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## Transverse Lusternik–Schnirelmann category of foliated manifolds

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### Abstract

The purpose of this paper is to develop a transverse notion of Lusternik–Schnirelmann category in the field of foliations. Our transverse category, denoted by  $\text{cat}_\perp(M, \mathcal{F})$ , is an invariant of the foliated homotopy type which is finite on compact manifolds. It coincides with the classical notion when the foliation is by points. We prove that for any foliated manifold  $\text{cat} M \leq \text{cat} L \text{cat}_\perp(M, \mathcal{F})$ , where  $L$  is a leaf of maximal category, thus generalizing a result of Varadarajan for fibrations. Also we prove that  $\text{cat}_\perp(M, \mathcal{F})$  is bounded below by the index of  $k^*H_b^+(M)$ , the latter being the image in  $H_{DR}(M)$  of the algebra of basic cohomology in positive degrees. In the second part of the paper we prove that  $\text{cat}_\perp(M, \mathcal{F})$  is a lower bound for the number of critical leaves of any basic function provided that  $\mathcal{F}$  is a foliation satisfying certain conditions of Palais–Smale type. As a consequence, we prove that the result is true for compact Hausdorff foliations and for foliations of codimension one. This generalizes the classical result of Lusternik and Schnirelmann about the number of critical points of a smooth function. © 2000 Elsevier Science Ltd. All rights reserved.

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The Lusternik–Schnirelmann category, in further LS category, of a space  $X$  is the least integer  $k$  such that  $X$  may be covered by  $k$  open subsets which are contractible in  $X$  [7,9–11]. This concept was introduced in the context of the calculus of variations in order to give a lower bound for the number of critical points of a smooth function.

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In this paper we introduce the notion of *transverse category*  $\text{cat}_\circ(M, \mathcal{F})$  of a foliated manifold. We consider foliated homotopies which are compatible with the foliation and we replace the contractibility in the space by requiring that the inclusion of an open set factors through a leaf up to the foliated homotopy. In this way we obtain that transverse category is an invariant of the foliated homotopy type which is finite on compact manifolds.

We generalize to any foliation Varadarajan's theorem [19] about the relationship between the category of the total space, the base space and the fiber of a fibration, by proving that  $\text{cat } M \leq \text{cat } L \text{cat}_\circ(M, \mathcal{F})$ , where  $L$  is a leaf of maximal category.

Basic cohomology has been widely studied since its definition by Reinhart (see [16]). In Section 2 we consider the complex of relative basic forms as a natural generalization of the mapping cone. Then we prove that  $\text{cat}_\circ(M, \mathcal{F}) \geq \text{nil } k^* H_b^+(M)$ , the latter being the index of the image in  $H_{DR}(M)$  of the basic cohomology algebra in positive degrees.

Among other examples, we discuss for smooth actions the relationship of transverse category with the equivariant category introduced by Fadell [6].

In the second part of the paper we state our version of Lusternik–Schnirelmann's main result. Namely, if  $f: M \rightarrow \mathbf{R}$  is a basic function, we show that, under certain conditions of Palais–Smale type, the transverse category gives a lower bound for the number of critical leaves of  $f$ . Then we prove that these conditions are verified by compact-Hausdorff foliations on a compact manifold, by considering the gradient of the basic function as a foliated vector field. Finally, we prove that the bound below remains true for any foliation  $\mathcal{F}$  of codimension one, by showing that when the number of critical leaves is finite then  $\mathcal{F}$  is a compact Hausdorff foliation.

## 1. Transverse category

Our setting will be (Hausdorff paracompact)  $C^k$ -manifolds endowed with  $C^k$ -foliations. Let  $(M, \mathcal{F}), (M', \mathcal{F}')$  be two foliated manifolds. A  $C^k$ -homotopy  $H: M \times \mathbf{R} \rightarrow M'$  is said to be *foliated* if for all  $t \in \mathbf{R}$  the map  $H_t$  sends each leaf  $L$  of  $\mathcal{F}$  into another leaf  $L'$  of  $\mathcal{F}'$  (notation:  $\simeq_{\mathcal{F}}$ ).

We describe an open subset  $U \subset M$  as *transversely categorical* if there is a foliated homotopy  $H: U \times \mathbf{R} \rightarrow M$  such that  $H_0: U \rightarrow M$  is an inclusion and the image of  $H_1: U \rightarrow M$  is contained in a single leaf of  $\mathcal{F}$ . Here  $U$  is regarded as a foliated manifold with the foliation induced by  $\mathcal{F}$ . In other words, the open subset  $U$  of  $M$  is transversely categorical if the inclusion  $(U, \mathcal{F}_U) \hookrightarrow (M, \mathcal{F})$  factors through a leaf up to foliated homotopy.

