On an inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane

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Abstract. We explain the construction and main properties of certain inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane. This is the first such example in a Riemannian symmetric space different from a sphere.

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1. INTRODUCTION

A hypersurface of a Riemannian manifold is said to be *isoparametric* if it and its nearby equidistant hypersurfaces have constant mean curvature. This notion was introduced and studied in the first decades of the 20th century by Somigliana, Segre, Levi-Civita and Cartan. The initial motivation for the study of these objects was the following problem of Geometric Optics: consider waves whose wavefronts are time-independent and parallel; then those wavefronts of codimension one are precisely isoparametric hypersurfaces. For a survey on isoparametric hypersurfaces and related topics, as well as for further references, see [11].

Cartan characterized isoparametric hypersurfaces in spaces of constant curvature as those hypersurfaces with *constant principal curvatures*. He also classified these objects in real hyperbolic spaces, while Segre did so for Euclidean spaces. In these cases, all such hypersurfaces are open parts of *(extrinsically) homogeneous hypersurfaces*, that is, open parts of orbits of isometric actions on the ambient space. However, in spheres the situation is different, since they admit inhomogeneous isoparametric hypersurfaces. All known examples were constructed by Ferus, Karcher and Münzner using representations of Clifford algebras [9]. In spite of several major advances in the last years, the classification of isoparametric hypersurfaces in spheres is still open nowadays.

In ambient spaces of nonconstant curvature, an isoparametric hypersurface does not need to have constant principal curvatures. On the one hand, the study of hypersurfaces with constant principal curvatures has been a fruitful area of research in the last decades, particularly in nonflat complex space forms (that is, complex projective spaces and complex hyperbolic spaces); see for example the recent work [5] on this topic, or the survey [8]. On the other hand, with the exception of homogeneous isoparametric hypersurfaces, there has not been much research on isoparametric hypersurfaces in manifolds of nonconstant curvature.

Recently, Díaz-Ramos and the author have found many new isoparametric hypersurfaces in complex hyperbolic spaces [6], most of which are not homogeneous and have nonconstant principal curvatures. In [7] we have developed a more general method of construction of isoparametric hypersurfaces in the so-called Damek-Ricci spaces. These spaces constitute a family of solvable Lie groups with left-invariant metric which includes the noncompact rank-one symmetric spaces as particular cases (i.e. real, complex and quaternionic hyperbolic spaces, and the Cayley hyperbolic plane).

The aim of this work is to explain a remarkable particular example that is obtained by our method: an isoparametric family of inhomogeneous hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane [7, §5.3]. This example is of particular interest because the only inhomogenous hypersurfaces with constant principal curvatures known so far (at least in Riemannian symmetric spaces, cf. [10, p. 7]) were the examples in spheres discovered by Ferus, Karcher and Münzner.

The isoparametric family will appear as the set of tubes around certain homogeneous focal submanifold of $\mathbb{O}H^2$. In order to define this focal submanifold we will use the fact that the Cayley hyperbolic plane can be seen as a solvable Lie group with left-invariant metric. We recall this fact in Section 2. Then, in Section 3 we will explain the construction of the new example and derive its main properties.

2. THE CAYLEY HYPERBOLIC PLANE AS A LIE GROUP

The Cayley hyperbolic plane $\mathbb{O}H^2$ (where $\mathbb{O}$ stands for the algebra of octonions) is the only simply connected complete noncompact Riemannian manifold with holonomy $Spin(9)$. It is a rank-one symmetric space of noncompact type: $\mathbb{O}H^2 = G/K$, where G is the exceptional Lie group F_4^{-20}, and $K = Spin(9)$ is the isotropy group at some point.

As for any other symmetric spaces of noncompact type, the Iwasawa decomposition of the Lie group G allows to identify G/K with a solvable Lie group AN equipped with a left-invariant metric. For noncompact rank-one symmetric spaces, one can give a direct construction of the Lie group structure and metric on AN. This approach, which is due to Damek and Ricci, provides many non-symmetric solvable Lie groups as well. We now briefly explain this construction; for further information, see [1, §5] and [4, Ch. 4].

