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Height functions on quaternionic Stiefel manifolds

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Abstract. In this note, we study height functions on quaternionic Stiefel manifolds and prove that all these height functions are Morse-Bott. Among them, we characterize the Morse functions and give a lower bound for their number of critical *values*. Relations with the Lusternik-Schnirelmann category are discussed.

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1. Introduction

The Lusternik-Schnirelmann category, $\operatorname{cat} X$, of a topological space X is defined as the least integer $m \geq 0$ such that X admits a covering by $m+1$ open sets which are contractible in X [1]. Category has proven useful in areas such as dynamical systems and symplectic geometry as well as homotopy theory, but it has also proven to be notoriously difficult to compute. A longstanding problem has been to compute the LS category of Lie groups. A significant step forward was Singhof's computations $\operatorname{cat} \mathrm{U}(n) = n$ and $\operatorname{cat} \mathrm{SU}(n) = n - 1$

[19] of the unitary and special unitary groups, accomplished through a clever use of eigenvalues. Such an approach cannot be carried out for the symplectic groups, $\mathrm{Sp}(n)$, [10]. Some progress has been made, however, for small n and in obtaining lower bounds. Namely, $\operatorname{cat}\mathrm{Sp}(2) = 3$ [17], $\operatorname{cat}\mathrm{Sp}(3) = 5$ [3], and $\operatorname{cat}\mathrm{Sp}(n) \geq n+2$ when $n \geq 3$ [7].

The quaternionic Stiefel manifolds, $X_{n,k} = \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$, have proved somewhat more amenable to category calculations in certain ranges. For instance, $\operatorname{cat} X_{n,k} = k$ when $n \geq 2k$. This result was proved by Nishimoto [15] using the number of eigenvalues of a complex matrix in a fashion similar to Singhof's approach for the unitary group [3]. This has also been proved by Kadzisa and Mimura [9] by a method based on the cone decomposition induced by the Morse-Bott function defined by Frankel in [4]. The analysis of Morse-Bott functions on Lie groups and homogeneous spaces has a long history (see, for instance, [6,12,20]). In this paper, we focus our attention on the properties of height functions themselves and relate these to the LS-category when the critical points are isolated [16]. With this method, it was proved in [11] that the category of the symplectic group $\mathrm{Sp}(n)$ is bounded above by $\binom{n+1}{2}$.

We extend this approach to quaternionic Stiefel manifolds and prove that any height function on $X_{n,k}$ is of Morse-Bott type. In order to do this, we collect basic properties on general height functions on $X_{n,k}$, giving an explicit expression for the gradient and an explicit determination of the critical set. We also prove that any Morse height function on $X_{n,k}$ is perfect and has at least $1+\binom{k+1}{2}$ critical *values*. Moreover, this bound is reached by some height function. This gives an upper bound for the LS-category of $X_{n,k}$, which is not the best possible but shows the limit of the method.

2. Height functions on quaternionic Stiefel manifolds

2.1 Stiefel manifolds

Let $\mathbb{H}^n$ be the quaternionic n-space (with the structure of a right $\mathbb{H}$-vector space) endowed with the hermitian product $\langle u, v\rangle = u^*v$. For $0 \leq k \leq n$, let $X_{n,k}$ be the Stiefel manifold of linear maps $\phi\colon \mathbb{H}^k \to \mathbb{H}^n$ which preserve the Hermitian product. The map ϕ can be identified with a matrix x of size $n \times k$ such that $x^*x = I_k$, so the columns of x form an orthonormal k-frame of $\mathbb{H}^n$. We shall represent any element $x \in X_{n,k}$ by two blocks,

$$x = \begin{pmatrix} T \\ P \end{pmatrix},$$

where T, P are quaternionic matrices of order $(n-k)\times k$ and $k\times k$, respectively, which (due to $x^*x = I_k$) verify the relation

$$T^*T + P^*P = I_k. \tag{1}$$

Let $\mathrm{Sp}(n)$ be the Lie group of $n \times n$ matrices A such that $A^*A = I_n$. The linear left action of $\mathrm{Sp}(n)$ on $X_{n,k}$ is transitive and the isotropy group of $x_0 = \binom{0}{I_k}$ is isomorphic to $\mathrm{Sp}(n-k)$, so $X_{n,k}$ is diffeomorphic to $\mathrm{Sp}(n)/\mathrm{Sp}(n-k)$.