**Definition 1.** The transverse LS category (or, briefly, transverse category) of a foliated manifold  $(M, \mathcal{F})$  is the least number  $\text{cat}_\circ(M, \mathcal{F})$  of transversely categorical open sets required to cover  $M$ . If no such covering exists, let  $\text{cat}_\circ(M, \mathcal{F}) = \infty$ .

If  $M$  is a compact manifold, then  $\text{cat}_\circ(M, \mathcal{F})$  is finite since foliated open sets — that is, the domains of small adapted charts — are transversely categorical. When  $\mathcal{F}$  is the foliation by points, an open subset is transversely categorical iff it is categorical, and we have  $\text{cat}_\circ(M, \mathcal{F}) = \text{cat } M$ .

**Proposition 2.** *Transverse category is an invariant of foliated homotopy type.*

**Proof.** Let  $f: (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$  be a foliated homotopy equivalence with foliated homotopic inverse  $g$ . If  $U \subset M$  is a  $\mathfrak{h}$ -categorical open subset then there is a foliated homotopy between  $f|_U$  and a map which has image in a single leaf. Thus  $g^{-1}(U) \subset M'$  is open and  $\mathfrak{h}$ -categorical because  $f$  and  $g$  are foliated maps and  $fg \simeq_{\mathcal{F}} id$ .  $\square$

Varadarajan’s theorem [19] about category and fibrations can be generalized to any foliation as follows:

**Theorem 3.** Let  $L$  be a leaf of maximal category in the foliated manifold  $(M, \mathcal{F})$ . Then  $cat M \leq cat L \ cat_{\mathfrak{h}}(M, \mathcal{F})$ .

**Proof.** Let  $\{U_1, \dots, U_s\}$  be a  $\mathfrak{h}$ -categorical open covering of  $(M, \mathcal{F})$ . For each  $U = U_i$  let  $H: U \times \mathbf{R} \rightarrow M$  be a foliated homotopy such that  $H_0 = i_U$  and  $H_1(U) \subset L'$  where  $L'$  is a leaf. We consider on  $L'$  the leaf topology. Since  $L'$  is a weakly imbedded submanifold of  $M$ , the map  $H_1: U \rightarrow L'$  is continuous. Let  $\{V_1, \dots, V_k\}$ ,  $k = k(L')$  be a categorical open covering of  $L'$ . For each  $V = V_j$  let  $G: V \times \mathbf{R} \rightarrow L'$  be a contraction to  $*$ .

Consider  $W = H_1^{-1}(V) \subset U$  which is open in  $M$ . To prove that  $W$  is categorical (if not void) we consider  $F: W \times \mathbf{R} \rightarrow M$  given by

$$F(x, t) = \begin{cases} H(x, 2t) & \text{if } t \leq \frac{1}{2}, \\ G(H_1(x), 2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Clearly  $F$  is well defined and it is continuous from the continuity of  $H_1|_W: W \rightarrow V$ . Also  $F_0 = H_0|_W = i_W$  and  $F_1 = G_1 H_1|_W = *$ . Remark that for a Hausdorff paracompact manifold the classical definition of LS category does not change if one only considers smooth maps instead of continuous maps because any  $C^0$  map is homotopic to a  $C^\infty$  map.

Thus each  $U_i$  can be covered by  $k = k(i)$  categorical open sets. We have

$$cat M \leq \sum_{i=1}^s cat L_i,$$

where  $L_i$  is the leaf into which  $U_i$  is deformed. Hence  $cat M \leq s \ cat L$  where  $L$  is a leaf of maximal category.  $\square$

## 2. Category and basic cohomology

The cohomology ring structure of a space classically serves to estimate its category. To obtain lower bounds for transverse category we consider basic cohomology [16].

Recall that a smooth form  $\omega$  on  $M$  is said to be basic if it satisfies

$$i_Y \omega = 0 \quad \text{and} \quad i_Y d\omega = 0$$

for all vector fields  $Y$  tangent to  $\mathcal{F}$ . We denote by  $\Omega_b^r(M)$  the space of basic  $r$ -forms,  $0 \leq r \leq n = \text{codim } \mathcal{F}$ , and  $H_b(M) = \bigoplus H_b^r(M)$  the *basic cohomology algebra*, endowed with the exterior product.

Let  $U \subset M$  an open set. In order to define a *relative basic cohomology* we consider the complex [3]

$$\Omega_b^r(M, U) = \Omega_b^r(M) \oplus \Omega_b^{r-1}(U)$$

with differential  $d(\omega, \theta) = (d\omega, \omega|_U - d\theta)$ . In this way we have a long exact sequence in basic cohomology associated to the pair  $(M, U)$ .

