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Some interesting families of non-Hopf real hypersurfaces in complex space forms

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Abstract. This article contains a survey on the construction of some interesting non-Hopf real hypersurfaces in nonflat complex space forms. Specifically, we explain the construction of homogeneous hypersurfaces and inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces, and of real hypersurfaces with two nonconstant principal curvatures in the complex projective and hyperbolic planes.

1 Introduction

Originally, the interest of studying real hypersurfaces in Kähler manifolds appeared in the field of Complex Analysis. In the theory of several complex variables, an important problem is to understand the relation between holomorphic functions defined on a domain of the complex space $\mathbb{C}^n$ and the boundary of such domain. When this boundary is smooth, it becomes a real hypersurface, that is, a submanifold of the Euclidean space $\mathbb{R}^{2n}$ with real codimension one. Note that the term *real* is used to distinguish these objects from complex hypersurfaces, i.e. submanifolds with complex codimension one. See [17] for a survey on real hypersurfaces from the viewpoint of Complex Analysis.

From the point of view of Differential Geometry, a problem that has attracted the attention of many mathematicians over the last few decades is to classify real hy-

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persurfaces in terms of different geometric conditions. The case of real hypersurfaces in nonflat complex space forms, that is, in the complex projective and hyperbolic spaces ($\mathbb{C}P^n$ and $\mathbb{C}H^n$, respectively), deserves special attention, for these spaces are the nonflat Kähler manifolds with simplest curvature tensor.

Traditionally, geometers interested in real hypersurfaces in complex space forms have mostly focused on studying the so-called Hopf examples. Given a real hypersurface M in a complex space form with complex structure J, the hypersurface M is said to be *Hopf* if its Reeb vector field (or structure vector field) $J\xi$ is an eigenvector of the shape operator of M at every point. Hopf real hypersurfaces provide a large list of examples, among which we can find for example all homogeneous hypersurfaces in complex projective spaces (see [13], [19], [24]). Recall that a submanifold of any Riemannian manifold is called *homogeneous* if it is an orbit of an isometric action on the ambient manifold.

However, some interesting families of non-Hopf real hypersurfaces have been discovered over the last few years. Our purpose in this paper is to describe some of them. The examples we present are related to homogeneous hypersurfaces, isoparametric hypersurfaces and hypersurfaces with two principal curvatures. Let us recall here that a hypersurface of any Riemannian manifold is called *isoparametric* if it and its close-by equidistant hypersurfaces have constant mean curvature.

The first important class of examples we will describe is provided by some non-Hopf homogeneous hypersurfaces in the complex hyperbolic spaces $\mathbb{C}H^n$. We review their construction, due to Berndt and Brück [1], in Section 2.

The previous construction can be generalized to produce inhomogeneous non-Hopf isoparametric hypersurfaces in $\mathbb{C}H^n$. This generalization is due to Díaz-Ramos and the author [9], and is described in Section 3.

Finally, in Section 4 we explain how one can construct new real hypersurfaces with two (nonconstant) principal curvatures in the complex projective and hyperbolic planes $\mathbb{C}P^2$ and $\mathbb{C}H^2$. These new examples are motivated by an open question proposed by Niebergall and Ryan in [22], and their construction is part of an ongoing joint project with Díaz-Ramos and Vidal-Castiñeira [11].

2 Non-Hopf homogeneous hypersurfaces in complex hyperbolic spaces

As we mentioned before, whereas all homogeneous hypersurfaces in $\mathbb{C}P^n$ are Hopf, this is not the case for $\mathbb{C}H^n$. The purpose of this section is to review the construction of the non-Hopf homogeneous hypersurfaces in $\mathbb{C}H^n$.

The description of these examples makes use of a Lie group theoretic model of the complex hyperbolic space. This model is related to the Iwasawa and root space decompositions of the isometry group of $\mathbb{C}H^n$, but can also be described directly from the point of view of the so-called Damek-Ricci spaces. Here we will content ourselves with a succinct description of this model, and we refer the reader to [2], [13] and [15, §1.7.3] for detailed information and references.