2.2 Height functions

Let $\mathbb{H}^{n\times k}$ be the vector space of quaternionic matrices of size $n \times k$. As a real vector space, it is isomorphic to $\mathbb{R}^{4(n\times k)}$ with *euclidean* norm given by $|x|^2 = \mathrm{Tr}(x^*x)$. It follows that the euclidean inner product is given by $[y, x] = \Re\mathrm{Tr}(y^*x)$, where $\Re\mathrm{Tr}$ is the real part of the trace, and the height of x with respect to the hyperplane orthogonal to a given matrix ω^* is given by $a\,\Re\mathrm{Tr}(\omega x) + b$, where a and b are real constants. Let $h_\omega\colon \mathbb{H}^{n\times k} \to \mathbb{R}$ be the function

$$h_\omega(x) = \Re\mathrm{Tr}(\omega x).$$

Since h_ω is $\mathbb{R}$-linear, its gradient at any point is ω^*. If we denote ω by a block matrix $(\delta|D)$, with blocks δ, D of size $k \times (n-k)$ and $k \times k$ respectively, then

$$h_\omega\left(\begin{pmatrix} T \\ P \end{pmatrix}\right) = \Re\mathrm{Tr}(\omega x) = \Re\mathrm{Tr}(\delta T + DP). \tag{2}$$

When $n = k$, the notation above means $\omega = D \in \mathbb{H}^{n\times n}$, $x = P$ and the Stiefel manifold is the group $\mathrm{Sp}(n)$. The corresponding height function is that considered in [11].

2.3 Height functions on $X_{n,k}$

Let us now consider the restriction $f_\omega\colon X_{n,k} \to \mathbb{R}$ of the height function h_ω to the Stiefel manifold. We want to choose the matrix ω in such a way that f_ω has isolated critical points and few critical *levels.* In order to determine the critical points, we first need expressions for the gradient and the Hessian of f_ω.

Proposition 2.1. *The gradient of $f_\omega\colon X_{n,k} \to \mathbb{R}$ at the point $x \in X_{n,k}$ is given by*

$$(\mathrm{grad} f_\omega)_x = \omega^* - (1/2)x(\omega x + x^*\omega^*).$$

Proof. Begin with $x_0 = \binom{0}{I_k}$. The tangent space, $T_{x_0}(X_{n,k})$, is obtained as the image of the tangent space at $I_n \in \mathrm{Sp}(n)$ by the canonical projection $\mathrm{Sp}(n) \to X_{n,k}$. It is therefore formed by the matrices $\binom{X}{Y}$ such that $Y + Y^* = 0$. (This may also be seen by differentiating a curve of matrices satisfying (1).) Moreover, the orthogonal subspace $(T_{x_0}(X_{n,k}))^\perp$ is formed by the matrices

$\binom{0}{Z}$ such that $Z - Z^* = 0$. So we see that the projection of an arbitrary matrix $y = \binom{X}{Y} \in \mathbb{H}^{n\times k}$ onto $T_{x_0}(X_{n,k})$ is given by

$$\pi_0(y) = \begin{pmatrix} X \\ (1/2)(Y - Y^*) \end{pmatrix}. \tag{3}$$

Now, let $x = \binom{T}{P} \in X_{n,k}$ and $\pi\colon \mathbb{H}^{n\times k} \to T_x(X_{n,k})$ be the projection. The gradient of $f_\omega\colon X_{n,k} \to \mathbb{R}$ at x is the projection $\pi(\omega^*)$ of the gradient ω^* of h_ω. Since the action of $\mathrm{Sp}(n)$ on $X_{n,k}$ is transitive, there is some $A \in \mathrm{Sp}(n)$ such that $x = Ax_0$, so $T_x(X_{n,k}) = A \cdot T_{x_0}(X_{n,k})$. Thus, we may consider ω^* as a vector in the tangent space of $\mathbb{H}^{n\times k}$ at x, translate it via $A^* = A^{-1}$ to the tangent space of $\mathbb{H}^{n\times k}$ at x_0, where the projection is already determined by (3), and return to the tangent space at x via A. As the action is an isometry, we obtain

$$(\operatorname{grad} f_\omega)_x = A\pi_0(A^*\omega^*). \tag{4}$$

The matrix A can be written as $A = \begin{pmatrix} \alpha & T \\ \beta & P \end{pmatrix} \in \mathrm{Sp}(n)$, with blocks $\alpha \in \mathbb{H}^{(n-k)\times(n-k)}$ and $\beta \in \mathbb{H}^{k\times(n-k)}$, the condition $AA^* = I_n$ being equivalent to

$$\alpha\alpha^* + TT^* = I_{n-k},\ \beta\alpha^* + PT^* = 0,\ \beta\beta^* + PP^* = I_k. \tag{5}$$

We replace A, x, ω in (4), by their decomposition in blocks. Using (3) and (5) we obtain,

$$(\operatorname{grad} f)_{\binom{T}{P}} = \begin{pmatrix} \delta^* - \frac{1}{2}T\big((\delta T + DP) + (\delta T + DP)^*\big) \\ D^* - \frac{1}{2}P\big((\delta T + DP) + (\delta T + DP)^*\big) \end{pmatrix}, \tag{6}$$

which is the expression of the statement. □

The study of height functions can be simplified by using the singular value decompositions, see [21, Theorem 7.2]. Let $\omega^* = U\binom{0}{S}V^*$ be a singular value decomposition of $\omega^* = (\delta|D)^*$. In this decomposition, we have $U \in \mathrm{Sp}(n)$, $V \in \mathrm{Sp}(k)$ and $S \in \mathbb{H}^{k\times k}$ is a block diagonal matrix

$$S = \begin{pmatrix} 0_{n_0} & & & \\ & s_1 I_{n_1} & & \\ & & \ddots & \\ & & & s_r I_{n_r} \end{pmatrix}, \quad n_0 + n_1 + \cdots + n_r = k, \tag{7}$$

with real numbers $0 < s_1 < \cdots < s_r$. The next result reduces the determination of the gradient and the Hessian to the case $\omega = \omega_0$.