Recall that a ring  $A$  is said to be nilpotent if  $A^k = 0$  for some positive integer  $k$ . The least such integer  $k$  is called the index or degree of nilpotence of  $A$ , and written  $\text{nil } A$ .

**Proposition 4.**  $\text{cat}_\flat(M, \mathcal{F}) \geq \text{nil } k^* H_b^+(M)$ , where  $k^*$  is the morphism induced by the inclusion  $k: \Omega_b(M) \subset \Omega(M)$ .

**Proof.** Consider the  $\flat$ -categorical open subset  $U$  of  $M$ . We have the following commutative diagram, where  $r > 0$ :

$$\begin{array}{ccc} H_b^r(M, U) & \xrightarrow{b} & H^r(M, U) \\ \downarrow a & & \downarrow c \\ H_b^r(M) & \xrightarrow{k^*} & H^r(M) \\ \downarrow i^* & & \\ H_b^r(U) & & \end{array}$$

The homomorphism  $i^*$  is trivial since it factors through  $H_b(L) = H_{DR}(* ) = 0$  for some leaf  $L$ . Hence  $a$  is onto by exactness. Let  $\{U_1, \dots, U_s\}$  be a  $\flat$ -categorical open covering of  $M$  and  $x_1, \dots, x_s$  elements of  $k^* H_b^+(M)$ . For each  $x_i$  there exists  $y_i \in H_b^r(M)$  such that  $k^*(y_i) = x_i$  and  $z_i \in H_b^r(M, U_i)$  such that  $a(z_i) = y_i$ . Thus  $x_i = k^* a(z_i) = c b(z_i)$  and the product

$$x_1 \cdots x_s = c(b(z_1) \bullet \cdots \bullet b(z_s))$$

vanishes since  $b(z_1) \bullet \cdots \bullet b(z_s) \in H(M, M) = 0$ .

Here we are considering the product of relative differential forms [1,3]

$$\bullet: \Omega^p(M, U) \times \Omega^q(M, V) \rightarrow \Omega^{p+q}(M, U \cup V)$$

which induces the usual cup product in relative singular cohomology [18].  $\square$

### 3. Saturated transverse category

Sometimes it is interesting to consider open sets which are saturated, that is a union of leaves.

**Definition 5.** The saturated transverse category  $\text{cat}^s M$  of a foliated manifold  $(M, \mathcal{F})$  is defined to be the least number of transversely categorical *saturated* open sets required to cover  $M$ . If no such covering exists, let  $\text{cat}^s M = \infty$ .

Obviously, the transverse category cannot exceed the saturated transverse category:  $\text{cat}_h \leq \text{cat}_h^s$ . On the other hand, a linear foliation  $\mathcal{F}_\alpha$  on the torus  $T^2$  with  $\alpha \notin \mathbf{Q}$  shows that transverse category can be finite even when saturated transverse category is infinite.

Let  $\pi: M \rightarrow M/\mathcal{F}$  be the projection onto the space of leaves. If  $V$  is a transversely categorical open saturated subset of  $M$ , then its image  $\pi(V)$  is a categorical open subset of  $M/\mathcal{F}$ . It follows at once that  $\text{cat}_h^s M$  is bounded below by the category of the leaf space. The equality holds for locally trivial bundles.

**Example 6.** Locally trivial bundles.

If  $\pi: M \rightarrow M/\mathcal{F}$  is a locally trivial  $C^\infty$ -bundle then  $\text{cat}_h^s M = \text{cat} M/\mathcal{F}$ .

We begin with an open covering  $\{V_1, \dots, V_k\}$  of  $M/\mathcal{F}$  by categorical subsets and then show that  $\{U_1, \dots, U_k\}$ , where  $U_i = \pi^{-1}(V_i)$ , is a transversely categorical open saturated covering of  $M$ . In fact, let  $\bar{H}_t: V_i \rightarrow M/\mathcal{F}$  be a homotopy joining the inclusion with a contraction  $\bar{H}_1$  of  $V_i$  onto a point  $\bar{x}_0 \in M/\mathcal{F}$ . Consider  $H = \bar{H} \circ (\pi|_{U_i} \times id)$ . Note that the inclusion of  $U_i$  into  $M$  gives a foliated lifting of  $H_0$ . By the homotopy lifting property, there exists a lifting  $\tilde{H}_t: U_i \rightarrow M$  of  $H_t$ . Since  $\tilde{H}_1(U) \subset L_0 = \pi^{-1}(\bar{x}_0)$  we have that  $\tilde{H}$  is a foliated homotopy.

**Example 7.** Equivariant category of actions.

Let  $M$  be a manifold foliated by the orbits of a locally free action of a compact Lie group  $G$ . Then  $\text{cat}_h^s M \leq \text{cat}_G^s M$ .

