

The constant osculating rank of the Wilking manifold V_3 ¹

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Abstract

We prove that the osculating rank of the Wilking manifold $V_3 = (SO(3) \times SU(3))/U^\bullet(2)$, endowed with the metric \tilde{g}_1 , equals 2. The knowledge of the osculating rank allows us to solve the differential equation of the Jacobi vector fields. These results can be applied to determine the area and the volume of geodesic spheres and balls. *To cite this article: E. Macías-Virgós, A. M. Naveira, A. Tarrío, C. R. Acad. Sci. Paris, Ser. I xxx (yyyy).*

Résumé

Le rang osculateur constant de la variété de Wilking V_3 . Nous prouvons que le rang osculateur de la variété de Wilking $V_3 = (SO(3) \times SU(3))/U^\bullet(2)$ vaut 2, lorsque on considère la métrique \tilde{g}_1 . La connaissance du rang osculateur nous permet de résoudre l'équation différentielle des champs de vecteurs de Jacobi. Ces résultats peuvent être appliqués pour déterminer l'aire et le volume des sphères et boules géodésiques. *Pour citer cet article : E. Macías-Virgós, A. M. Naveira, A. Tarrío, C. R. Acad. Sci. Paris, Ser. I xxx (yyyy).*

Introduction. Solving the Jacobi equation on a Riemannian manifold can be a rather difficult task. For the manifolds $V_1 = Sp(2)/SU(2)$ and $V_2 = SU(5)/Sp(2) \times S^1$ a partial solution was obtained by I. Chavel in [5,6]. It is well known that these manifolds are non-symmetric normal homogeneous spaces of rank 1 [3, p. 237]. A theorem from Berger [3] establishes that a simply connected, normal homogeneous space of positive sectional curvature is diffeomorphic either to a compact rank-one symmetric space or to one of the manifolds V_1 or V_2 . In [15] Wilking proves that this theorem is not correct because there is a third exception, the manifold $V_3 = (SO(3) \times SU(3))/U^\bullet(2)$, equipped with a one-parameter family \tilde{g}_λ of bi-invariant metrics.

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In [14] Tsukada concludes that the Jacobi tensor field over an n.r.h.s. (naturally reductive homogeneous space, see the definition in [6], [10, p. 202]) has constant osculating rank $r \in \mathbb{N}$. Therefore, for the Jacobi operator one has

$$R_t = R_0 + a_1(t)R_0^1 + \cdots + a_r(t)R_0^r.$$

T. Arias-Marco and the second author have extended this interesting geometric result to g.o. spaces [1]. In [11], two of the authors, always working with the Levi-Civita connection and using the result of Tsukada, have proved that the manifold V_1 has constant osculating rank 2. Also, they solved the Jacobi equation along a unit geodesic of arbitrary direction for this manifold. The solutions are applied to obtain the area of the geodesic spheres and the volume of the geodesic spheres and balls.

Given its generality, we have thought that this method could also be applied to solve the Jacobi equation in several other examples of n.r.h.s. In this note we are going to do the same study for the Wilking's manifold (V_3, \tilde{g}_λ) . We prove that it has osculating rank 2 when $\lambda = 1$. This result allows us to solve the Jacobi equation in such case.

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Preliminaries. Let M be an n -dimensional, connected, real analytical Riemannian manifold, $m \in M$. Let $\gamma : J \rightarrow M$ be a geodesic defined on some open interval $J \subset \mathbb{R}$ with $0 \in J$, $m = \gamma(0)$. The associated Jacobi operator R_t is the self-adjoint tensor field along $\gamma = \gamma(t)$ defined by $R_t(\cdot) := R(\cdot, \gamma')\gamma'(t)$. For the curvature tensor we follow the notations of [10]. The covariant derivative R_t^i of the Jacobi operator R_t along γ is the self-adjoint tensor field defined by

$$R_t^i(\cdot) := (\nabla_{\gamma'} \cdots \nabla_{\gamma'} R)(\cdot, \gamma')\gamma'(t),$$

where ∇ is the Levi-Civita connection associated to the metric. Its value at $\gamma(0)$ will be denoted by $R_0^i(\cdot)$.