The complex hyperbolic space $\mathbb{C}H^n$ can be identified with a solvable Lie group AN endowed with a left-invariant Riemannian metric. Here AN is a semidirect product of an abelian 1-dimensional Lie group A and a $(2n-1)$-dimensional nilpotent Lie group N. Let $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of A and N, respectively. The center $\mathfrak{z}$ of $\mathfrak{n}$ is 1-dimensional, and the orthogonal complement $\mathfrak{v} = \mathfrak{n} \ominus \mathfrak{z}$ of $\mathfrak{z}$ in $\mathfrak{n}$ turns out to admit a complex structure J (which can be related to the complex structure of $\mathbb{C}H^n$). Thus, we will identify the vector space $\mathfrak{v}$ with the complex Euclidean space $\mathbb{C}^{n-1}$. Furthermore, the Lie bracket of the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ satisfies the following properties: $[\mathfrak{a}, X] \subset \mathbb{R}X$, for all $X \in \mathfrak{v}$ and all $X \in \mathfrak{z}$, and $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$.

Let now $\mathfrak{w}$ be a proper vector subspace of $\mathfrak{v} \cong \mathbb{C}^{n-1}$; here proper simply means that $\mathfrak{w} \neq \mathfrak{v}$. Denote by $\mathfrak{w}^\perp$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$. We say that $\mathfrak{w}^\perp$ has *constant Kähler angle* $\varphi \in [0, \pi/2]$ if for every unit vector $\xi \in \mathfrak{w}^\perp$, the angle between $J\xi$ and $\mathfrak{w}^\perp$ is φ. In particular, $\mathfrak{w}^\perp$ is a complex subspace of $\mathfrak{v}$ if and only if $\mathfrak{w}^\perp$ has constant Kähler angle 0, and $\mathfrak{w}^\perp$ is a totally real subspace of $\mathfrak{v}$ if and only if $\mathfrak{w}^\perp$ has constant Kähler angle $\pi/2$. However, if $n \geq 3$, there are subspaces of $\mathfrak{v} \cong \mathbb{C}^{n-1}$ with constant Kähler angle φ, for each $\varphi \in (0, \pi/2)$. In fact, these subspaces can be classified (see [1, Proposition 7]).

We proceed now with the definition of the non-Hopf homogeneous hypersurfaces. Assume that the subspace $\mathfrak{w}$ of $\mathfrak{v}$ is such that $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has nonzero constant Kähler angle $\varphi \in (0, \pi/2]$. Then

$$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$$

is a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. Denote by $S_\mathfrak{w}$ the connected subgroup of AN with Lie algebra $\mathfrak{s}_\mathfrak{w}$. From the Riemannian viewpoint, this subgroup $S_\mathfrak{w}$ is a homogeneous austere (and hence minimal) submanifold of $AN \cong \mathbb{C}H^n$. It turns out that the distance tubes of any radius around $S_\mathfrak{w}$ are homogeneous hypersurfaces of the complex hyperbolic space. If $S_\mathfrak{w}$ has dimension $2n-1$ (that is, $\dim \mathfrak{w}^\perp = 1$), then $S_\mathfrak{w}$ is a homogeneous hypersurface itself. These two claims follow from the work of Berndt and Brück [1], where these submanifolds were first introduced. However, more is true: every non-Hopf homogeneous hypersurface of $\mathbb{C}H^n$ is either $S_\mathfrak{w}$ for some subspace $\mathfrak{w} \subset \mathfrak{v}$ of dimension $2n-1$, or a tube around a submanifold $S_\mathfrak{w}$ such that $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has nonzero constant Kähler angle. This is a consequence of the classification of homogeneous hypersurfaces in complex hyperbolic spaces, due to Berndt and Tamaru [5].

The submanifolds $S_\mathfrak{w}$ are often denoted in the literature by W_φ^{2n-k}, where φ is the value of the constant Kähler angle of $\mathfrak{w}^\perp$, and $k = \dim \mathfrak{w}^\perp$. Their geometry, as well as the geometry of the corresponding homogeneous hypersurfaces, was studied in [2]. A partial characterization of these examples by means of the property of the constancy of the principal curvatures was obtained in [8].