The equivariant category of a manifold where a compact Lie group  $G$  acts has been introduced by Fadell [6] and studied by many authors [12]. We consider here a smooth version  $\text{cat}_G^s M$  of it, assuming that the action and the involved homotopies are smooth.

**Example 8.** Suspensions.

Let  $G = \pi_1(M, x_0)$  be a finite group, and let  $(E_h, \mathcal{F})$  be the suspension of a homomorphism  $h: \pi_1(M, x_0) \rightarrow \text{Diff}(F)$ . Then

$$\text{cat}(E_h/\mathcal{F}) \leq \text{cat}_h^s(E_h) \leq \text{cat}_G^s F.$$

Here  $M$  and  $F$  are connected manifolds. Call  $\tilde{M}$  the universal cover of  $M$  and let  $\varphi$  be the action of  $G$  on  $\tilde{M} \times F$  given by  $\varphi(g, (x, y)) = (gx, h(g)(y))$ . We denote the set of orbits of this action by  $E_h = \tilde{M} \times_h F$ . Then the map  $p: E_h \rightarrow M$  induced by the first projection  $\tilde{M} \times F \rightarrow \tilde{M}$  is a locally trivial smooth bundle, while the map  $\pi: E_h \rightarrow F/G$  induced by the second projection  $\tilde{M} \times F \rightarrow F$  determines a foliation  $\mathcal{F}$  on  $E_h$  which is called the *suspension* of the homomorphism  $h$ .

Let  $U$  be a  $G$ -categorical open subset of  $F$  and let  $H_t: U \rightarrow F$  be a  $G$ -homotopy joining the inclusion with a map  $H_1$  sending  $U$  onto an orbit  $Gx \subset F$ . Define a homotopy  $\tilde{H}_t: \tilde{M} \times U \rightarrow \tilde{M} \times F$  by the formula  $\tilde{H}_t(x, y) = (x, H_t(y))$ . Then the image of  $\tilde{M} \times U$  by the canonical projection onto the orbit space  $E_h$  is transversely categorical and saturated by an obvious argument.

#### 4. Compact-Hausdorff foliations

A foliation  $\mathcal{F}$  on a manifold  $M$  is called a compact-Hausdorff foliation if every leaf is compact and the space of leaves is Hausdorff.

The structure of compact-Hausdorff foliations was studied by Epstein [5], Edwards et al. [4] among others: for any leaf  $L$  there exists a finite subgroup  $G$  of the orthogonal group  $O(n)$ ,  $n = \text{codim } \mathcal{F}$ ; a homomorphism  $h: \pi_1(L) \rightarrow G$ ; and a leaf preserving diffeomorphism of a neighborhood of  $L$  onto  $\tilde{L} \times_G D$ , where  $D$  is an open ball in  $\mathbf{R}^n$  and  $\tilde{L}$  is the covering space associated to the kernel of  $h$ .

This local model implies that  $M/\mathcal{F}$  is a Satake manifold [17]. The open sets  $U \subset M/\mathcal{F}$  diffeomorphic to  $D/G$  such that  $\pi^{-1}(U) \approx \tilde{L} \times_G D$  will be called  $\mathcal{F}$ -trivial open sets.

In order to obtain an upper bound for saturated transverse category we need to have some form of foliated homotopy lifting property.

**Lemma 9.** *Let  $H_1: D \rightarrow D$  be an equivariant homotopy and let  $f: \tilde{L} \times D \rightarrow \tilde{L} \times D$  be an equivariant map such that  $pr_D \circ f = H_0 \circ pr_D$ . Denote  $\bar{f}, \bar{H}$  the induced maps in the quotient spaces. Then there exists a foliated homotopy*

$$\tilde{H}: (\tilde{L} \times_G D) \times \mathbf{R} \rightarrow \tilde{L} \times_G D$$

such that  $\pi \circ \tilde{H} = \bar{H} \circ (\pi \times id_{\mathbf{R}})$  and  $\tilde{H}_0 = \bar{f}$  (we are identifying  $\pi$  with  $\tilde{L} \times_G D \rightarrow D/G$ ).

**Proof.** Let  $[l, d]$  denote the image of  $(l, d) \in \tilde{L} \times D$  in the quotient space  $\tilde{L} \times_G D$ . Define  $\tilde{H}$  by the formula

$$\tilde{H}([l, d], t) = [pr_{\tilde{L}} \circ f(l, d), H(d, t)]$$

which is well defined because  $H$  and  $f$  are equivariant maps. Then

$$\begin{aligned} \pi \tilde{H}([l, d], t) &= p H(d, t) = \bar{H}(p \times id)(d, t) = \bar{H}(p(d), t) \\ &= \bar{H}(\pi[l, d], t) = \bar{H}(\pi \times id)([l, d], t). \end{aligned}$$