Theorem 1 ([11]) For $n \geq 1$ we have

$$\nabla_{\gamma'} \cdots \nabla_{\gamma'} R(X, \gamma')\gamma' = \sum_{i=0}^n \binom{n}{i} R_t^{n-i} (\nabla_{\gamma'} \cdots \nabla_{\gamma'} X).$$

Let G be a Lie group, H a closed subgroup. The Lie algebras of G and H will be denoted by \mathfrak{g} and \mathfrak{h} , respectively. We identify the vector space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ with the tangent space to G/H at the origin o .

From now on we shall assume that G/H is an n.r.h.s. If we define $\Lambda : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ by $\Lambda(u)v = (1/2)[u, v]_{\mathfrak{m}}$ for $u, v \in \mathfrak{m}$, we can identify Λ with the Levi Civita connection ∇ [10, Ch. X, p. 201]. Accordingly to [13] and [10, Vol. II, Ch. X], for each $v \in \mathfrak{m}$ the (1,3)-tensor R_t on \mathfrak{m} obtained by the parallel translation of the Jacobi operator along γ is given as

$$R_t = e^{t\Lambda(v)} R_0,$$

where R_0 denotes the Jacobi operator at the origin o .

Proposition 2 ([6],[10, Vol. II, p. 202]) Let $\gamma(t)$ be a geodesic with $\gamma(0) = o$, $\gamma'(0) = v \in \mathfrak{m}$. If X is a differentiable vector field along γ , then

$$R_0(X) = -[[X, v]_{\mathfrak{h}}, v] - (1/4)[[X, v]_{\mathfrak{m}}, v]_{\mathfrak{m}}.$$

Proposition 3 ([11]) For $n > 0$,

$$(-1)^{n-1} 2^n R_0^n(X) = \sum_{i=0}^n (-1)^i \binom{n}{i} [[X, v]_{\mathfrak{m}}, \dots, v]_{\mathfrak{h}}^{i+1}, \dots, v]_{\mathfrak{m}},$$

where for each term of the sum we have $n + 2$ brackets and the exponent $i + 1$ means the position of the bracket valued on \mathfrak{h} .

An explicit form for the Jacobi operator. In [15] the author proves that Berger's classification is not totally correct. In fact, he shows that there is a third example $V_3 = (SO(3) \times SU(3))/U^\bullet(2)$ equipped with a one-parameter family \tilde{g}_λ of bi-invariant metrics, where $U^\bullet(2)$ is the image under the embedding $(\pi, i): U(2) \rightarrow SO(3) \times SU(3)$ given by the natural inclusion $i(A) = \text{diag}(A, \det A^{-1})$ and the projection $\pi: U(2) \rightarrow U(2)/S^1$. Using the natural isomorphism between $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$, we can consider the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$ of $SO(3) \times SU(3)$ as the subalgebra of $\mathfrak{su}(5)$ of matrices of the form $X = \text{diag}(X_1, X_2)$ with $X_1 \in \mathfrak{su}(2)$, $X_2 \in \mathfrak{su}(3)$. Then an element of the Lie algebra $\mathfrak{u}^\bullet(2)$ of $U^\bullet(2)$ may be expressed as a diagonal block matrix $\text{diag}(\pi_* A, A, -\text{trace } A)$ with $A \in \mathfrak{u}(2)$, where π_* denotes the differential of π . We shall consider the bi-invariant metrics $\langle \cdot, \cdot \rangle_c$, $c > 0$, on $SO(3) \times SU(3)$ given by

$$\langle X, Y \rangle_c = (-1/2)(c \text{trace } X_1 Y_1 + \text{trace } X_2 Y_2), \quad X, Y \in \mathfrak{so}(3) \oplus \mathfrak{su}(3).$$

They induce a one parametric family of metrics on V_3 . The corresponding Wilking's metrics are $\tilde{g}_\lambda = 12 \langle \cdot, \cdot \rangle_c$, $c = 3\lambda/2$. Then (V_3, \tilde{g}_λ) is isometric to the Aloff-Wallach space (M_{11}^7, g_t) [2], for $t = -3/(2\lambda + 3)$.