3 New isoparametric hypersurfaces

The previous definition of the non-Hopf homogeneous hypersurfaces due to Berndt and Brück admits a noteworthy generalization to the realm of isoparametric hypersurfaces in $\mathbb{C}H^n$. This was first pointed out by Díaz-Ramos and the author in [9]. More recently, in [10] we generalized the construction to the harmonic Damek-Ricci spaces and, in particular, to the quaternionic hyperbolic spaces and the Cayley hyperbolic plane. In this section we review the main result in [9] about the construction of new isoparametric hypersurfaces in complex hyperbolic spaces. We will also make use of the notation and results concerning the Lie group theoretic description of $\mathbb{C}H^n$ explained in the previous section.

In the definition of the non-Hopf homogeneous hypersurfaces above, we had to consider subspaces $\mathfrak{w}$ of $\mathfrak{v}$ such that $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has constant Kähler angle. However, the definition of the submanifold $S_\mathfrak{w}$ makes perfect sense for any choice of the subspace $\mathfrak{w}$ of $\mathfrak{v}$. What can we say about the tubes around $S_\mathfrak{w}$? The surprising answer is that all these tubes have constant mean curvature. Since by definition the tubes are equidistant to each other, it follows that they are isoparametric hypersurfaces.

Theorem 3.1. **[9]** *Identify the complex hyperbolic space $\mathbb{C}H^n$ with the solvable Lie group AN as in Section 2. Let $\mathfrak{w}$ be any proper vector subspace of $\mathfrak{v}$ and define the Lie subalgebra $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ of $\mathfrak{a} \oplus \mathfrak{n}$. Let $S_\mathfrak{w}$ be the connected subgroup of AN with Lie algebra $\mathfrak{s}_\mathfrak{w}$.*

Then, the tubes around the submanifold $S_\mathfrak{w}$ of $AN \cong \mathbb{C}H^n$ are isoparametric hypersurfaces of $\mathbb{C}H^n$. Moreover, the following conditions are equivalent:

- *The tubes around $S_\mathfrak{w}$ are homogeneous.*
- *The tubes around $S_\mathfrak{w}$ have constant principal curvatures.*
- *$\mathfrak{w}^\perp$ has constant Kähler angle as a subspace of $\mathfrak{v}$.*

An important consequence of the previous result is the existence of uncountably many inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces. Moreover, the tubes around $S_\mathfrak{w}$ are Hopf hypersurfaces if and only if $\mathfrak{w}$ is a complex subspace of $\mathfrak{v}$; in this case $S_\mathfrak{w}$ is a totally geodesic complex hyperbolic subspace of $\mathbb{C}H^n$ and the tubes around it are well-known homogeneous Hopf hypersurfaces. Therefore, most of the examples produced by the method above are non-Hopf. In fact, the number of nontrivial projections of their structure vector field onto the principal curvature spaces may vary from 1 to 3.

Let us conclude this section with a short remark on isoparametric hypersurfaces in complex projective spaces $\mathbb{C}P^n$. Whereas the classification of isoparametric hypersurfaces in $\mathbb{C}H^n$ is still elusive (in spite of the amount of examples provided above), the classification in the projective case has been almost completely settled by the author in [14]. As well as for $\mathbb{C}H^n$, every Hopf isoparametric hypersurface in $\mathbb{C}P^n$ is an open part of a homogeneous hypersurface, but there exist many non-Hopf isoparametric examples with nonconstant principal curvatures. All of them are obtained by projecting isoparametric hypersurfaces in odd-dimensional spheres by means of the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$. However, two interesting phenomena arise in this context. On the one hand, there can be noncongruent isoparametric hypersurfaces in $\mathbb{C}P^n$ that pullback under the Hopf fibration to congruent isoparametric hypersurfaces in S^{2n+1}. On the other hand, there are inhomogeneous isoparametric hypersurfaces in $\mathbb{C}P^n$ that pullback under the Hopf map to homogeneous isoparametric hypersurfaces in S^{2n+1}. See [14] for more details on these interesting properties.

4 Niebergall and Ryan's open problem and new real hypersurfaces with two principal curvatures

In the influential survey [22] by Niebergall and Ryan, the authors included a list of open problems in the field of real hypersurfaces in complex space forms. This list has motivated many investigations over the last years. One of the problems that was still outstanding, in spite of the efforts of several geometers, is Question 9.2 in [22]. Below, we explain this problem and announce a surprising answer that has been obtained in the preprint [11] by J. Carlos Díaz-Ramos, Cristina Vidal-Castiñeira and the author.