Corollary 2.2. *Let f_ω and f_{ω_0} be the height functions on $X_{n,k}$ with respect to $\omega = (\delta|D)$ and $\omega_0 = (0|S)$. Then the gradient and Hessian of f_ω are related to those of f_{ω_0} as follows.*

a) *For any $x \in X_{n,k}$, we have* $(\operatorname{grad} f_\omega)_x = U(\operatorname{grad} f_{\omega_0})_{U^*xV} V^*$.
b) *For any $x \in X_{n,k}$ and $W \in T_x(X_{n,k})$, we have*

$$(\mathcal{H} f_\omega)_x(W) = U\big((\mathcal{H} f_{\omega_0})_{U^*xV}(U^*WV)\big)V^*.$$

Proof. The equality in a) follows from Proposition 2.1. Using this equality, the Hessian, $(\mathcal{H} f_\omega)_x \colon T_x(X_{n,k}) \to T_x(X_{n,k})$ can be computed as

$$\begin{aligned}(\mathcal{H} f_\omega)_x(W) &= \left(\frac{d}{dt}(\operatorname{grad} f_\omega)_{x+tW}\right)_{|t=0} \\ &= U\left(\frac{d}{dt}(\operatorname{grad} f_{\omega_0})_{U^*(x+tW)V}\right)_{|t=0} V^* \\ &= U\big((\mathcal{H} f_{\omega_0})_{U^*xV}(U^*WV)\big)V^*.\end{aligned}$$

□

2.4 Critical points

We now characterize the critical points of the height function on $X_{n,k}$. Let $x = \binom{T}{P} \in X_{n,k}$ and $\omega = (\delta|D)$, with $\delta \in \mathbb{H}^{k\times(n-k)}$ and $D \in \mathbb{H}^{k\times k}$ as above.

Proposition 2.3. *The point x is critical for the height function, $f_\omega\colon X_{n,k} \to \mathbb{R}$ if, and only if, $\omega^* = x\omega x$. In this case, the matrix $\omega x = \delta T + DP$ is Hermitian.*

Proof. From Proposition 2.1 we deduce that x is a critical point of f_ω if, and only if, $2\omega^* = x(\omega x + x^*\omega^*)$. Since $x^*x = I_k$, we have $2x^*\omega^* = \omega x + x^*\omega^*$, then $(\omega x)^* = \omega x$. Moreover $2\omega^* = x(2\omega x)$, hence $\omega^* = x\omega x$.

Conversely, if $\omega^* = x\omega x$, it is $x^*\omega^* = \omega x$, so ωx is Hermitian. Then, $x(\omega x + x^*\omega^*) = 2x\omega x = 2\omega^*$, that is, $(\operatorname{grad} f)_x = 0$. □

Corollary 2.4. *If the point $x = \binom{T}{P}$ is critical for the height function, $f_\omega\colon X_{n,k} \to \mathbb{R}$, then the following formulae hold:*

$$\delta^* = T(\delta T + DP), \tag{8}$$

$$D^* = P(\delta T + DP), \tag{9}$$

$$\delta\delta^* + DD^* = (\delta T + DP)^2. \tag{10}$$

Proof. The first two formulae follow from Equation (6) and the fact that $\omega x = \delta T + DP$ is Hermitian, as proved in Proposition 2.3. The third one is $\omega\omega^* = (\omega x)^2$, which is also an immediate consequence of Proposition 2.3. □

In the case $\omega_0 = (0|S)$, Proposition 2.3 can be simplified as follows.

Corollary 2.5. *Let $\omega_0 = (0|S)$ be as in (7) and $\omega = U\omega_0 V^*$, with $U \in \mathrm{Sp}(n)$, $V \in \mathrm{Sp}(k)$. Then, the following properties hold.*

a) *The point $x = \binom{T}{P}$ is critical for the height function, f_{ω_0}, if, and only if, $TSP = 0$ and $PSP = S$. In this case, the matrix SP is Hermitian.*
b) *The point x is a critical point of f_ω if, and only if, $U^* x V$ is a critical point of f_{ω_0}.*

Proof. The statement a) is a rewriting of Proposition 2.3 in this particular case. Property b) is a direct consequence of Corollary 2.2.a. □

