Ganea and Whitehead definitions for the tangential LS category of a foliation

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(joint work with Jean Paul Doeraene et Daniel Tanré)

Seminario Vidal Abascal, November 16–17, 2007

Outline

- Tangential LS-category for foliations

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- The closed model category of stratified spaces

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- Relative Whitehead and Ganea constructions in a closed model category

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- Relative Whitehead and Ganea constructions in a closed model category
- Open LS-category of a stratified space
- Main result: equivalence of the definitions for a foliated manifold

Lusternik-Schnirelmann category of a topological space X

Definition

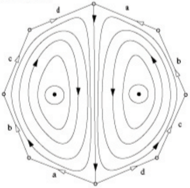
$\mathrm{cat}(X)$ is the least natural number n such that there exists a cover of X by $n+1$ open sets, each of them being contractible to a point inside X.

Categorical open sets.

1934

Critical points of functions on closed surfaces.

On a sphere with handles there exists a smooth function f with only three critical points: the minimum, the maximum, and a degenerate saddle. See the level sets of a function with only three critical points on a sphere with two handles



Problem. Show that the function constructed above cannot be realized as a height function.

Figure: For a smooth manifold $\mathrm{cat}(M) + 1$ it is a lower bound of the number of critical points for any smooth function.

Cornea, O.; Lupton, G.; Oprea, J.; Tanré, D.
Lusternik-Schnirelmann category.
Mathematical Surveys and Monographs. 103. Providence, RI: American Mathematical Society (2003)

Equivalent definitions

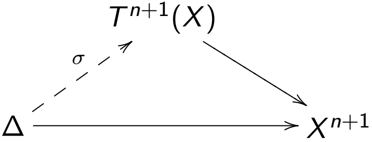
- Whitehead procedure, using the fat wedge (1956)
- Ganea construction (1967)

Whitehead

- The n-th fat wedge of a space X, noted $T^n(X)$, is the subset of X^n whose elements have at least one coordinate equal to the base point.

Whitehead

- The n-th fat wedge of a space X, noted $T^n(X)$, is the subset of X^n whose elements have at least one coordinate equal to the base point.
- $\mathrm{cat}(X) \leq n$ iff the diagonal map Δ factors through $T^{n+1}(X)$ up to homotopy.



Idea of proof $\mathrm{cat} = \mathrm{Whcat}$

- If X is a normal space, the contractions of the categorical open sets can be extended to the whole space, this gives the global map $\sigma\colon \Delta \to T^{n+1}(X)$.

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- If X is a normal space, the contractions of the categorical open sets can be extended to the whole space, this gives the global map $\sigma\colon \Delta \to T^{n+1}(X)$.
- If the base point is non-degenerate, i.e. has a contractible neighbourhood W, then the reciprocal images of W by σ give a categorical covering.

Ganea

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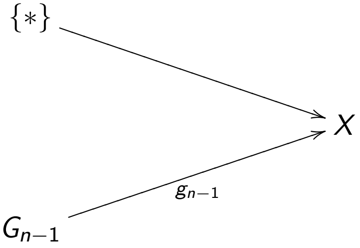

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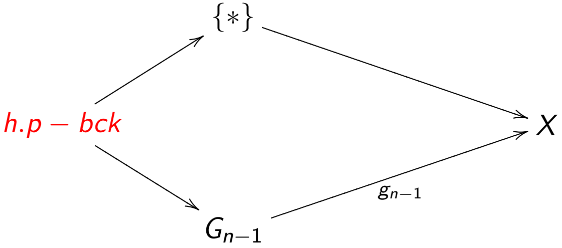
- 

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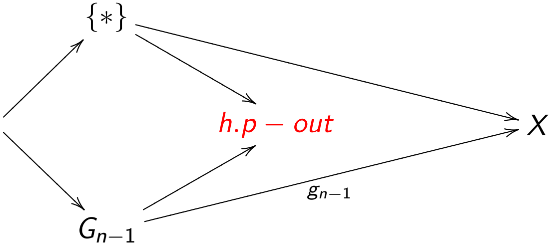
- 

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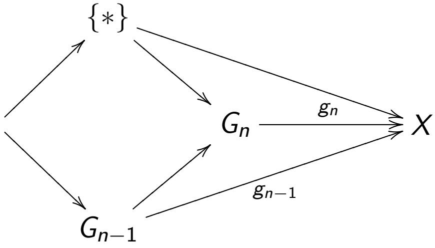
- 

- $\mathrm{cat}(X) \leq n$ iff $G^n(X)$ admits a homotopic section.

