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LL-FOLIATIONS

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Let F be a G-Lie foliation on a compact manifold M. We say that F is a LL-foliation if the closure $K = \overline{\Gamma}$ of the holonomy group $\Gamma \subset G$ is a normal subgroup of G. In this paper we prove that for any Lie foliation F the ambient manifold M fibers over some G-homogeneous space in such a way that: (1) on each fiber N the induced foliation is a LL-foliation; (2) F is a LL-foliation if and only if $N = M$. Examples are given.

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INTRODUCTION

Roughly speaking, a LL-foliation is a Lie foliation ([2], [7], [9]) such that the basic fibration is a Lie foliation, too. More precisely, we have

Definition 1. Let F be a G-Lie foliation on the compact manifold M. We say that F is a *LL-foliation* if the closure $K = \overline{\Gamma}$ of the holonomy group $\Gamma \subset G$ is a normal subgroup of G.

This implies that the basic manifold $W = K\backslash G$ is a compact Lie group. In this paper we shall prove the following result.

THEOREM 2. *Let (M, F) be any Lie foliation on a compact manifold. Then there exists a locally trivial bundle $M \to B$ with generic fiber N such that*

1. *F induces on N a LL-foliation;*
2. *F is a LL-foliation iff B is a point.*

The proof will be done in several steps. The interest of LL-foliations comes from the fact that its basic basic cohomology $H(M/F)$ is isomorphic to the cohomology $H(\mathfrak{G})$ of the Lie algebra of G, as was proved by Llabrés and Reventós in [3].

1. THE NORMALIZER OF A CLOSED LIE SUBGROUP

Let G be a connected Lie group, K a *closed* Lie subgroup of G, $N(K) = \{g \in \in G \mid gKg^{-1} = K\}$ the normalizer of K in G. Then $N(K)$ is closed too, and $N(K)$, $N(K)/K$ are Lie groups. In general $N(K)$ may not be connected, even when K is.

Let K_e denote the connected component of the identity. It is straightforward to prove that $N(K) \subset N(K_e)$ and $(K \cap N(K)_e)_e = K_e$.

PROPOSITION 3. *The connected component of the identity of $N(K)/K$ is*

$$\left(\frac{N(K)}{K}\right)_e = \frac{N(K)_e}{K \cap N(K)_e} = \frac{KN(K)_e}{K}.$$

Proof. Remark that $K \cap N(K)_e$ is a normal subgroup of $N(K)_e$. The quotient group $N(K)_e/(K \cap N(K)_e)$ is connected and has the same dimension that $N(K)_e/K_e$, hence that of $N(K)/K$. Moreover it equals (not only is diffeomorphic to) the subgroup $KN(K)_e/K$ of $N(K)/K$ because normality implies that any element of $KN(K)$ can be written in the form kg with $k \in K$, $g \in N(K)$. □

As a matter of fact, the connected component of the identity in $KN(K)_e$ equals $N(K)_e$, then $KN(K)_e/N(K)_e$ is a discrete group.

PROPOSITION 4. *There is a covering of Lie groups $N(K)_e/K_e \to KN(K)_e/K$.*

Proof. Since H_e is a normal subgroup of $N(K)_e$, the map $N(K)_e/K_e \to \to N(K)_e/(K \cap N(K)_e)$ is a morphism of Lie groups. Its kernel $(K \cap N(K)_e)/K_e$ is contained in K/K_e, hence it is a discrete subgroup. □

The following result is well known [6]

PROPOSITION 5. *Let G be a simply connected group. Let K be a normal subgroup of G. Then K_e is closed and simply connected.*

2. LIE FOLIATIONS

Let F be a Lie foliation on the compact manifold M, transversely modeled by the connected simply connected Lie group G [9].

The so-called *holonomy group* Γ is a finitely generated subgroup of G, such that the *basic* manifold $W = K\backslash G$ is compact—where we denote $K = \overline{\Gamma}$ the closure of Γ in G.