Recall that $\langle , \rangle$ denotes the Hermitian product in $\mathbb{H}^n$. Even if the matrix ω is not a square matrix, we have $\langle \omega\omega^* v, v\rangle = \langle \omega^* v, \omega^* v\rangle = |\omega v|^2 \geq 0$, for all $v \in \mathbb{H}^k$. Therefore, the matrix

$$\Delta = \omega\omega^* = \delta\delta^* + DD^*$$

is *positive semi-definite.*

Proposition 2.6. *The point x is critical for the height function, $f_\omega\colon X_{n,k} \to \mathbb{R}$ if, and only if, there exists a Hermitian square root Y of Δ such that $\omega^* = xY$. In this case $Y = \omega x$.*

Proof. First, let $\omega^* = xY$ with $Y = Y^*$ and $Y^2 = \Delta$. Then $x\omega x = x(Y^* x^*)x = xY(x^* x) = xY = \omega^*$, so x is a critical point by Proposition 2.3. Moreover from $x^* x = I_k$ it follows $x^*\omega^* = Y$, so $Y = Y^* = \omega x$.

Conversely, if x is a critical point, then $\omega^* = x(\omega x)$, and the matrix $Y = \omega x$ is Hermitian. Moreover $(\omega x)^2 = (\omega x)(\omega x)^* = \omega(x^* x)\omega^* = \omega\omega^* = \Delta$. □

Proposition 2.7. *Let $\omega_0 = (0|S)$, $x = \binom{T}{P} \in X_{n,k}$, $x_0 = \binom{0}{I_k}$ and $W \in T_x X_{n,k}$. Let $A \in \mathrm{Sp}(n)$ such that $x = Ax_0$ and $W = AW_0$, with $W_0 = \binom{X}{Y} \in T_{x_0} X_{n,k}$. Then, if x is a critical point of f_{ω_0}, the Hessian of f_{ω_0} is given by,*

$$(\mathcal{H} f_{\omega_0})_x(W) = -(1/2)A\begin{pmatrix} 2XSP \\ YSP + SPY \end{pmatrix}.$$

Proof. We first observe that the existence of A comes from the transitivity of the action of $\mathrm{Sp}(n)$. For the determination of the Hessian, we follow a classical procedure (see [5]) which consists of three steps.

(i) First, we extend the gradient computed in Proposition 2.1 to a vector field on $\mathbb{H}^{n\times k}$, by

$$(\widetilde{\mathrm{grad}} f_{\omega_0})_M = \omega_0^* - (1/2)M(\omega_0 M + M^*\omega_0^*).$$

(ii) Secondly, we determine the covariant derivative $\nabla_W(\widetilde{\mathrm{grad}} f_{\omega_0})$. Since the covariant derivative in $\mathbb{H}^{n\times k}$ is the usual derivative, we find after computation,

$$\left(\frac{d}{dt}(\widetilde{\mathrm{grad}} f_{\omega_0})_{M+tW}\right)_{|t=0}$$
$$= (-1/2)[M(\omega_0 W + W^*\omega_0^*) + W(\omega_0 M + M^*\omega_0^*)]. \qquad (11)$$

(iii) Finally, the Hessian consists of the projection of the expression (11) onto the tangent space, $T_x X_{n,k}$. We denote $A = \begin{pmatrix} \alpha & T \\ \beta & P \end{pmatrix} \in \mathrm{Sp}(n)$ and recall that in $W_0 = \begin{pmatrix} X \\ Y \end{pmatrix} \in T_{x_0} X_{n,k}$, the matrix X is arbitrary and Y is anti-hermitian, i.e., $Y + Y^* = 0$. Set

$$(\Gamma_{\omega_0})_x(W) = \left(\frac{d}{dt}(\widetilde{\mathrm{grad}} f_{\omega_0})_{M+tW}\right)_{|t=0}.$$

Replacing M by x and ω_0, W by their values in (11) gives,

$$(\Gamma_{\omega_0})_x\begin{pmatrix} U \\ V \end{pmatrix} = -\frac{1}{2}\left\{\begin{pmatrix} T \\ P \end{pmatrix}(SV + V^*S) + \begin{pmatrix} U \\ V \end{pmatrix}(SP + P^*S)\right\}.$$

So

$$A^*\,(\Gamma_{\omega_0})_x\begin{pmatrix} U \\ V \end{pmatrix} = -\frac{1}{2}\left\{\begin{pmatrix} 0 \\ I \end{pmatrix}(SV + V^*S) + \begin{pmatrix} X \\ Y \end{pmatrix}(SP + P^*S)\right\}$$
$$= -\frac{1}{2}\left\{\begin{pmatrix} 0 \\ SV + V^*S \end{pmatrix} + \begin{pmatrix} XSP + XP^*S \\ YSP + YP^*S \end{pmatrix}\right\}.$$

As x is a critical point, we have $SP = P^*S$ and we obtain

$$A^*\,(\Gamma_{\omega_0})_x\begin{pmatrix} U \\ V \end{pmatrix} = -\frac{1}{2}\left\{\begin{pmatrix} 0 \\ SV + V^*S \end{pmatrix} + \begin{pmatrix} 2XSP \\ 2YSP \end{pmatrix}\right\}.$$

So, the image by the projection onto $T_{x_0} X_{n,k}$, recalled in (3), equals,

$$-\frac{1}{2}\begin{pmatrix} 2XSP \\ YSP + SPY \end{pmatrix}.$$

Finally, a left translation by A gives the projection onto $T_x X_{n,k}$ and we get the formula of the statement. □

2.5 Critical sets

In this paragraph, we characterize the critical sets of any height function on a quaternionic Stiefel manifold.