A classical result by Tashiro and Tachibana [25] asserts that there are no umbilical real hypersurfaces in nonflat complex space forms, that is, there are no real hypersurfaces with only one principal curvature in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. Later, Cecil and Ryan [6] showed that a real hypersurface with exactly two principal curvatures in $\mathbb{C}P^n$, $n \geq 3$, has constant principal curvatures and is an open part of a geodesic sphere. An analogous result for $\mathbb{C}H^n$, $n \geq 3$, was obtained by Montiel [21], who showed that a real hypersurface with two distinct principal curvatures must be an open part of a geodesic sphere, a tube around a totally geodesic $\mathbb{C}H^{n-1}$ in $\mathbb{C}H^n$, a tube of radius $\frac{1}{\sqrt{-c}}\ln(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n$, or a horosphere; here c is the constant sectional curvature of $\mathbb{C}H^n$. In both cases, the examples that arise in these classifications are open parts of homogeneous Hopf hypersurfaces, which are sometimes referred to as the standard examples.

The problem appears when one tries to generalize the results by Cecil and Ryan, and Montiel, to the case $n = 2$, since the methods used by these authors fail for this case. Niebergall and Ryan stated the problem in [22, Question 9.2] as follows:

Problem. *Do the above results by Cecil, Ryan and Montiel extend to $n = 2$? Are there hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ that have 2 principal curvatures, other than the standard examples?*

In [11] we show that the answer to the second question is affirmative: there are non-Hopf real hypersurfaces with two nonconstant principal curvatures in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Moreover, in [11] we obtain a geometric description of all such examples. Below, we explain the main ideas of our construction.

We will denote by $\bar{M}^2(c)$ the complex projective plane $\mathbb{C}P^2$ of constant holomorphic sectional curvature $c > 0$, or the complex hyperbolic plane $\mathbb{C}P^2$ of constant holomorphic sectional curvature $c < 0$. We also need to recall here the notion of polar action. Given a group H of isometries of a given Riemannian manifold $\bar{M}$, its action on $\bar{M}$ is said to be *polar* if there exists a submanifold Σ of $\bar{M}$ that intersects all the orbits of the action, and always orthogonally. Such submanifold Σ must then be totally geodesic, and is called a *section* of the action. Given a polar action on $\bar{M}$, there is a section through every point in $\bar{M}$. The investigation of polar actions and their classification, particularly on symmetric spaces, is an active field of research nowadays. We refer the reader to [7], [16], [20] and [26] for more information on polar actions.

In order to construct our new hypersurfaces, we start by recalling that the group $U(3)$ of unitary linear transformations of $\mathbb{C}^3$ acts transitively on the complex projective plane $\mathbb{C}P^2$ by isometries. For the case of the complex hyperbolic plane $\mathbb{C}H^2$, we consider the group $U(1,2)$ of complex 3×3 matrices that preserve a Hermitian metric on $\mathbb{C}^3$ of signature $(1,2)$. Similarly as for $\mathbb{C}P^n$, its action on the anti-de Sitter space of dimension 5 descends to a transitive isometric action on $\mathbb{C}H^2$.