Next, we choose an orthonormal basis of $\mathfrak{so}(3) \oplus \mathfrak{su}(3)$ adapted to the reductive decomposition $\mathfrak{u}^\bullet(2) \oplus \mathfrak{m}$, see also [7]. For $1 \leq k, l \leq 5$, let E_{kl} denote the 5×5 matrices $(\delta_{ak} \delta_{bl})_{1 \leq a, b \leq 5}$. We introduce the matrices

$$Q_{kl} = E_{kl} - E_{lk}, \quad R_{kl} = \sqrt{-1}(E_{kl} + E_{lk}), \quad P_k = \sqrt{-1}(E_{kk} + E_{55}).$$

An orthonormal basis $\{M_1, \dots, M_7\}$ for \mathfrak{m} is given by

$$\begin{aligned} M_1 &= 1/\sqrt{c(1+c)}(P_1 - P_2 - c(P_3 - P_4)), & M_4 &= Q_{35}, & M_7 &= R_{45}, \\ M_2 &= 1/\sqrt{c(1+c)}(R_{12} - cR_{34}), & M_5 &= Q_{45}, \\ M_3 &= 1/\sqrt{c(1+c)}(Q_{12} - cQ_{34}) & M_6 &= R_{35}, \end{aligned}$$

while a basis for $\mathfrak{u}^\bullet(2)$ is given by

$$\begin{aligned} M_8 &= 1/\sqrt{1+c}(Q_{12} + Q_{34}), & M_{10} &= 1/\sqrt{1+c}(P_1 - P_2 + P_3 - P_4), \\ M_9 &= 1/\sqrt{1+c}(R_{12} + R_{34}), & M_{11} &= 1/\sqrt{3}(P_3 + P_4). \end{aligned}$$

Now it is necessary to compute the matrices of the Jacobi operator R_0 and its derivatives $R_0^1, R_0^2, R_0^3, \dots$ in order to know if there exists any dependence relation among them.

Lemma 4 *For the metric \tilde{g}_1 (i.e. $c = 2/3$), at $\gamma(0)$ we have:*

- (i) $5R_0^3 = -2\|\gamma'\|^2 R_0^1 = -2R_0^1$;
- (ii) $5R_0^4 = -2\|\gamma'\|^2 R_0^2 = -2R_0^2$.

In order to prove the Lemma it is necessary to compute all the brackets between the vectors of the basis $\{M_1, \dots, M_{11}\}$ and to use the formula given in Proposition 3. The results and an explicit computation with *Mathematica* can be seen at the electronic address <http://xtsunxet.usc.es/macias/wilkingV3>.

Remark 1 In fact we have computed the derivatives R_0^1 and R_0^3 for any value of λ (see the website above). These matrices are tensors on the coordinates x_1, \dots, x_7 in \mathfrak{m} . When computing R^5 we found that the product $x^1 \cdots x^7$ appears with a coefficient which is null if and only if $\lambda = 1$. Hence, in general it is not possible to write a dependence relation like

$$R^5 = a\|\gamma'\|^2 R^3 + b\|\gamma'\|^4 R^1,$$

where $\|\gamma'\|^2 = x_1^2 + \cdots + x_7^2 = 1$ is the module of the geodesic. This obstruction suggests that the osculating rank is greater than 5 for any of the metrics \tilde{g}_λ , with $\lambda \neq 1$.

Remark 2 The metric \tilde{g}_1 is the natural one induced by the Killing form on $\mathfrak{so}(3) \times \mathfrak{su}(3)$. It is an Einstein metric and gives the optimal pinching constant among the positively curved Aloff-Wallach metrics on M_{11}^7 [12].

Now, in a similar way that in [11], by using the induction method we have for the metric \tilde{g}_1 :

Proposition 5 *At $\gamma(0)$ we have:*

$$(i) R_0^{2n} = (-1)^{n-1} (2/5)^{n-1} R_0^2;$$

$$(ii) R_0^{2n+1} = (-1)^n (2/5)^n R_0^1.$$

Corollary 6 *Along the geodesic γ the Jacobi operator can be written as*

$$R_t = R_0 + \frac{5}{2} R_0^2 + \frac{\sin(\sqrt{2/5} t)}{\sqrt{2/5}} R_0^1 - \frac{\cos(\sqrt{2/5} t)}{2/5} R_0^2.$$

The proof follows from the Taylor's development of R_t at $t = 0$.

Remark 3 The solution of the Jacobi equation on the manifold V_3 allows us to determine the volume of geodesic spheres and balls in this manifold. We use for the computations the standard methods given in [4,8,9]. An analogous study was done for V_1 in [11].

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