Theorem 2.8. *The critical set of the height function* $f_\omega \colon X_{n,k} \to \mathbb{R}$ *is diffeomorphic to the product*

$$X_{n-k+n_0,n_0} \times \Sigma(n_1) \times \cdots \times \Sigma(n_r),$$

where $X_{n-k+n_0,n_0} = \mathrm{Sp}(n-k+n_0)/\mathrm{Sp}(n-k)$ *is a Stiefel manifold and* $\Sigma(m)$ *denotes a disjoint union* $G_{0,m} \sqcup G_{1,m} \sqcup \cdots \sqcup G_{m,m}$ *of Grassmannians,* $G_{i,m} = \mathrm{Sp}(m)/(\mathrm{Sp}(i) \times \mathrm{Sp}(m-i))$.

Proof of Theorem 2.8. According to Corollary 2.5, it is sufficient to consider the case f_{ω_0} with $\omega_0 = (0|S)$ and $S = \mathrm{diag}(0_{n_0}, s_1 I_{n_1}, \ldots, s_r I_{n_r})$ a non-negative real diagonal block matrix. From Corollary 2.5 we know that $x = \binom{T}{P}$ is a critical point of f_{ω_0} if, and only if, $TSP = 0$ and $PSP = S$. Moreover, we have also that SP is Hermitian and $S^2 = (SP)^2$.

The latter equality implies that S is the modulus of SP and [2, Theorem 5.5] implies the existence of $U \in \mathrm{Sp}(k)$ such that $SP = US$. As $SP = US$ is Hermitian, we get $SU = U^*S$, hence

$$S = USU.$$

A direct computation (see [12, Lemma 6]) implies that the unitary matrix U is of the form $U = \mathrm{diag}(U_0, U_1, \ldots, U_r)$ with $U_i U_i^* = I_{n_i}$ for $0 \le i \le k$, and $U_i = U_i^*$ for $1 \le i \le k$.

Now, we decompose T in $T = (T_0, \ldots, T_r)$, where T_j is a block of size $(n-k) \times n_j$. The equality $TSP = 0$ implies $s_j T_j U_j = 0$ and thus $T_j = 0$ for $j \ge 1$.

Secondly, we decompose P in $P = \begin{pmatrix} P_{00} & P_{01} & \ldots & P_{0r} \\ P_{10} & P_{11} & \ldots & P_{1r} \\ \vdots & & & \\ P_{r0} & P_{r1} & \ldots & P_{rr} \end{pmatrix}$, where P_{ij} is a block of size $n_i \times n_j$. The equality $S = PSP = PUS$ gives $P_{ij} = 0$ if $i \neq j$ and $P_{ii} = U_i$ for $1 \le i \le r$.

In conclusion, $\binom{T}{P}$ has for its first columns $\begin{pmatrix} T_0 \\ P_{00} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and the other columns are of the form

$$\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ U_1 & & \\ & \ddots & \\ & & U_r \end{pmatrix}.$$

The equality $T^*T + P^*P = I_k$ implies $T_0^*T_0 + P_{00}^*P_{00} = I_{n_0}$, so these first columns represent an element of the Stiefel manifold X_{n-k+n_0,n_0}. Finally each Hermitian matrix $U_i \in \mathrm{Sp}(n_i)$ verifies $U_i^2 = I_{n_i}$, so it can be written as $U_i = VD_iV^*$, where D_i is a diagonal matrix with ± 1 on the diagonal. The orbit for the action of $\mathrm{Sp}(n_i)$ by conjugation of such D_i is diffeomorphic to some Grassmannian, $\mathrm{Sp}(n_i)/(\mathrm{Sp}(l) \times \mathrm{Sp}(n_i - l))$. □

2.6 Height functions as Morse functions: Indices and critical values

By definition, a function on a compact manifold is Morse-Bott if the Hessian is non-degenerate in the directions transverse to the critical set. Moreover, such a function is Morse if it has a finite number of critical points. The next result characterizes the height functions that are Morse on a quaternionic Stiefel manifold. (Note that the case of real Stiefel manifolds is developed in [18, Theorem 1.2].)