Since $N(K)$ is closed in G, $N(K)/K$ will be closed in W. Then, according to Proposition 3, the identity connected component of $N(K)/K$ is the *compact connected*

Lie group

$$W_0 = \frac{N(K)_e}{K \cap N(K)_e} = \frac{KN(K)_e}{K}.$$

Moreover (Proposition 4), a covering of W_0 is the unimodular Lie group $\tilde{W}_0 = N(K)_e/K_e$. Also, by computing the space of connected components of the compact Lie group $N(K)/K$, we obtain that $N(K)/(KN(K)_e)$ is a finite group and $KN(K)_e$ is a closed subgroup of $N(K)$, hence of G. Remark that, in general, the smallest subgroup containing two closed subgroups may not be closed.

LEMMA 6. *1.* $\Gamma \cap K_e$ *is dense in* K_e*; 2.* $K = \Gamma K_e$*; 3.* $KN(K)_e = \bigcup_{\gamma \in \Gamma} \gamma N(K)_e$.

Proof. Let $x \in K_e$. Take a sequence in Γ such that $\gamma_n \to x$. Since K/K_e is discrete, we have that $\gamma_n \in K_e$ for $n > n_0$.

Analogously, let $x \in K$. Then there is some $\gamma \in \Gamma$ such that $xK_e = \gamma K_e$, so $x \in \Gamma K_e$. Remark that $\gamma K_e = K_e \gamma$ because $\Gamma \subset N(K_e)$.

Indeed, we have proved that any $x \in K$ can be written as γy with $y \in K_e$ and $\gamma \in \Gamma$. □

As a consequence, $K/K_e = \Gamma/(\Gamma \cap K_e)$ and $\Gamma N(K)_e = KN(K)_e$ —the latter group may not be $N(K)$, see the examples in Section 4.

3. THE INTERMEDIATE FIBRATION $N(F)$

We know from Section 2 that the basic manifold $W = K\backslash G$ contains the *connected compact* Lie group W_0 which is the identity component of $N(K)/K$.

As it is well known, on the homogeneous space $K\backslash G$ there is a right action G, and a left action of $N(K)$. In this line we consider the *left* action of W_0 on W given by $[h]Kg = Khg$, for $[h] \in W_0$. This action has no isotropy, and the space of orbits is the *Hausdorff* manifold $B = W_0\backslash W = (KN(K)_e)\backslash G$.

PROPOSITION 7. *The submersion* $M \to B$ *defines on* M *a transversely homogeneous foliation* $N(F)$ *with compact leaves.*

See the paper of Blumenthal [1] for the definition and structure of transversely homogeneous foliations. The fibres of $N(F)$ are connected submanifolds because the basic fibration $M \to W$ is a locally trivial bundle.

Remark 8. It would seem more natural to consider the submersion $W = K\backslash G \to N(K)\backslash G$, but it has non-connected fibres. The leaves of the corresponding foliation are precisely the orbits of the action of W_0. □

Now, let us consider the developping map $D: \tilde{M} \to G$ of the Lie foliation F, where $p: \tilde{M} \to M$ is the Galois covering associated to the holonomy group Γ (see [7] for details). It is Γ-equivariant, that is $D(\gamma(\tilde{x})) = \gamma D(\tilde{x})$, and its fibers are (diffeomorphic to) the leaves of F.

LEMMA 9. *Let L be a leaf of F with closure $\overline{L}$: 1. If $L = p(D^{-1}(g))$, then $p^{-1}(\overline{L}) = D^{-1}(Kg)$; 2. $D^{-1}(K_e g)$ is a (connected, onto) covering of $\overline{L}$.*

Proof. This is an application of the results of Fédida and Molino [9]. Let $p(\tilde{x}) \in \overline{L}$ be the limit of a sequence $(x_n) \subset L$. Since p is a local diffeomorphism, we have a sequence $(\tilde{x}_n) \to \tilde{x}$ in $\tilde{M}$ with $p(\tilde{x}_n) = x_n$. Let $\tilde{L}_n$ be the fibre of D passing through $\tilde{x}_n$. Then $D(\tilde{L}_n) = \gamma_n g$ with $\gamma_n \in \Gamma$. This means that $(\gamma_n g) \to D(\tilde{x}) \in Kg$. Conversely, let $D(\tilde{x}) \in Kg$, that is $(\gamma_n g) \to D(\tilde{x})$. Since D is a locally trivial bundle, we can choose a local section in order to obtain a sequence $(\tilde{x}_n) \to \tilde{x}$ with $D(\tilde{x}_n) = \gamma_n g$. Then $p(\tilde{x}_n) \in L$ converges to $p(\tilde{x}) \in \overline{L}$.

