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Continuous Cohomology of Linear Foliations on T^2

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RIASSUNTO – *Si calcolano i gruppi di coomologia di T^2 a coefficienti nel fascio dei germi delle funzioni continue reali che sono costanti sulle foglie di una foliazione lineare.*

ABSTRACT – *We compute the cohomology groups of T^2 with coefficients in the sheaf of germs of continuous real maps which are constant on the leaves of a linear foliation.*

KEY WORDS – *Foliations - Continuous cohomolgy.*

A.M.S. CLASSIFICATION: 57R30 - 42A

1 – Introduction

The aim of this note is to give an explicit calculation of $H^{\cdot}(T^2;\underline{C})$, the cohomology groups of the torus with coefficients in the sheaf $\underline{C}$ of germs of *continuous* real maps which are constant on the leaves of a linear foliation F. Such a foliation is obtained by considering the lines with a constant slope $\alpha \in [0,1]$ in R^2. When we pass to the quotient we obtain a one-dimensional foliation on T^2 whose leaves are all compact ($\alpha \in Q$) or all dense ($\alpha \notin Q$).

Let us summarize our calculation:

PROPOSITION. *$H^0(T^2;\underline{C}) = H^1(T^2;\underline{C}) = C(S^1)$ when α is a rational number; otherwise, $H^0(T^2;\underline{C}) = R$ and* dim $H^1(T^2;\underline{C}) > \aleph_0$.

These results differ from the classical calculation [7] [4] of the cohomology groups of T^2 with coefficients in the shef of germs of *differentiable* maps which are constant on the leaves, as a reflection of the properties of Fourier series in the C^0 case.

Although similar results to those presented here are relatively well known, no explicit calculations have been published until now. It seems possible to extend them to other nilpotent Lie foliations on compact manifolds.

2 – Proof of the proposition

When α is 0 or 1 the foliation is a product, and one has

$$H^r(T^2;\underline{C}) = C(S^1)\otimes H^r(S^1,R) = \begin{cases} C(S^1) & \text{if } r=0,1 \\ 0 & \text{if } r\geq 2\,. \end{cases}$$

where $C(S^1) = \mathrm{Map}(S^1, R)$.

For $\alpha \in (0,1)$ the leaves are transversal to the fibers of the trivial fibering $\pi\colon T^2 \to S^1$, which has a Leray sheaf $\underline{H}^{\cdot}(\pi;\underline{C})$ with coefficients in $\underline{C}$ [1] generated by the presheaf $H^{\cdot}(\pi^{-1}(U);\underline{C}) = H^{\cdot}(U\times S^1;\underline{C})$. Clearly, S^1 with the discrete foliation is an integrable deformation retract of $U\times S^1$; because of the invariance of the continuous cohomology under integrable homotopies, one has $H^{\cdot}(U\times S^1;\underline{C}){=}H^{\cdot}(S^1;\underline{C}_d)$ where $\underline{C}_d$ is the sheaf of germs of continuous maps from S^1 to R.

So $\underline{H}^S(\pi;\underline{C}) = 0$ when $s>0$ and $\underline{H}^0(\pi;\underline{C}) = \underline{C(S^1)}$, a locally constant sheaf with $C(S^1)$ as stalks. The Leray spectral sequence is then on a line, and $H^r(T^2;\underline{C}) = H^r(S^1;\underline{C(S^1)})$.

Detailed proofs of these facts are given in [6] or [5].

Now $\underline{C(S^1)}$ is a local system of coefficients with an action of Z on $C(S^1)$ given by $1\cdot f = fr_\alpha$, where r_α is the rotation $r_\alpha(t) = t+\alpha$ and then $H^r(S^1;\underline{C(S^1)})$ is the cohomology of the group Z with coefficients in $C(S^1)$.

Let $\rho\colon C(S^1)\to C(S^1)$ be the homomorphism given by $\rho(f) = f -$

fr_α. We have finally [2]:

$$H^r(T^2;\underline{C}) = H^r(Z;C(S^1)) = \begin{cases} \operatorname{Ker}\rho & \text{if} \quad r=0 \\ \operatorname{Coker}\rho & \text{if} \quad r=1 \\ 0 & \text{if} \quad r \geq 2 \end{cases}$$

When α is an irrational number the image of r_α is a dense subset of S^1, and $f = fr_\alpha$ implies that f is a constant function. Then $H^0(T^2;\underline{C}) = R$.

Otherwise, if $\alpha = p/q$ is an irreducible fraction, choose $n, m \in Z$ such that $n\alpha + m = 1/q$. Then $f = fr_\alpha = \ldots = fr_\alpha^n$ implies $f(t+1/q) = f(t)$, and $\operatorname{Ker}\rho$ is the set of continuous functions with $1/q$ as period. So $H^0(T^2;\underline{C}) = C(S^1)$.

3 – Computation of $H^1(T^2;\underline{C})$

Calculating $H^1(T^2;\underline{C}) = C(S^1)/ < f - fr_\alpha >$ is a little more difficult.

3.1 – $\alpha \in Q$

We consider the homomorphism $\xi\colon C(S^1) \to C(S^1)$ with

$$(\xi g)(t) = \sum_{k=0}^{q-1} g(t+k/q)$$

and we verify easily that $\operatorname{Im}\xi$ is the set of functions with $1/q$ as period. Because g has 1 as period we obtain:

$$(\xi g)(t) = \sum_{k=0}^{q-1} g(t+k\alpha).$$

We shall prove $\operatorname{Ker}\xi = \operatorname{Im}\rho$:
"$\supset$" If $g \in \operatorname{Im}\rho$ then $g = f - fr_\alpha$; and $g(t) = f(t) - f(t+\alpha), \ldots, g(t+(q-1)\alpha) = f(t+(q-1)\alpha) - f(t+q\alpha)$ implies $(\xi g)(t) = 0$ because $f(t) = f(t+p)$.

