathbf{i} + 2\mathbf{k} & 0 \\ -1 - 3\mathbf{i} + 8\mathbf{j} - 6\mathbf{k} & 1 + 3\mathbf{i} - 3\mathbf{j} + \mathbf{k} & -5\mathbf{j} - 5\mathbf{k} \\ 11 + \mathbf{i} - 8\mathbf{j} - 8\mathbf{k} & -1 - \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} & -5\mathbf{j} + 5\mathbf{k} \end{bmatrix}.$$

Its polynomial characteristic map is

$$\begin{aligned} \mu_{B^{-1}}(\lambda) =& 10 - \lambda\mathbf{i} - 2\mathbf{i}\lambda \\ & - \frac{1}{10}\mathbf{i}\lambda(2\mathbf{i} - \mathbf{k})\lambda - \frac{1}{10}\lambda(1 + \mathbf{j} + 2\mathbf{k})\lambda - \frac{1}{100}\lambda(\mathbf{j} + \mathbf{k})\lambda(2\mathbf{i} - \mathbf{k})\lambda. \end{aligned} \tag{14}$$

On the other hand, the rational characteristic map for B is

$$\mu_B(\lambda) = -\lambda R(\lambda) = -\lambda \left[1 + \mathbf{k} + \mathbf{i}\lambda + \lambda\mathbf{i} - \lambda^2 + (\mathbf{j} + \mathbf{k} + \lambda\mathbf{i})\lambda^{-1}(2\mathbf{i} - \mathbf{k} + \lambda) \right]$$

and one can check that $\lambda R(\lambda^{-1})\lambda$ equals (up to a constant) the map (14).

Example 7.2 An eigenvalue of rank 3. Let

$$A = \begin{bmatrix} \mathbf{j} & 1 & 0 \\ 2\mathbf{i} & -\mathbf{k} & 1 \\ 2 - \mathbf{i} - 2\mathbf{j} & -1 - \mathbf{j} + \mathbf{k} & -\mathbf{i} - \mathbf{k} \end{bmatrix}.$$

The characteristic map is

$$\mu(\lambda) = 2 - \mathbf{i} - 2\mathbf{j} + (1 + \mathbf{j} - \mathbf{k})(\mathbf{j} - \lambda) + (\mathbf{i} + \mathbf{k} + \lambda)\,(2\mathbf{i} + (\mathbf{k} + \lambda)(\mathbf{j} - \lambda)).$$

For the left eigenvalue $\lambda = 0$ the differential is

$$\mu_{*0}(X) = \mathbf{k}X + X\mathbf{i} + (\mathbf{i} + \mathbf{k})X\mathbf{j},$$

whose real associated matrix $M = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$ has rank 3.

Example 7.3 A matrix which can not be reduced to the polynomial case $c = 0$. We shall exhibit a matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{H})$ such that for any real invertible matrix P all entries in the matrix PAP^{-1} outside the diagonal are not null. Let $A = T + \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z$, with $T, X, Y, Z \in \mathcal{M}_{3\times 3}(\mathbb{R})$. Then

$$PAP^{-1} = PTP^{-1} + \mathbf{i}PXP^{-1} + \mathbf{j}PYP^{-1} + \mathbf{k}PZP^{-1},$$

which means that the matrix PAP^{-1} has a null entry if and only if the same happens for the real matrices PTP^{-1}, PXP^{-1}, PYP^{-1} and PZP^{-1}. Moreover, from Remark 4 we can suppose that the null entry is at the place $(1, 3)$. Now, recall that A is the matrix associated to a linear map $\mathbb{H}^3 \to \mathbb{H}^3$ with respect to the canonical basis e_1, e_2, e_3 and that the real matrix $P = (p_{ij})$ represents the change to another basis v_1, v_2, v_3, that is, $v_j = \sum_i p^{ij} e_i$ where $P^{-1} = (p^{ij})$. Assume that T, X are the matrices associated to two rotations $\mathbb{R}^3 \to \mathbb{R}^3$ with the same rotation axis $\mathcal{L}$ and rotation angles $+\pi/2$ and $-\pi/2$, respectively. Then the nullity of the entry (1,3) means that $Tv_3, Xv_3 \in \langle v_2, v_3\rangle$, which implies that either v_3 is in the direction of the axis, i.e. $\mathcal{L} = \langle v_3\rangle$, or it is orthogonal to the axis, in which case $\langle v_2, v_3\rangle = \mathcal{L}^{\perp}$ (otherwise it is impossible that v_3, Tv_3 and Xv_3 lie in the same plane). Now, suppose that Y, Z are two other rotations with axis $\mathcal{L}'$ and rotation angles $\pm\pi/2$. If $\mathcal{L}$ and $\mathcal{L}'$ are different and not perpendicular, then it is impossible that $Yv_3, Zv_3 \in \langle v_2, v_3\rangle$.

Example 7.4 A continuous rational characteristic map. The pole $\pi_A = 1 + \mathbf{j}$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 0 & -\mathbf{j} & \mathbf{i} \\ -1+\mathbf{j} & \mathbf{j} & \mathbf{k} \\ p & q & r \end{bmatrix},$$

with $p, q, r \in \mathbb{H}$ arbitrary, $p, q \neq 0$. In fact,

$$\mu(\pi_A) = (q - (r - 1 - \mathbf{j})\mathbf{k})\,(-1 + \mathbf{j} + \mathbf{j}(-1 - \mathbf{j})) = 0.$$

Example 7.5 Generic polynomial case. Let

$$A = \begin{bmatrix} \mathbf{k} & 0 & 0 \\ 3\mathbf{i} - \mathbf{j} & -\mathbf{i} & \mathbf{i} \\ 1 - 2\mathbf{k} & \mathbf{j} & -\mathbf{j} \end{bmatrix}.$$

The characteristic map is $\mu(\lambda) = (-1 - \mathbf{k} + \lambda\mathbf{i})\lambda(\mathbf{k} - \lambda)$ hence $\sigma_l(A) = \{\mathbf{k}, 0, -\mathbf{i} - \mathbf{j}\}$. The differential of μ at each eigenvalue is

$$\mu_{*\mathbf{k}}(X) = (-1 - \mathbf{i} + \mathbf{k})X,$$
$$\mu_{*0}(X) = -(1 + \mathbf{k})X\mathbf{k},$$
$$\mu_{*(-\mathbf{i}-\mathbf{j})}(X) = X(1 + 2\mathbf{i} + \mathbf{k}).$$

Then the matrix A has three different eigenvalues, all of them with maximal rank.

Example 7.6 Two eigenvalues, one of null rank, the other one of maximal rank. Let

$$A = \begin{bmatrix} -\mathbf{i}-\mathbf{j} & 0 & 0 \\ \mathbf{k} & -\mathbf{i} & \mathbf{i} \\ 1-\mathbf{i} & \mathbf{j} & -\mathbf{j} \end{bmatrix}.$$

This time $\mu(\lambda) = (1 + \mathbf{k} - \lambda\mathbf{i})\lambda(\mathbf{i} + \mathbf{j} + \lambda)$ so $\sigma_l(A) = \{0, -\mathbf{i} - \mathbf{j}\}$. We have

$$\mu_{*0}(X) = (1 + \mathbf{k})X(\mathbf{i} + \mathbf{j}),$$
$$\mu_{*(-\mathbf{i}-\mathbf{j})}(X) = 0.$$

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