Gcat = Whcat

This is in fact an abstract process, valid in any closed model category satisfying an additional property called the cube lemma.

The cube lemma is the key stone in the equivalence between the two definitions of Lusternik Schnirelmann category.

Jean-Paul Doeraene. L.S.-category in a model category. *J. Pure Appl. Algebra*, 84(3):215–261, 1993.

Tangential category of a foliation, $\mathrm{cat}_{\mathcal{F}}(M)$

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- W. Singhof and E. Vogt. Tangential LS-category of $K(\pi, 1)$-foliations. preprint 2006.

Definition of tangential LS-category.

Definition

An open set $U \subset M$ (endowed with the induced foliation $\mathcal{F}_U$) is *tangentially categorical* if there exists a tangential homotopy $H \colon U \times I \to M$ such that $H(-, 0)$ is the inclusion and $H(-, 1)$ is constant along the leaves of U, that is $H(-, 1)$ maps each leaf of $\mathcal{F}_U$ onto some point. We call H a *tangential contraction*.

An integrable homotopy is exactly a homotopy for which all of the "traces" are leafwise curves.

Definition

The *tangential LS-category* of the foliated manifold $(M, \mathcal{F})$ is the least integer n such that there exists a covering of M by $(n+1)$ open sets which are tangentially categorical. We denote it by $\mathrm{cat}_{\mathcal{F}}(M)$

Known results

- The tangential category is less than or equal to the dimension of the foliation plus one.

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- Computations on examples.

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- The tangential category is less than or equal to the dimension of the foliation plus one.
- Methods for determining the tangential category, as the use of the length of the cup product in foliated cohomology and the other terms of the spectral sequence of the foliation.
- Computations on examples.
- A new definition using measure theory is given in recent Carlos Meniño's work.

Aim of this talk

To give Ganea and Whithehead definitions for the tangential category, equivalent to the original one.

Steven Hurder
FOLIATION GEOMETRY/TOPOLOGY PROBLEM SET
Kyoto 2003
PROBLEM 6.6. Give a homotopy-theoretic interpretation of $cat_{\mathcal{F}}(M)$ corresponding to the Whitehead and Ganea definitions of category. This is one of the most important open problems in the subject.

First step: Stratified spaces

We work in the topological framework, by considering topological spaces endowed with a partition.

Definition

A stratified space $(X, \mathcal{F})$ is a topological space together with a partition $\mathcal{F}$ whose elements $S \subseteq X$ are path-connected for the induced topology.

S -Top is the category of stratified spaces and stratified maps.

Second step: Closed model Categories

[Quillen 1976]
A category $\mathcal{M}$ with three types of morphisms (fibrations, cofibrations, weak equivalences) verifying:

NB. Acyclic (or trivial cofibration) $=$ cofibration $+$ weak equivalence

- $\mathcal{M}$ is closed for finite inductive and projective limits

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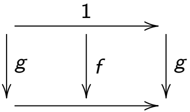
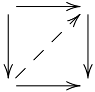
- If f, g are morphisms such that $g \circ f$ is defined, and any two of them are weak equivalences, then so is the third
- A retract of a fibration (resp. cofibration, resp. weak equivalence) is a fibration (resp. cofibration, resp. weak equivalence)



- Acyclic cofibrations (resp. cofibrations) have the left lifting property with respect to fibrations (resp. acyclic fibrations)



- Every morphism f in $\mathcal{M}$ factors in two ways:
    - $f = p \circ i$ with p fibration, i acyclic cofibration;