Let $\mathfrak{z} = \mathbb{R}^m$ and denote by $\langle\cdot,\cdot\rangle$ the usual inner product on $\mathfrak{z}$. Let q be the quadratic form on $\mathfrak{z}$ defined by $q(Z) = -\langle Z, Z\rangle$, and $J\colon Z \in \mathrm{Cl}(\mathfrak{z},q) \mapsto J_Z \in \mathrm{End}(\mathfrak{v})$ a real representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{z},q)$ on some real vector space $\mathfrak{v}$. It is possible to extend $\langle\cdot,\cdot\rangle$ to an inner product on $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ such that: $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular, and for every unit $Z \in \mathfrak{z}$, J_Z is an orthogonal (and hence skew-symmetric) endomorphism of $\mathfrak{v}$ with respect to $\langle\cdot,\cdot\rangle$. Now we define a skew-symmetric bilinear map $[\cdot,\cdot]\colon \mathfrak{n}\times\mathfrak{n} \to \mathfrak{n}$ by $\langle [U+X, V+Y], W+Z\rangle = \langle J_Z U, V\rangle$, for all $U,V,W \in \mathfrak{v}$ and $X,Y,Z \in \mathfrak{z}$. Then $\mathfrak{n}$ becomes a two-step nilpotent Lie algebra with center $\mathfrak{z}$, which is called a *generalized Heisenberg algebra*. The corresponding simply connected Lie group N with Lie algebra $\mathfrak{n}$, equipped with the induced left-invariant metric, is called a *generalized Heisenberg group*.

Now we construct a solvable extension of $\mathfrak{n}$. Let $\mathfrak{a} = \mathbb{R}$ and $B \in \mathfrak{a}$. We define $[B,V] = \frac{1}{2}V$ and $[B,Z] = Z$, for all $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$, so that we obtain a Lie algebra structure on $\mathfrak{a}\oplus\mathfrak{n}$ that extends the one on $\mathfrak{n}$. Since $[\mathfrak{a}\oplus\mathfrak{n}, \mathfrak{a}\oplus\mathfrak{n}] = \mathfrak{n}$, the extension is solvable. We also extend $\langle\cdot,\cdot\rangle$ to an inner product on $\mathfrak{a}\oplus\mathfrak{n}$ such that both factors are orthogonal and

B has unit length. The corresponding simply connected solvable Lie group AN endowed with the induced left-invariant metric is called a *Damek-Ricci space*.

Finally we particularize this construction to obtain the Cayley hyperbolic plane. Choose $\mathfrak{z} = \mathbb{R}^7$. Then there are two inequivalent irreducible Clifford algebra representations J of $\mathrm{Cl}(\mathfrak{z}, q)$. But with independence of the choice of the irreducible representation J, we have that $\dim \mathfrak{v} = 8$, and AN becomes isometric to the Cayley hyperbolic plane $\mathbb{O}H^2$ with minimal sectional curvature -1 (see [4, §4.1.9]).

Before proceeding with the definition of the new isoparametric family of hypersurfaces in Section 3, we need to recall here the notion of *generalized Kähler angle* introduced by Díaz-Ramos and the author in [7].

Let $\mathfrak{z} = \mathbb{R}^m$, consider a real representation $J\colon Z \in \mathrm{Cl}(\mathfrak{z}, q) \mapsto J_Z \in \mathrm{End}(\mathfrak{v})$ of the Clifford algebra $\mathrm{Cl}(\mathfrak{z}, q)$ on a real vector space $\mathfrak{v}$, and endow $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with the structure of generalized Heisenberg algebra as above. Let $\mathfrak{w}^\perp$ be any vector subspace of $\mathfrak{v}$, and let $\xi \in \mathfrak{w}^\perp$ be of unit length. Let us denote by $(v)^\perp$ the orthogonal projection of a vector $v \in \mathfrak{v}$ onto $\mathfrak{w}^\perp$. Since the map $Z \in \mathfrak{z} \mapsto (J_Z\xi)^\perp \in \mathfrak{w}^\perp$ is linear, we have that

$$Q_\xi : Z \in \mathfrak{z} \mapsto \langle (J_Z\xi)^\perp, (J_Z\xi)^\perp \rangle \in \mathbb{R}$$