Also observe that  $\tilde{H}$  is a foliated homotopy. Finally,

$$\begin{aligned} \tilde{H}_0([l, d]) &= [pr_{\tilde{L}} \circ f(l, d), H(d, 0)] = [pr_{\tilde{L}} \circ f(l, d), H(pr_D(l, d), 0)] \\ &= [pr_{\tilde{L}} \circ f(l, d), pr_D f(l, d)] = [f(l, d)] \\ &= \bar{f}[l, d]. \quad \square \end{aligned}$$

**Proposition 10.** *If  $U \subset M/\mathcal{F}$  is an  $\mathcal{F}$ -trivial open set, then  $\pi^{-1}(U)$  is a transversely categorical open saturated set.*

**Proof.** Let  $U$  be diffeomorphic to  $D/G$  and  $\pi^{-1}(U) \approx \tilde{L} \times_G D$ . Let  $H: D \times \mathbf{R} \rightarrow D$  be an equivariant contraction to the origin. Then the existence of  $\tilde{H}: (\tilde{L} \times_G D) \times \mathbf{R} \rightarrow \tilde{L} \times_G D$  follows from Lemma 9 because  $pr_D \circ f = H_0 \circ pr_D$ . If  $L_0 = \pi^{-1}(p(0))$  then  $\tilde{H}[l, d] \in L_0$  for all  $[l, d] \in \tilde{L} \times_G D$  because  $\pi \tilde{H}_1[l, d] = p(0)$ .  $\square$

The above statement shows that any leaf of a compact-Hausdorff foliation has a transversely categorical open saturated neighborhood.

**Corollary 11.** *If  $M/\mathcal{F}$  can be covered by  $k$   $\mathcal{F}$ -trivial open sets, then*

$$\text{cat } M/\mathcal{F} \leq \text{cat}_h^s M \leq k.$$

1. For example, let us consider the compact-Hausdorff foliation on  $S^3$  with projection  $\pi: S^3 \rightarrow S^2$  given by the following  $\mathcal{F}$ -trivialization. If  $\{U, V\}$  is the covering of  $S^2$  with  $U = S^2 - \{p_S\}$  and  $V = S^2 - \{p_N\}$ , and  $(a, b)$  is a coprime pair of integers, we take

$$(r \exp it, \exp is) \mapsto r \exp i(tp + sq),$$

where  $(p, q) = (-a, b)$  in  $U$  and  $(p, q) = (b, -a)$  in  $V$ . Then the category of the leaf space is 2 and  $k = 2$  too, so  $\text{cat}_h^s S^3 = 2$  for this foliation.

2. However, the following example shows that in general  $\text{cat } M/\mathcal{F} \neq \text{cat}_h^s M$ . Let  $K$  be the Klein bottle and  $\mathcal{F}$  the compact-Hausdorff foliation by circles on  $K$  with projection  $\pi: K \rightarrow I$ , where  $I \subset \mathbf{R}$  is a compact interval. Then the leaf space is contractible whereas  $\text{cat}_h^s K = k = 2$ .

### 5. Critical leaves

In this section we show how transverse category is useful in the study of the number of critical leaves of basic functions, by generalizing the classical result of Lusternik–Schnirelmann.

#### 5.1. The transverse Lusternik–Schnirelmann method

Let  $f: M \rightarrow \mathbf{R}$  denote a basic function, that is a smooth function which is constant along the leaves. Let  $K$  be the critical set of  $f$ . It is saturated. Any leaf  $L \subset K$  is called critical. For  $c \in \mathbf{R}$  we write  $M_c = f^{-1}(-\infty, c]$  and  $K_c = K \cap f^{-1}(c)$ .

Our aim is to determine the number of critical leaves. Certain assumptions on  $M$  and  $f$  are required.

**Definition 12.** If  $A$  and  $B$  are saturated subsets of  $M$ , we say that  $A$  is  $\mathfrak{h}$ -deformable into  $B$  if there exists a foliated homotopy  $H: M \times \mathbf{R} \rightarrow M$  such that  $H_0|_A$  is the inclusion and  $H_1(A) \subset B$ .

Consider the following *transverse conditions*:

- C1. Every leaf of  $\mathcal{F}$  has a saturated neighborhood which is transversely categorical.
- C2. For any regular value  $c$  of  $f$  there is an  $\varepsilon > 0$  such that  $M_{c+\varepsilon}$  is  $\mathfrak{h}$ -deformable into  $M_{c-\varepsilon}$ .
- C3. For any critical value  $c$  of  $f$  and any neighborhood  $U$  of  $K_c$  there is an  $\varepsilon > 0$  such that  $M_{c+\varepsilon} - U$  is  $\mathfrak{h}$ -deformable into  $M_{c-\varepsilon}$ .