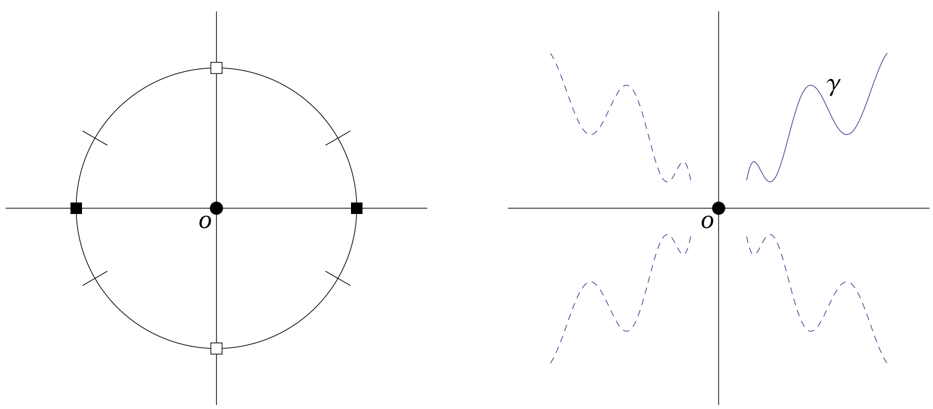
Now we consider the product group $H = U(1)\times U(1)\times U(1)$ embedded as a subgroup of $U(3)$, and also of $U(1,2)$, in the canonical way (i.e., by means of diagonal matrices). It is easy to show that the action of H on $\mathbb{C}P^2$ is polar and its sections are totally geodesic real projective planes $\mathbb{R}P^2$; this also follows from the classification of polar actions on complex projective spaces due to Podestà and Thorbergsson [23]. Similarly, the classification of polar actions on the complex hyperbolic plane, due to Berndt and Díaz-Ramos [4], shows that H also acts polarly on $\mathbb{C}H^2$, and the sections are totally geodesic real hyperbolic planes $\mathbb{R}H^2$. It is worthwhile to point out that, although in general there does not exist a one-to-one correspondence between polar actions on $\mathbb{C}P^2$ and on $\mathbb{C}H^2$, it does exist for the action of compact groups [12], as is the case of $H = U(1)\times U(1)\times U(1)$.

We describe now the structure of orbits of the action of H on $\bar{M}^2(c)$, $c\neq 0$. The situation is quite similar for the cases $c>0$ and $c<0$, although there are some differences due to the compactness of $\bar{M}^2(c)=\mathbb{C}P^n$ when $c>0$.

We start considering the subgroup $K = U(1)\times U(2)$ of $U(3)$ and of $U(1,2)$, embedded in the canonical way. The group K acts isometrically on $\bar{M}^2(c)$ with cohomogeneity one and with a fixed point, say o. The principal orbits of this action are all geodesic spheres centered at o. If $c<0$, all such geodesic spheres, together with the point o, fill the whole $\mathbb{C}H^n$, since $\mathbb{C}H^n$ is a Hadamard manifold. However, in the case $c>0$, the conjugate locus of o is not empty: it consists of a totally geodesic complex projective line $\mathbb{C}P^1$ inside $\mathbb{C}P^2$. Hence, the orbits of the H-action on $\mathbb{C}P^2$ are: the point o, the geodesic spheres centered at o, and the conjugate locus $\mathbb{C}P^1$ of o.

The group $H = U(1)\times U(1)\times U(1)$ is contained in the subgroup $K = U(1)\times U(2)$. Hence, the orbits of the H-action are contained in the orbits of the K-action. The orbits of H contained in a fixed geodesic sphere around o turn out to be equidistant, flat, totally real tori $S^1\times S^1$, and two singular orbits isometric to circles S^1. In the projective case (i.e. when $c>0$), we have to describe the orbits contained in the conjugate locus of o. It turns out that the restriction of the H-action to this conjugate locus (which is a $\mathbb{C}P^1\cong S^2$) is equivalent to the standard action of $SO(2)$ on the sphere S^2. In particular, whereas the H-action on $\mathbb{C}H^n$ has only one fixed point o, the action on $\mathbb{C}P^n$ has exactly three fixed points. In both cases, the remaining orbits are either circles or tori.

A convenient way of visualizing the action of H is by geometrically interpreting each one of the points of a section of the action (see Figure 1(a)). This interpretation follows for example from the theory of polar actions and Weyl groups, see [26]. So, let Σ be a section of the polar action of H. Recall that Σ is a totally geodesic $\mathbb{R}P^2$ if $c > 0$, or a totally geodesic $\mathbb{R}H^2$ if $c < 0$. By definition, Σ intersects all the orbits of the H-action at least once, but maybe more than once. The fixed points of the action (in particular, the point o) must then be contained in Σ. The intersections of the 1-dimensional orbits contained in geodesic spheres around o with the section Σ lie in two orthogonal geodesics of Σ, which we will call axes. On the other hand, the intersections of the geodesic spheres around o with Σ are circles in Σ centered at o. Fix a given geodesic sphere around o. Then one of the two 1-dimensional orbits in this geodesic sphere intersects Σ in exactly two points, which are symmetric with respect to the center o. Each of the tori contained in the geodesic sphere intersects Σ in exactly four points, which are symmetric with respect to the two axes in Σ.