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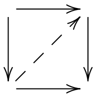
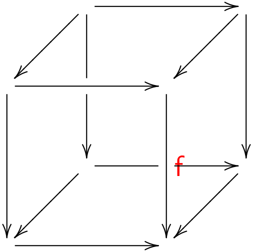


- Every morphism f in $\mathcal{M}$ factors in two ways:
  - $f = p \circ i$ with p fibration, i acyclic cofibration;
  - $f = q \circ j$ with q acyclic fibration, j cofibration

Cube Lemma

has: homotopic amalgamated sum
hfp: homotopic fibered product



If $\begin{cases} \text{Bottom face } \textit{has}, \\ \text{Vertical faces } \textit{hfp}. \end{cases}$ then Top face *has*

Top is a closed model category
fibrations =Serre fibrations,
weak equivalences =weak homotopy equivalences
cofibrations =continuous maps having the left lifting property relatively to trivial fibrations

The category **Top** satisfies the cube lemma [M. Mather 1976]

S-Top

S -**Top** is the category of stratified spaces and stratified maps.

Theorem

*The category $\mathcal{S}$-**Top** is a closed model category with*
– fibrations the Serre fibrations $f : X \to Y$ in the stratified sense
– weak equivalences the weak homotopy equivalences which induce weak equivalences between the strata,
– cofibrations the continuous maps having the left lifting property relatively to trivial fibrations.

*Moreover, the category $\mathcal{S}$-**Top** satisfies the cube lemma.*

Relative Whitehead and Ganea constructions in a closed model category

We consider a morphism $\iota_X \colon A \to X$ of $\mathcal{M}$ and define the Whitehead and Ganea constructions of ι_X which give the Ganea and the Whitehead category, $\mathrm{Gcat}(\iota_X)$ and $\mathrm{Whcat}(\iota_X)$.

The source A of ι_X will play the role of the transverse structure in the case of a foliated manifold.

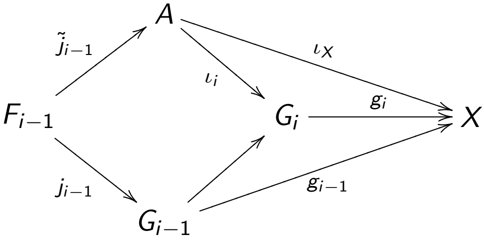
Related works:

- Mónica Clapp and Dieter Puppe. Invariants of the Lusternik-Schnirelmann type and the topology of critical sets. *Trans. Amer. Math. Soc.*, 298:603620, 1986.
- Donald Yau, Clapp-Puppe type LS category in a model category (2001)

Ganea construction.

Definition

For any map $\iota_X \colon A \to X$ of $\mathcal{M}$, the *Ganea construction* of ι_X is the following sequence of homotopy commutative diagrams ($i > 0$):



where the outside square is a homotopy pullback, the inside square is a homotopy pushout and the map $g_i : G_i \to X$ is the map induced by this homotopy pushout. The induction starts with $g_0 = \iota_X : A \to X$.

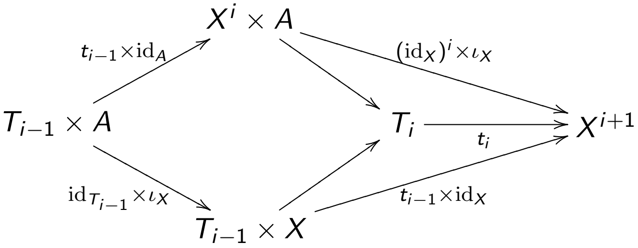
Definition

Let $\iota_X : A \to X$ be a map of $\mathcal{M}$.
The Ganea LS-category of ι_X is the least integer n such that the map $g_n\colon G_n \to X$ has a homotopy section.

We denote it by $\mathrm{Gcat}(\iota_X)$, or $\mathrm{Gcat}(X, A)$ if there is no ambiguity.
It is an invariant of the homotopy type.