**Theorem 13.** *Let  $f: M \rightarrow \mathbf{R}$  be a basic function on a compact manifold  $M$  endowed with a foliation  $\mathcal{F}$  satisfying the transverse conditions above. Then  $f$  has at least  $\text{cat}_h^s M$  critical leaves.*

**Proof.** All arguments in the proof of the classical version [11,13] can be adapted to the foliated case by defining

$$c_m(f) = \inf\{c \in \mathbf{R} \mid \text{cat}_h^i M_c \geq m\}$$

for every positive integer  $m \leq \text{cat}_h^i M$ .  $\square$

In the next section we shall show that any compact-Hausdorff foliation  $\mathcal{F}$  on a compact manifold  $M$  verifies the transverse conditions above.

### 5.2. Compact-Hausdorff foliations

Let  $f: M \rightarrow \mathbf{R}$  be a basic function. It is known [3] that compact-Hausdorff foliations admit bundle-like metrics [15]. Let  $g$  be a bundle-like metric and  $\|\cdot\|$  the norm defined by  $g$ . Then the gradient  $\nabla f$  of  $f$  is a foliated vector field, that is for all  $Y \in \chi(\mathcal{F})$  the Lie bracket  $[\nabla f, Y]$  also belongs to  $\chi(\mathcal{F})$ . Hence  $\|\nabla f\|^2$  is a basic function too.

We denote  $M^* = M - K$  the set of regular points of  $f$ .

**Lemma 14.** *Let  $Y$  be the vector field in  $M^*$  defined by  $Y = \nabla f / \|\nabla f\|^2$ . Then:*

- (i) *If  $U, V \subset M^*$  are saturated open sets such that  $\bar{V} \subset U$ , there is a foliated vector field  $X$  on  $M$  such that  $X = Y$  on  $V$  and  $X = 0$  on  $M - U$ .*
- (ii) *Let  $\varphi$  be the flow defined by  $X$ . Then the function  $f\varphi_p(t)$  is monotone non-decreasing in  $t$  for all  $p \in M$ . Moreover, if  $\varphi_p[s, 0] \subset V$  then  $f\varphi_p(t) = f(p) + t$  for all  $t \in [s, 0]$ .*

**Proof.** (i) We assume as known the existence of basic partitions of the unity subordinated to saturated open coverings [3]. Since  $\bar{V} \subset U$ , there exists a basic function  $h \geq 0$  on  $M$  such that  $h = 1$  on  $V$  and  $h = 0$  on  $M - U$ . Since  $Y$  is a foliated vector field on  $M^*$ , we can define the foliated vector field  $X$  on  $M$  by  $X = hY$ .

(ii) We have

$$\begin{aligned} \frac{d}{dt}(f\varphi_p(t)) &= X_{\varphi_p(t)}f \\ &= \langle X_{\varphi_p(t)}, \nabla f_{\varphi_p(t)} \rangle \\ &= \langle h\varphi_p(t)Y_{\varphi_p(t)}, \nabla f_{\varphi_p(t)} \rangle \\ &= h\varphi_p(t) \left\langle \frac{\nabla f_{\varphi_p(t)}}{\|\nabla f_{\varphi_p(t)}\|^2}, \nabla f_{\varphi_p(t)} \right\rangle \\ &= h\varphi_p(t) \geq 0 \end{aligned}$$

and the first statement follows. Moreover, if  $\varphi_p(t) \in V$  for all  $t \in [s, 0]$  we have  $d/dt(f\varphi_p(t)) = h(\varphi_p(t)) = 1$ , which completes our proof.  $\square$

**Theorem 15.** *Let  $M$  be a compact manifold endowed with a compact-Hausdorff foliation. Then any basic function  $f: M \rightarrow \mathbf{R}$  verifies the transverse conditions C1, C2 and C3 of Section 5.1.*



**Proof.** C1. It follows at once from Proposition 10.

C2. Let  $c$  be a regular value. There exists  $\varepsilon < \frac{1}{2}$  such that all points of  $[c - 4\varepsilon, c + 4\varepsilon]$  are regular values too. Therefore,  $U = f^{-1}(c - 3\varepsilon, c + 3\varepsilon)$  and  $V = f^{-1}(c - 2\varepsilon, c + 2\varepsilon)$  consist only of regular points. Also  $U$  and  $V$  are saturated open subsets of  $M^*$  such that  $\bar{V} \subset U$ . Then, by Lemma 14.1, there is a foliated vector field  $X$  on  $M$  such that  $X = f/\|\nabla f\|^2$  on  $V$  and  $X = 0$  on  $M - U$ . Define  $H: M \times \mathbf{R} \rightarrow M$  by

$$H(p, t) = \varphi_{-t}(p),$$

where  $\varphi$  is the flow defined by  $X$ . Since  $X$  is a foliated vector field,  $H_t$  is a foliated map. Since by Lemma 14  $f\varphi_p$  is monotone non-decreasing in  $t$  we have