(a) This figure represents the section Σ where two geodesics and one circle were drawn. The circle is the intersection of Σ with a geodesic sphere in $\mathbb{C}P^2$ centered at o. The square-shaped points are the intersections of Σ with the two orbits diffeomorphic to S^1 inside the geodesic sphere. The four short segments represent the intersection of Σ with a principal orbit.

(b) Σ_{reg} is diffeomorphic to the Euclidean plane minus both coordinate axes. We seek a connected curve γ in one of the four quadrants of Σ_{reg}. The dashed curves are obtained by reflecting γ with respect to the axes. The four resulting curves are the intersection of Σ with the hypersurface $H \cdot \gamma$.

Figure 1: Geometric interpretation of the polar action of H.

We will be interested only in the regular part Σ_{reg} of Σ, that is, the points in the section that belong to orbits of maximal dimension of the H-action, in this case, to the two-dimensional tori. This regular part is an open and dense subset of Σ diffeomorphic to the plane $\mathbb{R}^2$ minus two orthogonal lines. Given a curve

$\gamma\colon t \in (-\varepsilon, \varepsilon) \mapsto \gamma(t) \in \Sigma_{reg}$ in the regular part of Σ, the subset

$$H \cdot \gamma = \{h(\gamma(t)) : t \in (-\varepsilon, \varepsilon),\, h \in H\}$$

is a real 3-dimensional hypersurface in $\bar{M}^2(c)$. The tangent space to $H \cdot \gamma$ at a point $h(\gamma(t))$ is spanned by the velocity $\dot{\gamma}(t)$ of the curve and the tangent space to the torus $H \cdot \gamma(t)$. The real hypersurface $H \cdot \gamma$ is clearly foliated by equidistant tori, and perpendicularly, by the curves $h \cdot \gamma\colon t \in (-\varepsilon, \varepsilon) \mapsto (h \cdot \gamma)(t) = h(\gamma(t)) \in \Sigma_{reg}$, for each $h \in H$.

Now, our purpose is to determine which curves γ give rise to hypersurfaces with exactly two principal curvatures at every point.

The first observation is that the principal curvatures of any principal H-orbit (any torus) with respect to any nonzero normal vector are always different, that is, there are exactly two. This follows, after some calculations, from the explicit expression of the shape operator of these tori (see for example [18, p. 299]).

Fix now a point $p \in \Sigma_{reg}$ and a tangent vector $v \in T_p\Sigma_{reg}$. Consider a (locally defined) regular curve γ in Σ_{reg}, parametrized by arc-length and such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Fix a unit vector field ξ normal along γ, i.e. $\langle \xi_{\gamma(t)}, \dot{\gamma}(t)\rangle = 0$ for all t where γ is defined. Consider also a local chart $\mathcal{U}$ for Σ_{reg} around p, with coordinates (x_1, x_2). Let α, $\beta\colon T\mathcal{U} \to \mathbb{R}$ be the principal curvature functions of the tori intersecting $\mathcal{U}$ at the intersection points. Note that $T\mathcal{U}$ is a fiber bundle of normal spaces to the tori intersecting $\mathcal{U}$ at the points of $\mathcal{U}$. As explained above, we know that $\alpha(w) \neq \beta(w)$ for any vector $w \in T\mathcal{U}$.

We want to impose the condition that the hypersurface $H \cdot \gamma$ have two principal curvatures. It turns out that the shape operator of $H \cdot \gamma$ at $\gamma(t)$ with respect to the unit normal vector $\xi_{\gamma(t)}$ has the following eigenvalues: $\alpha(\xi_{\gamma(t)})$, $\beta(\xi_{\gamma(t)})$, and the curvature of the curve γ in $\bar{M}^2(c)$. The latter coincides with the curvature of γ (with respect to the orientation determined by the normal field ξ) as a curve in Σ, since Σ is totally geodesic.