For the second part, let $x \in \overline{L}$ and take a point $\tilde{x}_0$ such that $x_0 := p(\tilde{x}_0) \in L$ and $D(\tilde{x}_0) = g$. Since $\overline{L}$ is path connected, any path α in $\overline{L}$ with $\alpha(0) = x_0$ and $\alpha(1) = x$ can be lifted to a unique path $\tilde{\alpha}$ in $\tilde{M}$ with $\tilde{\alpha}(0) = \tilde{x}_0$, which is neccesarily contained in $D^{-1}(Kg)$. then in $D^{-1}(K_e g)$. □

As a consequence, the foliation $(\overline{F}, F)$ induced by F in each closure $\overline{L}$ is a Lie foliation modeled by K_e (this is well known).

PROPOSITION 10. *The foliation $(N(F), F)$ induced by F in each fiber of $N(F)$ is a LL-foliation modeled on $N(K)_e$.*

Proof. Let $N(L)$ be a leaf of $N(F)$ projecting onto $KN(K)_e g \in W_0 \backslash W$. Then $\tilde{N}(L) := D^{-1}(N(K)_e g)$ is a covering of $N(L)$. In fact, $p(\tilde{N}(L)) = p(D^{-1}(\gamma N(K)_e g))$ for any $\gamma \in \Gamma$, by the equivariance of D; thus $p(\tilde{N}(L)) = N(L)$ by Lemma 6.3.

By composing with a right translation we obtain the developping map $\tilde{N}(L) \to N(K)_e$. Then the holonomy group is the group $\Gamma \cap N(K)_e$ of deck transformations.

Finally, the basic manifold is W_0 because the closure of $\Gamma \cap N(K)_e$ is $K \cap N(K)_e$. In fact, let $x \in K \cap N(K)_e$, which we can write as $x = \gamma y$ with $\gamma \in \Gamma \cap N(K)_e$, $y \in K_e$ because $K = \Gamma K_e$ (Lemma 6.2). Now, from Lemma 6.1, there is a sequence $\gamma_n \to y$ with $\gamma_n \in \Gamma \cap K_e$, hence $(\gamma\gamma_n)$ is a sequence in $\Gamma \cap N(K)_e$ converging to x. □

Remark that the fibration $N(F)$ is related with the normalizer of K in a non-obvious way. In fact, one could consider the foliation determined in G by the cosets of $N(K)_e$, but the lifted foliation in $\tilde{M}$ is not Γ-invariant, hence it does not determine any foliation on M.

4. EXAMPLES

Let us consider the the Lie group $G_8^0 = SO(2) \ltimes \mathbf{R}^2$ of matrices

$$\begin{pmatrix} \cos t & -\sin t & x \\ \sin t & \cos t & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, t \in \mathbf{R}$. Its Lie algebra is given as $[e_1, e_2] = 0$, $[e_1, e_3] = e_2$, $[e_2, e_3] = -e_1$. Topologically, the universal covering of G_8^0 is $\mathbf{R}^3$.

1. This example is due to P. Molino. On $M = T^2 \times T^2$ we have the flow $F = F_\alpha \times \{\text{point}\}$ associated to an irrational linear flow F_α on the first torus. Then F is a Lie foliation modeled on G_8^0. The subgroup $\Gamma \subset G_8^0$ is the given by $x = a + \alpha b$, $y = c$, $t = 2\pi d$, with $a, b, c, d \in \mathbf{Z}$. Then K is a disjoint union of infinitely many straight lines parallel to K_e, the OX axis. Here, $K/K_e = \mathbf{Z} \oplus \mathbf{Z}$. Moreover $N(K)$ is the family of planes $t = \pi n$, $n \in \mathbf{Z}$, and $N(K)_e$ equals $N(K_e)$, the XY plane.