Theorem 2.9. *Height functions, $f_\omega\colon X_{n,k} \to \mathbb{R}$, on quaternionic Stiefel manifolds satisfy the following properties.*

a) *Any height function is Morse-Bott.*
b) *A height function, f_ω, is a Morse function if, and only if, the singular values of $\omega\omega^*$ are positive and pairwise different. In this case, the critical set is diffeomorphic to the set of points $\binom{0}{E}$ where $E = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_k) \in \mathrm{Sp}(k)$, with $\varepsilon_i = \pm 1$.*

Proof. Property b) is a direct consequence of a) and Theorem 2.8. We begin the proof of Property a) for the height function f_{ω_0}, with $\omega_0 = (0|S)$, see (7). Observe that the definition of Morse-Bott means that the kernel of the Hessian equals the tangent space to the critical set. Let $A \in \mathrm{Sp}(n)$ such that $x = Ax_0$ with $x_0 = \binom{0}{I}$ and x a critical point.

The *tangent space to the critical set* Σ can be computed as usual (here, the coordinates correspond to the lines of the matrix) from the equations established in Corollary 2.5.a, i.e., we have

$$W = \begin{pmatrix} U \\ V \end{pmatrix} \in T_x\Sigma \;\text{ if, and only if, }\; \begin{cases} USP = 0 \\ VSP + PSV = 0. \end{cases} \tag{12}$$

Let $W = \binom{U}{V} = AW_0$ with $W_0 = \binom{X}{Y}$. According to Proposition 2.7, the *kernel of the Hessian of* f_{ω_0} is characterized by,

$$W = \begin{pmatrix} U \\ V \end{pmatrix} \in \ker(\mathcal{H} f_{\omega_0})_x \text{ if, and only if, } \begin{cases} XSP = 0 \\ YSP + SPY = 0. \end{cases} \tag{13}$$

We observe

$$\begin{cases} XSP = 0 \\ YSP + SPY = 0 \end{cases} \iff \begin{pmatrix} X \\ Y \end{pmatrix} SP + \begin{pmatrix} 0 \\ SPY \end{pmatrix} = 0$$

$$\iff A\begin{pmatrix} X \\ Y \end{pmatrix} SP + A\begin{pmatrix} 0 \\ SPY \end{pmatrix} = 0.$$

$$\iff \begin{pmatrix} U \\ V \end{pmatrix} SP + A\begin{pmatrix} 0 \\ SPY \end{pmatrix} = 0.$$

So the equivalence of the systems (12) and (13) is a consequence of

$$\textit{Claim:}\ A\begin{pmatrix} 0 \\ SPY \end{pmatrix} = \begin{pmatrix} 0 \\ PSV \end{pmatrix}. \tag{14}$$

To establish this equality, we first deduce from Theorem 2.8 that

$$\begin{pmatrix} T \\ P \end{pmatrix} = \begin{pmatrix} T_0 & 0 \\ P_{00} & 0 \\ 0 & U_{1\ldots r} \end{pmatrix},$$

where $U_{1\ldots r} = \operatorname{diag}(U_1, \ldots, U_r)$ with $U_i \in \operatorname{Sp}(n_i)$. We decompose the matrix A as

$$A = \begin{pmatrix} \alpha & T_0 & 0 \\ \beta_0 & P_{00} & 0 \\ \beta_{1\ldots r} & 0 & U_{1\ldots r} \end{pmatrix}$$

with $\beta_0 \in \mathbb{H}^{n_0\times(n_0-k)}$, $\beta_{1,\ldots,r} \in \mathbb{H}^{(k-n_0)\times(n-k)}$. The equality $AA^* = I_n$ implies $\beta_{1\ldots r}U^*_{1\ldots r} = 0$, so $\beta_{1\ldots r} = 0$. Finally we have

$$S\beta = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ & s_1 I & & \\ & & \ddots & \\ & & & s_r I \end{pmatrix} \begin{pmatrix} \beta_{00} & \beta_{01} & \ldots & \beta_{0r} \\ 0 & 0 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & 0 \end{pmatrix} = 0.$$

Now, we compute, by using $TSP = 0$ and $PSP = S$, see Corollary 2.5.a,

$$A\begin{pmatrix} 0 \\ SPY \end{pmatrix} = \begin{pmatrix} \alpha & T \\ \beta & P \end{pmatrix} \begin{pmatrix} 0 \\ SPY \end{pmatrix} = \begin{pmatrix} TSPY \\ PSPY \end{pmatrix} = \begin{pmatrix} 0 \\ SY \end{pmatrix}.$$

From $\binom{U}{V} = \begin{pmatrix} \alpha & T \\ \beta & P \end{pmatrix}\binom{X}{Y}$, we get $V = \beta X + PY$ and

$$PSV = PS\beta X + PSPY = PSPY = SY.$$

The claim (14) is proved and the critical point x is non-degenerate.

So, Property a) is proved for f_{ω_0}. The general case of f_ω follows from the transformation rule of the Hessian (Corollary 2.2.b) and the determination of critical points (see Corollary 2.5.b). □

If we denote by α_i the number of critical points of index i and by b_i the i-th Betti number of $X_{n,k}$, we know, from classical Morse theory, that $\alpha_i \geq b_i$, for all i. A Morse function is called *perfect* if we have $\alpha_i = b_i$, for all i.