$$fH_1(p) = f\varphi_{-1}(p) \leq f\varphi_0(p) \leq f(p),$$

hence  $H_1(M_{c-\varepsilon}) \subset M_{c-\varepsilon}$ . And if  $p \in f^{-1}(c - \varepsilon, c + \varepsilon]$  suppose  $H_1(p)$  were not in  $M_{c-\varepsilon}$ , then  $f\varphi_{-1}(p) > c - \varepsilon$ . It follows that  $\varphi_p[-1, 0] \subset f^{-1}(c - \varepsilon, c + \varepsilon] \subset V$  and by the same Lemma,  $f\varphi_p(t) = f(p) + t \quad \forall t \in [-1, 0]$ . In particular  $f\varphi_p(-1) = f(p) - 1$ , hence

$$c - \varepsilon < f(p) - 1 \leq c + \varepsilon - 1.$$

whereas  $\varepsilon < \frac{1}{2}$ . This contradiction proves  $H_1(M_{c+\varepsilon}) \subset M_{c-\varepsilon}$ .

C3. For each positive integer  $k$ , define

$$V_k = \left\{ p \in M \mid \|\nabla f(p)\|^2 < \frac{1}{k} \right\},$$

so  $V_k$  is a saturated open neighborhood of  $K$  and  $\bar{V}_{k+1} \subset V_k$ . Then  $W = M - \bar{V}_{k+1}$  is an open set containing the closed set  $C = M - V_k$ , and since  $M$  is a normal space there exists an open set  $A$  such that  $C \subset A \subset \bar{A} \subset W$ . Hence  $A_k = \text{sat } A$  is a saturated open set such that  $C \subset A_k \subset \bar{A}_k \subset W$ .

By Lemma 14, there is a foliated vector field  $X_k$  on  $M$  such that  $X_k = Y$  on  $A_k$  and  $X_k = 0$  on  $\bar{V}_{k+1}$ . Let  $c$  be a critical value. Now define

$$U_k = \left\{ p \in M : |f(p) - c| < \frac{1}{k} \text{ and } \varphi_p(t) \in V_k \text{ for some } t \in \left[ -\frac{1}{k}, 0 \right] \right\}.$$

We will prove that each neighborhood  $U$  of  $K_c$  includes some  $U_k$ . Suppose that this assertion is false. Then for each  $k$  we can choose  $p_k \in U_k$  such that  $p_k \notin U$  and there exists a subsequence  $\{p_n\}$  of  $\{p_k\}$  convergent to some  $q \in M$ . Then  $f(q) = \lim f(p_n) = c$ . Since  $\varphi_{p_n}(t_n) \in V_n$  for some  $t_n \in [-1/n, 0]$ , we have  $\lim \|\nabla f(\varphi_{p_n}(t_n))\|^2 = 0$  and it follows that  $\nabla f(q) = 0$ . Hence  $q \in K_c \subset U$ , which is a contradiction because  $\{p_n\} \subset M - U$  implies  $q \in M - U$ .

Then we choose  $k > 0$  such that  $U_k \subset U$  and  $\varepsilon > 0$  such that  $\varepsilon < 1/(2k)$ . Define  $H: M \times \mathbf{R} \rightarrow M$  by

$$H(p, t) = \varphi_{-t}^k(p),$$

where  $\varphi^k$  is the flow defined by  $X_k$ . Since  $X_k$  is a foliated vector field,  $H_t$  is a foliated map. By Lemma 14 we have

$$fH_1(p) = f\varphi_{-1}^k(p) \leq f\varphi_{-1/k}^k(p)$$

hence  $H_1(p) \in M_{c-\varepsilon}$  when  $p \in M_{c-\varepsilon}$ . If  $p \in M_{c+\varepsilon} - U$  and  $f(p) > c - \varepsilon$  then  $|f(p) - c| < \varepsilon$  so, by definition of  $U_k$ ,  $\varphi_p(t) \notin V_k$  for all  $t \in [-1/k, 0]$ . Then by the same Lemma  $f\varphi_p^k(-1/k) = f(p) - 1/k$  and  $fH_1(p) \leq c - \varepsilon$ .  $\square$

**Theorem 16.** *Let  $f: M \rightarrow \mathbf{R}$  be a basic function on a compact manifold  $M$  endowed with a compact-Hausdorff foliation. Then  $f$  has at least  $\text{cat}_h^s M$  critical leaves.*

When we consider the foliation by points we obtain the classical Lusternik–Schnirelmann theorem stating that the category of the manifold is a lower bound for the number of critical points of any smooth function.