Now, $N(K)/K$ are two disjoint circles. Then, the compact Lie group $W_0 = S^1$ acts naturally on the basic manifold $W = T^2$, and the fibration $N(F)$ has leaves $T^2 \times S^1$. Finally, the basic fibration $(N(F), \overline{F})$ is the trivial bundle $T^2 \times S^1 \to S^1$.

2. The following example is due to G. Meigniez and B. Herrera [4], [5].

Let Γ be the subgroup of G_8^0 given by $x = a + \alpha b$, $y = c$, $t = \pi d$, with $a, b, c, d \in \mathbf{Z}$. Call S the Lie group $\mathbf{R}^5 = \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}$ with the product

$$(v, w, t) * (v', w', t') = (v + A(t)v', w + A(t)w', t + t'),$$

where $A(t)$ is the rotation of angle t. Then S contains the discrete cocompact subgroup $\Gamma_S = \mathbf{Z}^4 \oplus \pi\mathbf{Z}$.

Let $\varphi: S \to G_8^0$ be the map $\varphi(v_1, v_2, w_1, w_2, t) = (v_1 + \alpha w_1, v_2 + w_2, t)$. Then we obtain a G_8^0-Lie foliation in the quotient manifold $\Gamma_S \backslash S$ with holonomy $\Gamma = \varphi(\Gamma_S)$.

In this case, K is again a family of straight lines, with K_e the OX axis. But $N(K)$ is the family of lines $y = m/2$, $t = n\pi$, $m, n \in \mathbf{Z}$, while $N(K_e)$ is the family of planes $t = n\pi$, $n \in \mathbf{Z}$, parallel to the XY plane. Then $N(K)/K$ are two points in the Klein bottle W, W_0 is the trivial group, and $N(\Gamma)_e = K_e$. In other words, $N(F) = \overline{F}$.

5. BASIC COHOMOLOGY

LL-foliations have the following common property with dense Lie foliations, as was proved by Llabrés and Reventós.

THEOREM 11. [3] *Let (M, F) be a LL-foliation. Then the basic cohomology $H(M/F)$ is isomorphic to the cohomology $H(\mathfrak{G})$ of the Lie algebra of G.*

Then, accordingly with [8], LL-foliations are minimalizable iff the Lie group G is unimodular.

COROLLARY 12. *Let $\mathfrak{N}$ be the Lie algebra of $N(K)$. Then the basic cohomology $H(N(F)/F)$ is isomorphic to the Lie algebra cohomology $H(\mathfrak{N})$.*

Proof. We know (Proposition 10) that the closure of the holonomy group of the Lie foliation $(N(F), F)$ is $K \cap N(K)_e$, whose identity connected component is K_e. This is coherent with the fact that the foliation $(\bar{F}, F)$ is yet known to have structural Lie group K_e.

Moreover, $K \cap N(K)_e$ is a normal subgroup of $N(K)_e$ (see Proposition 3). Then we can apply the theorem above. □

Remark 13. The transverse Lie algebra $\mathfrak{G}$ corresponds to G, while the structural Lie algebra $\mathfrak{H}$ is associated to the identity component K_e. It is easy to convince oneself that $\mathfrak{N}$ is *not* the normalizer of $\mathfrak{H}$ in $\mathfrak{G}$. □

Since the foliation $(N(F), \overline{F})$ is a fibre bundle over the compact Lie group W_0 whose Lie algebra is $\mathfrak{N}/\mathfrak{H}$ we have

PROPOSITION 14. *The basic cohomology $H(N(F)/\overline{F})$ is isomorphic to $H_{DR}(W_0)$, that is the Lie algebra cohomology $H(\mathfrak{N}/\mathfrak{H})$.*

Then, the Hoschild-Serre spectral sequence can be written as $H(W_0; H(\mathfrak{H})) \Rightarrow \Rightarrow H(\mathfrak{N})$. As particular cases we obtain

1. K is a normal subgroup of G if and only if $W_0 = W$, that is $N(F)$ is the foliation of M with a unique leaf. We reobtain in this case the result $H(M/F) = H(\mathfrak{G})$.
2. $W_0 = 0$ if and only if $N(K)/K$ is finite, that is $N(F) = \overline{F}$. We obtain $H(\overline{F}/F) = = H(\mathfrak{H})$, which is well known (leaves are dense).

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