Corollary 2.10. *Each Morse height function on $X_{n,k}$ is a perfect Morse function.*

Proof. From their description in Theorem 2.9, it is clear that a Morse height function has exactly 2^k critical points. On the other side, we know [14, Theorem 3.10] that, for $k > 0$, the cohomology of $X_{n,k}$, with $\mathbb{Z}$ coefficients, is an exterior algebra,

$$H^*(X_{n,k}) = \wedge(y_{n-k+1}, \dots, y_n), \tag{15}$$

with y_i of degree $4i - 1$. Its Poincaré series is $P(t) = (1 + t^{4(n-k)+3}) \cdots (1 + t^{4n-1})$. The sum of coefficients is 2^k and we get $\sum_{i\geq 0} \alpha_i = \sum_{i\geq 0} b_i$. With the Morse inequalities, $\alpha_i \geq b_i$, this implies $b_i = \alpha_i$, for all i. □

We have proved the previous Corollary without the determination of the index of a critical point. The next result specifies this index.

Proposition 2.11. *Let $x = \binom{0}{E}$ be a critical point of a Morse height function, with $E = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_k)$, $\varepsilon_i = \pm 1$. Suppose that $\varepsilon_{i_1}, \dots, \varepsilon_{i_p}$ are the positive entries in E, with $i_1 < \cdots < i_p$. Then the index of x is given by*

$$\operatorname{Ind}(x) = p(4(n-k) - 1) + 4(i_1 + \cdots + i_p).$$

Proof. According to Theorem 2.9, the matrix defining the height function can be assumed to have the form $\omega_0 = (0|S)$, with $S = \operatorname{diag}(s_1, \dots, s_k)$ and $0 < s_1 < \cdots < s_k$. A critical point of the associated height function, f_{ω_0}, is of the form $x = \binom{0}{\operatorname{diag}(\varepsilon_1,\dots,\varepsilon_k)}$, with $\varepsilon_i = \pm 1$.

We refer to Proposition 2.7. The point $x = \binom{0}{E}$ is the image of $x_0 = \binom{0}{I}$ by the left action of $\operatorname{Sp}(n)$, so there is some A such that $Ax = x_0$. Since the isotropy of x_0 is $\operatorname{Sp}(n-k)$, the matrix A is not unique but we can choose $A = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} \in \operatorname{Sp}(n)$ and we have

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ EY \end{pmatrix}.$$

Therefore, the result of Proposition 2.7 can be written as

$$
\begin{aligned}
(\mathcal{H} f_{\omega_0})_x \begin{pmatrix} U \\ V \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} 2XSE \\ YSE + SEY \end{pmatrix} \\
&= -\frac{1}{2}\begin{pmatrix} 2XSE \\ EYSE + ESEY \end{pmatrix} \\
&= -\frac{1}{2}\begin{pmatrix} 2USE \\ VSE + ESV \end{pmatrix}
\end{aligned}
$$

The Hessian is a self-adjoint linear map, whose eigenvalues are of two types:

(i) $-\varepsilon_j s_j$, for $1 \leq j \leq k$. The corresponding eigenvectors are the matrices all of whose terms are zero, except the j^{th}-column of U and the element v_{jj} of V. As EV is skew-hermitian, the associated eigenspace is of (real) dimension $4(n-k)+3$.

(ii) $(-1/2)(\varepsilon_i s_i + \varepsilon_j s_j)$, with $1 \leq i < j \leq k$. The corresponding eigenvectors are the matrices all of whose terms are zero, except v_{ij} and $v_{ji} = -v_{ij}^*$. The associated eigenspace is of (real) dimension 4.

Recall that the index of the critical point $x = \binom{0}{E}$, with $E = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_k)$, is the total dimension of the eigenspaces associated to negative eigenvalues. Let $\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}$ be the positive entries, with $i_1 < \cdots < i_p$. In case (i), negative eigenvalues are exactly the ε_{i_k}'s, with $k = 1, \ldots, p$. In case (ii), they are the pairs (i, j), with $i < j$ and $\varepsilon_j = +1$. In conclusion we get the formula

$$
\operatorname{Ind}(x) = p(4(n-k)+3) + 4((i_1 - 1) + \cdots + (i_p - 1)),
$$

which is equivalent to that of the statement. □

Finally, we give a lower bound for the number of critical *values* of a Morse height function which is of interest in the determination of the LS-category.