Definition

For any map $\iota_X : A \to X \in \mathcal{M}$ the Whitehead construction of ι_X is the following sequence of homotopy commutative diagrams ($i > 0$):



.

The outside square is a homotopy pullback, the inside square is a homotopy pushout, $t_i : T_i \to X^{i+1}$ is the map induced by this homotopy pushout. The induction starts with $t_0 = \iota_X : A \to X$.

The space T_i is called the *ith fat wedge* of ι_X (also denoted by $T_i(\iota_X)$ or by $T_i(X, A)$).

Definition

Let $\iota_X : A \to X$ be a map of $\mathcal{M}$. The Whitehead LS-category of ι_X is the least integer n such that the diagonal $\Delta : X \to X^{n+1}$ factorizes up to homotopy through the map $t_n : T_n \to X^{n+1}$.

We denote it by $\mathrm{Whcat}(\iota_X)$, or $\mathrm{Whcat}(X, A)$ if there is no ambiguity.

Theorem

For any map $\iota_X : A \to X$ of a closed model category $\mathcal{M}$, we have

$$\mathrm{Whcat}(\iota_X) \leq \mathrm{Gcat}(\iota_X).$$

Moreover, if $\mathcal{M}$ satisfies the cube axiom, one has the equality

$$\mathrm{Whcat}(\iota_X) = \mathrm{Gcat}(\iota_X).$$

Clapp-Puppe when $\mathcal{M} = \mathbf{Top}$. Will be used for $\mathcal{M} = \mathcal{S}$-$\mathbf{Top}$.

Third step: Open LS-category of a stratified space

We come back to to the category $\mathcal{S}$-**Top** of stratified spaces and stratified maps, with its structure of closed model category.

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Definition

A stratified space $(X, \mathcal{F})$ is a topological space together with a partition $\mathcal{F}$ whose elements $S \subseteq X$ are path-connected for the induced topology.

We have seen that it verifies the cube axiom, which implies the coincidence of the Ganea and of the Whitehead definitions of the LS-category of a map $\iota_X \colon A \to X$ in $\mathcal{S}$-**Top**.

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For instance, the ith fat wedge $T_i(X, A) \in \mathcal{S}$-**Top** is the stratified subspace of the product $(X, \mathcal{F})^{i+1}$ formed of the $(i+1)$-uples $(x_0, \ldots, x_i)$ having at least one x_k in A.
The product stratification $\mathcal{F}^{i+1}$ is defined by
$(x_0, \ldots, x_i) \sim (y_0, \ldots, y_i)$ if, and only if, $x_k \sim y_k$ for $k = 0, \ldots, i$.

A notion of LS-category for stratified pairs using open sets.

We provide a definition of LS-category by open sets and study its relationship with the two first ones when ι_X is a canonical injection.

Stratified pair $(X, A, \mathcal{F})$: a stratified space $(X, \mathcal{F})$ and A is a subset of X.

Stratified deformation A non-empy set U of X is said to be *A-categorical* if there is a stratified homotopy $H\colon (U, \mathcal{F}_U) \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for any $x \in U$.

Definition

The open LS-category of the stratified pair $(X, A, \mathcal{F})$ is the least integer n such that there exists a covering of X by $(n+1)$ open sets which are A-categorical.

We denote it by $\mathrm{Ocat}(X, A, \mathcal{F})$ or $\mathrm{Ocat}(X, A)$ if there is no ambiguity.

Basic facts

- $\mathrm{Ocat}(X, X) = 0$

Basic facts

- the finiteness of $\mathrm{Ocat}(X, A)$ implies that A cuts each stratum.

Basic facts

- Example. We take the plane $X = \mathbb{R}^2$ stratified by the orbits of the action of the rotation group $\mathrm{SO}(2)$. Let A be the half-ray Ox.
  We observe that any open subset U containing the singular orbit O contains also a circle C which cannot be contracted to some point of $A \cap C$ by a stratified deformation of U.
  Therefore $\mathrm{Ocat}(\mathbb{R}^2, Ox) = \infty$.