### 5.3. Codimension one foliations

The goal of this section is to show that every basic function on a compact manifold  $M$  endowed with a foliation  $\mathcal{F}$  of codimension one has at least  $\text{cat}_h^s M$  critical leaves.

Throughout this section  $M$  will be a compact manifold and  $\mathcal{F}$  a codimension one foliation on  $M$ . We know that when all leaves are compact,  $\mathcal{F}$  is a compact-Hausdorff foliation [5,7,14] so the result in this case follows from the preceding section.

**Proposition 17.** *If  $\mathcal{F}$  has a nonclosed leaf, then any basic function has an infinite number of critical leaves.*

In order to prove this proposition, we first review some results about minimal sets of foliations [2,8]. A subset  $\mu$  of  $M$  is called *minimal* if it is a minimal nonempty saturated closed subset of  $M$ . For example, every closed leaf is a minimal set. For any leaf  $L$  of  $\mathcal{F}$  the closure  $\bar{L}$  of  $L$  in the compact manifold  $M$  contains a minimal set. In particular, every foliation of a compact manifold has a minimal set.

Any minimal set  $\mu$  of  $M$  is of one and only one of the following types:

- (i)  $\mu$  is a compact leaf.
- (ii)  $\mu = M$ .
- (iii)  $\mu$  is a union of exceptional leaves. In this case we say that  $\mu$  is an exceptional minimal set.

Now, we will prove the following lemma. Let  $f: M \rightarrow \mathbf{R}$  be a basic function and  $K$  the set of critical points as before. For every minimal set  $\mu$ , define

$$A_\mu = \{L \in \mathcal{F} \mid \bar{L} \supset \mu \text{ and } L \text{ is not closed}\}.$$

Note that since we are supposing the existence of some nonclosed leaf then there exists a minimal set  $\mu$  such that  $A_\mu \neq \emptyset$ .

**Lemma 18.** *Let  $N = \bigcup \{\bar{L} \mid L \in A_\mu\}$  where  $\mu$  is a minimal set such that  $A_\mu \neq \emptyset$ . Then there exists a saturated open set  $W$  of  $M$  such that  $W \subset N$ . Moreover,  $N \subset K$ .*

**Proof.** The minimal set  $\mu$  is of one of the three types above. The first assertion follows immediately when  $\mu = M$  and it is proved in [8, part B, p. 94] in case 3. Thus we only prove the assertion for a minimal set  $\mu$  which is a compact leaf  $L_0$ .

Let  $L$  be a leaf in  $N$  and  $T$  a transversal passing through the point  $x \in L$ . Since the holonomy group of  $L_0$  cannot be finite, there exists a diffeomorphism  $g \in \text{Hol}(L_0)$  such that  $\lim g^n(x) = z$  where  $z = T \cap L_0$ . If  $g^i(x) < x < g^j(x)$ ,  $i, j \in \mathbf{Z}$ , we consider the segment  $J = (g^i(x), g^j(x)) \subset T$ . We have

$$g^n g^i(x) < g^n(y) < g^n g^j(x) \quad \text{for all } y \in J, n \in \mathbf{Z}$$

hence

$$\lim g^n(y) = z \quad \forall y \in J.$$

Moreover,  $J \subset N$  because if  $y \in L'$  then  $g^n(y) \in L'$  for all  $n \in \mathbf{Z}$  and  $\bar{L}' \supset L_0$ , hence  $y \in N$ . We set  $W = \text{sat } J$ . Clearly,  $W$  is a saturated subset of  $M$  and since  $N$  is saturated,  $W \subset N$ .

It only remains to prove that  $N \subset K$ . Since  $f$  is a basic function,  $f$  is constant on the saturated open subset  $W$ . Thus  $W \subset K$ .

If  $\mu$  is a compact leaf, then for every leaf  $L \subset N$  there is a saturated open neighborhood  $W \subset N$ . Hence  $L \subset W \subset K$  for all  $L \subset N$ .

If  $\mu = M$  then  $W = M$  and  $N \subset K$ .

Finally, if  $\mu$  is an exceptional minimal set,  $\mu \subset K$  since  $W$  is a neighborhood of  $\mu$ . Moreover, if  $\mu \subset \bar{L}$  for all leaf  $L \subset N$  then  $\bar{L} \cup K \neq \emptyset$  which implies that  $\bar{L} \subset K$  then  $L \subset K$  for all  $L \subset N$ .  $\square$

The statement of Proposition 17 follows from Lemma 18. Thus either all leaves are compact in which case Theorem 13 implies that  $f$  has at least  $\text{cat}^s M$  critical leaves, or else  $f$  has infinitely many critical leaves. In any case we have

**Theorem 19.** *Let  $F$  be a codimension one foliation on  $M$ . Then any basic function has at least  $\text{cat}^s M$  critical leaves.*

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