Proposition 2.12. *The number of critical values of a Morse height function on* $X_{n,k}$ *is at least* $1 + \binom{k+1}{2}$. *Moreover, there exists a height function for which the bound is sharp.*

Proof. Let $\omega_0 = (0|S)$, with $S = \operatorname{diag}(s_1, \ldots, s_k)$, $0 < s_1 < \cdots < s_k$. The value of f_{ω_0} at the critical point $x = \binom{0}{E}$, with $E = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_k)$, $\varepsilon_i = \pm 1$, is $f_{\omega_0}(x) = \sum_{i=0}^k \varepsilon_i s_i$. Therefore, the number of critical values is equal to the cardinality of the set $\left\{ \sum_{i=0}^k \varepsilon_i s_i \mid \varepsilon_i = \pm 1 \right\}$. We observe that

$$\#\left\{\sum_{i=0}^{k}\varepsilon_i s_i \mid \varepsilon_i=\pm 1\right\}=\#\left\{\sum_{i=0}^{k}\varepsilon_i s_i+s_1+\cdots+s_k \mid \varepsilon_i=\pm 1\right\}$$
$$=\#\left\{\sum_{i=0}^{k} f_i s_i \mid f_i\in\{0,2\}\right\}$$
$$=\#\left\{\sum_{i=0}^{k} g_i t_i \mid g_i\in\{0,1\}\right\},$$

with $t_i = 2s_i$. The latter set has a cardinality greater than, or equal to, $1+\binom{k+1}{2}$ because it contains the following distinct elements:

$$0, t_1, \ldots, t_k, t_k+t_1, \ldots, t_k+t_{k-1}, t_k+t_{k-1}+t_1, \ldots,$$
$$t_k+t_{k-1}+t_{k-2}, \ldots, t_k+\cdots+t_1.$$

This lower bound is reached in the case $s_i = i$, as a direct computation shows (see [11]). □

3. LS category of Stiefel manifolds

3.1 Properties of LS category

The definition of the LS-category of a topological space, X, has been recalled in the introduction. We list here some basic properties of it, referring to [1] for more details.

1. The LS-category is a homotopy type invariant.
2. By definition, the *cup length* of a space X is the largest integer ℓ such that there exists a product $x_1 \cdots x_\ell \neq 0$, with $x_i \in \tilde{H}^*(X; A)$. Here the coefficient ring A may vary and the cup length may be considered for any coefficients. Then

   $$\mathrm{cup}(X) \leq \mathrm{cat}\, X.$$

   For instance, from Equation (15), it appears that the cup length of $H(X_{n,k}; \mathbb{Z})$ equals k.
3. Let X be an $(n-1)$-connected CW-complex. Then

   $$\mathrm{cat}\, X \leq (\dim X)/n.$$

4. If M is a smooth compact manifold and $\mathrm{crit}(M)$ denotes the minimum number of critical points for any smooth function $f\colon M \to \mathbb{R}$, then

   $$\mathrm{cat}\, M + 1 \leq \mathrm{crit}(M).$$

In fact, we shall use a refined version of property (4) which was observed in [16].

Theorem 3.1. *Let M be a connected compact manifold and let $f\colon M \to \mathbb{R}$ be a smooth function with isolated critical points. Then*

$$\operatorname{cat} M + 1 \leq \#\{\text{critical values of } f\}.$$

This result is the reason we are interested in finding Morse functions with as few critical *values* as possible. In the opposite way, we know (see [13, Lemma 2.8, page 17]) that any Morse function with critical points, $x_1, \dots, x_\ell$, can be approximated by a Morse function, g, with the same critical points and such that $g(x_i) \neq g(x_j)$ if $i \neq j$.

3.2 LS category of Sp(n)

The case of $\mathrm{Sp}(n)$, which corresponds to $k = n$, was already studied in [11]. It was shown there that, when $\omega = D = \operatorname{diag}(1, 2, \dots, n)$ the height function f_ω has $\binom{n+1}{2} + 1$ different critical values. Then, by Theorem 3.1,

$$\operatorname{cat} \mathrm{Sp}(n) \leq \binom{n+1}{2}.$$

3.3 The Stiefel manifolds $X_{n,k}$

In Proposition 2.12, we proved that, for any Morse height function on $X_{n,k}$, the cardinality of the set of different critical values is always greater than, or equal to, $\binom{k+1}{2} + 1$. Theorem 3.1 implies

$$\operatorname{cat} X_{n,k} \leq \binom{k+1}{2}. \tag{16}$$

Recall from [8, Proposition 2.1, Page 15], that $X_{n,k}$ is of dimension $2k(2n - k + 1) - k = 4nk - 2k^2 + k$ and, of connectivity plus 1 equal to $4n - 4k + 3$. We want to compare the bound given by the dimension and connectivity to (16). For that, we study the sign of the difference

$$\begin{aligned} A(n,k) &= \frac{4nk - 2k^2 + k}{4n - 4k + 3} - \frac{k(k+1)}{2} \\ &= \frac{4nk - 4nk^2 + 4k^3 - 3k^2 - k}{8n - 8k + 6}. \end{aligned}$$

As $0 \leq k \leq n$, the sign of $A(n,k)$ is the same as the sign of

$$\begin{aligned} B(n,k) &= 4k^2 - k(4n+3) + 4n - 1 \\ &= 4(k-1)(k - (n - (1/4))). \end{aligned}$$

This quadratic polynomial in k has two zeros, $k_1 = 1$ and $k_2 = n - (1/4)$. Thus $B(n, k)$ is only positive when $k = n$ and this is the case of $\mathrm{Sp}(n)$ already studied in [11].

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