Proposition

Let $(X, A, \mathcal{F})$ be a stratified pair. If $f : (X, \mathcal{F}) \to (X', \mathcal{F}')$ is a stratified map with a right homotopical inverse g, then

$$\mathrm{Ocat}(X', f(A)) \leq \mathrm{Ocat}(X, A).$$

Definition

Let $(X, \mathcal{F})$ be a stratified space. A subspace $A \subseteq X$ is transverse to the stratification if, for any stratum $S \in \mathcal{F}$, the set $A \cap S$ is at most countable. We denote this property by $A \pitchfork \mathcal{F}$.

Definition

Let $(X, \mathcal{F})$ be a stratified space. The *open LS-category* of $(X, \mathcal{F})$ is the infimum of the integers $\mathrm{Ocat}(X, A, \mathcal{F})$, when A runs along the transverse subsets to $\mathcal{F}$:

$$\mathrm{Ocat}(X, \mathcal{F}) = \mathrm{Inf}\ \{\mathrm{Ocat}(X, A, \mathcal{F}) \mid A \pitchfork \mathcal{F}\}\,.$$

Lemma

*The image, by a homotopy equivalence of $\mathcal{S}$-**Top**, of a transverse set of $(X, \mathcal{F})$ is a transverse set of $(X', \mathcal{F}')$.*

Then the infimum gives a homotopy invariant.

Theorem

*In the closed model category $\mathcal{S}$-**Top**, the integer $\mathrm{Ocat}(-,-)$ is a homotopy invariant.*

Counter-examples

Possible definitions of transverse set that do not work

Subsets A of $(X, \mathcal{F})$ such that $A \cap S$ is totally disconnected for any $S \in \mathcal{F}$.

Subsets such that the property $A \cap S$has topological dimension 0.

Example

Let $f \colon X = S^1 \times \mathbb{R} \to X' = S^1$ be the canonical projection. We put on X and X' the trivial stratification with only one stratum. Let $\varphi \colon S^1 \to \mathbb{R}$ be an application that is discontinuous at each point and let A be the graph of φ.

Then f is a weak equivalence of $\mathcal{S}$-**Top**, A is totally disconnected and $f(A) = X'$ is not.

Relationship between Ocat and Whcat.

Let $(X, \mathcal{F})$ be a stratified space and A be a subspace of X.

Theorem

If the space X is normal, then we have:

$$\mathrm{Whcat}(X, A) \leq \mathrm{Ocat}(X, A).$$

Definition

Let $(X, \mathcal{F})$ be a stratified space. The subset A of X is called a *B-stratified neighborhood deformation* (in short B-SND) if A has some open neighborhood which is B-categorical.

The usual property used for classical LS-category is: the base point of X is non-degenerate (it has a contractible neighbourhood).

Theorem

Let $(X, \mathcal{F})$ be a stratified space and A be a subspace of X.
If A is a B-SND, then we have:

$$\mathrm{Ocat}(X, B) \leq \mathrm{Whcat}(X, A).$$

Coming back to the tangential LS-category of a foliated manifold.

- We consider a foliated manifold $(M, \mathcal{F})$ of class C^0 and prove that $\mathrm{cat}_{\mathcal{F}}(M) = \mathrm{Ocat}(M, \mathcal{F})$ as a stratified space.

Coming back to the tangential LS-category of a foliated manifold.

- We consider a foliated manifold $(M, \mathcal{F})$ of class C^0 and prove that $\mathrm{cat}_{\mathcal{F}}(M) = \mathrm{Ocat}(M, \mathcal{F})$ as a stratified space.
- When $(M, \mathcal{F})$ is a smooth closed manifold with a C^1 -foliation, the images of categorical open sets by integrable deformations have an adequate structure of B-SND (stratified neighborhood deformation), hence $\mathrm{Ocat}(M, \mathcal{F}) = \mathrm{Whcat}(M, \mathcal{F})$.

Outline: Equivalence with $\mathrm{Ocat}(X, \mathcal{F})$

- For a tangentially categorical open set U, the transverse space $H(U, 1)$ which receives the stratified deformation is not predetermined.