"$\subset$" Suppose $\alpha = 1/q$ (the general proof is essentially the same), and let g be in $\operatorname{Ker}\xi$. Define

$$f(t) = \begin{cases} -qg(0) & \text{if } t \in [0, \frac{1}{q}] \\ f(t-\nu/q) - \sum_{k=1}^{\nu} g(t - \frac{k}{q}) & \text{if } t \in [\frac{\nu}{q}, \frac{\nu+1}{q}],\ 1 \le \nu \le q-1; \end{cases}$$

it is immediate to test $f(0) = f(1)$ and the continuity of f at points $1/q, \ldots, (q-1)/q$.

But then $g \in \operatorname{Im}\rho$ because $g(t) = f(t) - f(t+\alpha)$ for all $t \in [0,1]$.

We have proved that

$$H^1(T^2; \underline{C}) = C(S^1)/\operatorname{Im}\rho = C(S^1)/\operatorname{Ker}\xi = \operatorname{Im}\xi = C(S^1)$$

when $\alpha = p/q$.

3.2 – $\alpha \notin Q$

We shall explicitly exhibit a non countable infinite set of linearly independent elements of $H^1(T^2; \underline{C})$.

The uniform continuity of the function $e^{2\pi it}$ on [0,1] implies that, for each $k \in Z$, there exists $\delta_k > 0$ such that

$$|e^{2\pi ix} - e^{2\pi iy}| < 1/k^2$$

when $|x-y| < \delta_k$; and we can suppose $\lim \delta_k = 0$ when $|k| \to \infty$.

There exists moreover a monotone decreasing sequence $\{n_j\alpha + m_j\} \to 0$, $m_j, n_j \in Z$, with $|n_j\alpha + m_j| < \min(1, \delta_j)$, because of the irrationality of α.

Fix $\mu \in [0,1)$ and consider the functional series

$$(*) \qquad \sum_{k\in Z} \alpha_k^{\mu} e^{2\pi ikt}$$

with $a_k^{\mu} = 0$ if $k \ne n_j$ for all j, and $\alpha_{n_j}^{\mu} = j^{\mu}(1 - e^{2\pi in_j\alpha})$.

Then, for all $t \in [0,1]$

$$|n_jt - (n_j(t+\alpha) + m_j)| = |n_j\alpha + m_j| < \delta_j$$

and so

$$\left|\alpha^{\mu}_{n_j} e^{2\pi i n_j t}\right| = j^{\mu}\left|e^{2\pi i n_j t} - e^{2\pi i(n_j(t+\alpha)+m_j)}\right| < 1/j^{2-\mu}$$

By the Weierstrass criterion, the convergence of $\sum 1/j^{2-\mu}$ implies that the series given at $(*)$ converges uniformly, so it defines a continuous function $g_\mu \in C(S^1)$ whose Fourier coefficients are precisely $\alpha^{\mu}_{k}, k \in Z$. See [3] for results about Fourier series.

We test now that $\{g_\mu/\mu \in [0,1]\}$ is a linearly independent set in $H^1(T^2;\underline{C}) = C(S^1)/\operatorname{Im}\rho$. For if $\sum_{1\leq s\leq N} C_s g_{\mu_s} = \rho(f) \in \operatorname{Im}\rho$ one has, for all $k \in Z$,

$$\sum_{1\leq s\leq N} C_s \alpha^{\mu_s}_{k} = \int_0^1 f(t)e^{-2\pi i k t}dt - \int_0^1 f(t+\alpha)e^{-2\pi i k t}dt =$$

$$= \int_0^1 f(t)e^{-2\pi i k t}(1 - e^{2\pi i k\alpha})dt.$$

So, for $k \neq 0$ the Fourier coefficients of f must be

$$\beta_k = (\sum_{1\leq s\leq N} C_s \alpha^{\mu_s}_{k}/(1 - e^{2\pi i k\alpha})$$

or in other words, $\beta_k = 0$ if $k \neq n_j$ for all j , and

$$\beta_{n_j} = \sum_{1\leq s\leq N} j^{\mu_s} C_s.$$

But the series $\sum |\beta_{n_j}|^2$ converges because of the continuity of f, so in particular

$$\lim_{j\in Z}\left|\sum_{1\leq s\leq N} j^{\mu_s} C_s\right| = 0$$

and this can only occur when all the C_s are zero. To see it, let us take $\mu = \max\{\mu_s\}$ and suppose $\mu = \mu_1$ for the sake of simplicity; then

$$\lim_{j\in Z}\left|\sum_{1\leq s\leq N} j^{\mu_s} C_s\right| / j^{\mu_1} = 0$$

and

$$\lim_{j\in Z}\left|\sum_{2\le s\le N} j^{\mu_s-\mu_1}C_s + C_1\right| = 0.$$

But $\mu_s - \mu_1 < 0$ implies $\lim j^{\mu_s-\mu_1} = 0$, and so $|C_1| = \lim|C_1| = 0$.

By the same procedure we shall arrive to $C_1 = \ldots = C_N = 0$ as expected. □

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