Hence, $H \cdot \gamma$ will have two principal curvatures at the points of γ if and only if the curvature of γ (with respect to ξ) as a curve in Σ coincides with one of the two functions $\alpha(\xi_{\gamma(t)})$ or $\beta(\xi_{\gamma(t)})$. One can then see that this condition determines two possible systems of ordinary differential equations of second order. If we write γ in local coordinates as $\gamma(t) = (x_1(t), x_2(t))$ and denote by $\bar{\nabla}$ the Levi-Civita connection of $\bar{M}^2(c)$, we have:

$$\begin{aligned}\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} &= x_1''\partial_1 + x_1'\bar{\nabla}_{\dot{\gamma}}\partial_1 + x_2''\partial_2 + x_1'\bar{\nabla}_{\dot{\gamma}}\partial_2 \\ &= (x_1'' + f(x_1, x_2, x_1', x_2'))\partial_1 + (x_2'' + g(x_1, x_2, x_1', x_2'))\partial_2,\end{aligned}$$

where f, g are differentiable functions of x_1, x_2, x_1', x_2' and of the Christoffel symbols of Σ. The requirement that the curvature of γ coincides with $\alpha(\xi_{\gamma(t)})$ means that $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \alpha(\xi_{\gamma(t)})\xi_{\gamma(t)}$ (similarly with β instead of α). Hence, there exist smooth functions F_α and G_α (depending on x_1, x_2, x_1', x_2', the Christoffel symbols of Σ and the function α) such that

$$\left.\begin{aligned} x_1'' &= F_\alpha(x_1, x_2, x_1', x_2') \\ x_2'' &= G_\alpha(x_1, x_2, x_1', x_2') \end{aligned}\right\}.$$

This is a second order system of ordinary differential equations written in normal form and with two unknowns, x_1 and x_2. Hence, it has a unique solution for given initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. A completely analogous argument applies for β instead of α. Therefore, the hypersurface $H \cdot \gamma$ has two principal curvatures at the points of γ if and only if γ is a solution to one of two possible systems of ODEs constructed as explained above. Using the fact that the action of H is polar, it is not difficult to show that, given a solution γ to one of the systems of ODEs, the hypersurface $H \cdot \gamma$ has two principal curvatures at all points, not only along γ.

Finally, one has to show that the examples we have just constructed are indeed new, that is, their two principal curvatures are nonconstant. We start fixing a point $p \in \Sigma_{reg}$. Then we know that for every unit $v \in T_p\Sigma_{reg}$ there is a locally defined curve γ_v such that $\gamma_v(0) = p$, $\dot{\gamma}_v(0) = v$ and $H \cdot \gamma_v$ has two principal curvatures. In fact, there are exactly two such curves, but the arguments that follow apply to any one of them. Assume that, for all v in an open subset of the unit sphere $S^1(T_p\Sigma_{reg})$ of $T_p\Sigma_{reg}$, the real hypersurfaces $H \cdot \gamma_v$ are Hopf at p. We will get a contradiction with this assumption. For each hypersurface, let ξ_v be a unit normal vector field along $H \cdot \gamma_v$, which we know is H-equivariant along the tori that foliate $H \cdot \gamma_v$. Note that the subindex v in ξ_v only denotes that the normal vector field depends on the initial value v for $\dot{\gamma}_v$; in particular, $\langle \dot{\gamma}_v(t), (\xi_v)_{\gamma_v(t)} \rangle = 0$ for each posible t. The assumption that $H \cdot \gamma_v$ is Hopf at p means that $(J\xi_v)_p$ is an eigenvector of the shape operator of $H \cdot \gamma_v$, and hence, $(J\xi_v)_p$ is also an eigenvector of the shape operator $S_{(\xi_v)_p}$ of the torus $H \cdot p$ with respect to the normal vector $(\xi_v)_p$. In particular, the map

$$v \in S^1(T_p\Sigma_{reg}) \mapsto \langle S_{(\xi_v)_p}(J\xi_v)_p, Jv \rangle \in \mathbb{R}$$