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- For a tangentially categorical open set U, the transverse space $H(U, 1)$ which receives the stratified deformation is not predetermined.
- For getting the tangential LS-category of foliations we have to take into account all the transverse subspaces associated to tangential deformations, for a categorical covering.

Outline: Equivalence with $\mathrm{Ocat}(X, \mathcal{F})$

- For a tangentially categorical open set U, the transverse space $H(U, 1)$ which receives the stratified deformation is not predetermined.
- For getting the tangential LS-category of foliations we have to take into account all the transverse subspaces associated to tangential deformations, for a categorical covering.
- Singhof and Vogt showed that those subspaces have a nice structure.

Lemma

Consider a tangentially categorical open set U with its tangential contraction $H\colon U \times I \to M$. Then $H(U,1)$ is contained in a set $A(U)$ which is a finite or countable union of compact $(n-p)$-submanifolds with boundary, each one transverse to $\mathcal{F}$.

$n = \dim M$, $p = \dim \mathcal{F}$.

Moreover, for each leaf L of $\mathcal{F}$ the intersection $L \cap A(U)$ is a countable set.

Proof. $\mathrm{cat}_{\mathcal{F}}(M) \geq \mathrm{Ocat}(M, \mathcal{F})$.

Suppose $\mathrm{cat}_{\mathcal{F}}(M) = n < \infty$.
Let $U_0, \ldots, U_n$ be a covering of M by tangentially categorical open sets. Denote by $H_i \colon U_i \times I \to M$ a tangential contraction of U_i and by A_i the set $A(U_i)$ containing $H_i(U_i, 1)$.

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Let $A = A_0 \cup \cdots \cup A_n$. Then the set A is transverse to $\mathcal{F}$, and the open sets U_i are A-categorical. Hence $\mathrm{Ocat}(M, \mathcal{F}) \leq n$.

Proof. $\mathrm{cat}_{\mathcal{F}}(M) \leq \mathrm{Ocat}(M, \mathcal{F})$.

Suppose $\mathrm{Ocat}(M, \mathcal{F}) = n < \infty$.
Then there exists a transverse subset A such that $\mathrm{Ocat}(M, A, \mathcal{F}) = n$.
Let U be an A-categorical open set and H be the corresponding stratified homotopy. Then for each leaf $L \in \mathcal{F}$, the intersection $L \cap A$ is a countable, thus totally disconnected set.
Hence the image by $H(-, 1)$ of each connected component of $L \cap U$ is reduced to a point. Therefore, U is tangentially categorical, which proves that $\mathrm{cat}_{\mathcal{F}}(M) \leq n$.

Equivalence of Gcat and Whcat

Definition

Let M be a closed manifold with a C^1-foliation $\mathcal{F}$. The *Ganea LS-category* of $(X, \mathcal{F})$ is the infimum of the integers $\mathrm{Gcat}(X, A, \mathcal{F})$, when A runs along the transverse subsets to $\mathcal{F}$, i.e.

$$\mathrm{Gcat}(M, \mathcal{F}) = \mathrm{Inf}\ \{\mathrm{Gcat}(M, A, \mathcal{F}) \mid A \pitchfork \mathcal{F}\}\,.$$

Similarly, we define the *Whitehead LS-category* of $(X, \mathcal{F})$ and denote it by $\mathrm{Whcat}(M, \mathcal{F})$.

idea of Proof. $\mathrm{Gcat} = \mathrm{Whcat} = \mathrm{Ocat}$

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On the other hand M is a normal space. So $\mathrm{Ocat} = \mathrm{Whcat}$.

Theorem

Let $(M, \mathcal{F})$ be a closed manifold with a C^1-foliation. Then, we have:

$$\mathrm{Gcat}(M, \mathcal{F}) = \mathrm{Whcat}(M, \mathcal{F}) = \mathrm{Ocat}(M, \mathcal{F}) = \mathrm{cat}_{\mathcal{F}}(M).$$