vanishes in an open subset of the unit sphere of $T_p\Sigma_{reg}$. Since this map is analytic, it vanishes identically, which means that $(J\xi_v)_p$ is an eigenvector of $S_{(\xi_v)_p}$, for every unit $v \in T_p\Sigma_{reg}$. Since Jv is perpendicular to $(J\xi_v)_p$, we have that Jv is an eigenvector of $S_{(\xi_v)_p}$ for each v, because $S_{(\xi_v)_p}$ is a self-adjoint endomorphism of a two-dimensional vector space. But now, if we fix any v and take unit normal vectors $\xi = (\xi_v)_p$ and $\eta = v$ at p, then $\{J\xi, Jv\}$ is a common basis of eigenvectors for the shape operators S_ξ and S_η of the torus $H \cdot p$ at p with respect to ξ and η. This means that the shape operators S_ξ and S_η commute. Using this and the fact that the torus $H \cdot p$ has flat normal bundle, the Ricci equation of $H \cdot p$ can be used to derive a contradiction. Therefore, the real hypersurfaces $H \cdot \gamma_v$ are Hopf at p for all v in a subset of $S^1(T_p\Sigma_{reg})$ with measure zero. So, generically, our hypersurfaces are non-Hopf. But since they have two principal curvatures, these cannot be constant, because all hypersurfaces in $\bar{M}^2(c)$ with two constant principal curvatures are Hopf, as follows from their well-known classification result (see [3]).

The arguments above imply the existence of non-Hopf real hypersurfaces with two nonconstant principal curvatures in the complex projective and hyperbolic planes. However, the most difficult part of our work [11] consists in proving a classification result for hypersurfaces with two principal curvatures in $\bar{M}^2(c)$, $c \neq 0$. Several ingredients are needed to prove such result, but the most complicated part consists in dealing with all the information provided by the fundamental equations of the hypersurfaces under consideration. This information can then be used to prove a structure result for such hypersurfaces, which turn out to be locally foliated by flat surfaces with parallel second fundamental form and, perpendicularly, by geodesic curves of the hypersurface. We include below the main result in [11], which first states the existence result for both $\mathbb{C}P^2$ and $\mathbb{C}H^2$, and then the characterization result for $\mathbb{C}P^2$.

Theorem 4.1. **[11]** *Let $\bar{M}^2(c)$ be a complex space form of complex dimension 2 and constant holomorphic curvature $c \neq 0$. Consider a polar action of the group $H = U(1) \times U(1) \times U(1)$ on $\bar{M}^2(c)$ with section Σ.*

Then, for any regular point $p \in \Sigma$ and any tangent vector $v \in T_p\Sigma$, there are exactly two different locally defined curves $\gamma_i \colon (-\varepsilon, \varepsilon) \to \Sigma$, $i = 1, 2$, with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v$, such that the set $H \cdot \gamma_i = \{h(\gamma_i(t)) : h \in H, t \in (-\varepsilon, \varepsilon)\}$ is a real hypersurface with two principal curvatures in $\bar{M}^2(c)$. Generically, such hypersurface is non-Hopf and with nonconstant principal curvatures.

Conversely for the case $c > 0$, let M be an analytic real hypersurface of $\mathbb{C}P^2(c)$ with two nonconstant principal curvatures and which is non-Hopf at every point. Then M locally congruent to an open part of a real hypersurface constructed as above.

The arguments in [11] also apply to the complex hyperbolic plane $\mathbb{C}H^2$. However, in this case there are even more examples. Roughly, the reason for this stems from the following fact. While in $\mathbb{C}P^2$ there is only one polar action of cohomogeneity two up to orbit equivalence (the action of $U(1) \times U(1) \times U(1)$), in $\mathbb{C}H^2$ there are three more polar actions of cohomogeneity two, as follows from the classification in [4]. The construction explained in this section may be adapted to deal with these other polar actions and, thus, obtain more examples of real hypersurfaces with two principal curvatures. For more details and the proofs, we refer the reader to [11].

As a final remark, I would like to point out the interest of obtaining new characterizations of the non-Hopf hypersurfaces described in this and the previous two sections. There exists a good number of characterization and classification results for Hopf hypersurfaces in nonflat complex space forms. Most of these results were included in the survey [22], or motivated by the open questions proposed in [22], and involve geometric conditions related to the structure Jacobi operator of real hypersurfaces, the pseudo-Einstein condition or the property of being a Ricci soliton, just to give a few examples. However, the problem of finding geometric properties that can be used to characterize non-Hopf real hypersurfaces is virtually unsettled and, thus, constitutes a broad field for future research in the context of Differential Geometry.

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