is a quadratic form. Let $\{Z_1, \dots, Z_m\}$ be an orthonormal basis of $\mathfrak{z}$ for which the quadratic form Q_ξ takes a diagonal form. Clearly, for each $i \in \{1, \dots, m\}$ we have that $Q_\xi(Z_i) = \cos^2(\varphi_i)$ for a unique real number $\varphi_i \in [0, \pi/2]$. Reorder the elements of $\{Z_1, \dots, Z_m\}$ if necessary so that $\varphi_1 \leq \cdots \leq \varphi_m$. Then, we define the *generalized Kähler angle* of $\xi \in \mathfrak{w}^\perp$ with respect to the subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ as the m-tuple $(\varphi_1, \dots, \varphi_m)$. If this m-tuple is independent of the unit vector $\xi \in \mathfrak{w}^\perp$, we say that the subspace $\mathfrak{w}^\perp$ of $\mathfrak{v}$ has *constant generalized Kähler angle* $(\varphi_1, \dots, \varphi_m)$.

3. THE EXAMPLE

In this section we construct the inhomogeneous isoparametric family of hypersurfaces with constant principal curvatures in $\mathbb{O}H^2$ announced before.

First let us proceed with the construction of the focal submanifold of the family. Consider the solvable model AN for the Cayley hyperbolic plane $\mathbb{O}H^2$ explained above; the corresponding Lie algebra is $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{a} = \mathbb{R}$, $\mathfrak{v} = \mathbb{R}^8$ and $\mathfrak{z} = \mathbb{R}^7$. Take a 3-dimensional subspace $\mathfrak{w}$ of $\mathfrak{v}$ and denote by $\mathfrak{w}^\perp$ the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{v}$. Then one can easily check that $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ is a solvable Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. The connected subgroup $S_\mathfrak{w}$ of AN with Lie algebra $\mathfrak{s}_\mathfrak{w}$ is clearly a homogeneous submanifold of $\mathbb{O}H^2$. Moreover, $S_\mathfrak{w}$ is a minimal (even austere) submanifold [7, p. 7]. One can also show that, if we choose any other 3-dimensional subspace $\mathfrak{w}'$ of $\mathfrak{v}$, the resulting submanifold $S_{\mathfrak{w}'}$ is congruent to $S_\mathfrak{w}$ (see [3, p. 3436]).

Fix an arbitrary unit vector ξ in $\mathfrak{w}^\perp$. A property of generalized Heisenberg algebras is that $\langle J_X U, J_Y U \rangle = \langle X, Y \rangle \langle U, U \rangle$, for every $X, Y \in \mathfrak{z}$ and $U \in \mathfrak{v}$ (see [4, p. 24]). From this, we obtain that the linear map $Z \in \mathfrak{z} = \mathbb{R}^7 \mapsto J_Z\xi \in \mathfrak{v} \ominus \mathbb{R}\xi = \mathbb{R}^7$ is an isometry (here $\mathfrak{v} \ominus \mathbb{R}\xi$ denotes the orthogonal complement of $\mathbb{R}\xi$ in $\mathfrak{v}$). Hence, we can find an orthonormal basis $\{Z_1, \dots, Z_7\}$ of $\mathfrak{z}$ such that $\mathfrak{w} = \mathrm{span}\{J_{Z_5}\xi, J_{Z_6}\xi, J_{Z_7}\xi\}$ and $\mathfrak{w}^\perp = \mathrm{span}\{\xi, J_{Z_1}\xi, J_{Z_2}\xi, J_{Z_3}\xi, J_{Z_4}\xi\}$. Then, the quadratic form Q_ξ defined in Section 2 assumes a diagonal form in the orthonormal basis $\{Z_1,\dots,Z_m\}$; moreover $Q_\xi(Z_i)=1$ for $i=1,2,3,4$, whereas $Q_\xi(Z_i)=0$ for $i=5,6,7$. By definition, the generalized Kähler angle of ξ with respect to $\mathfrak{w}^\perp$ is then $(0,0,0,0,\pi/2,\pi/2,\pi/2)$. Since this value is independent of ξ, we deduce that $\mathfrak{w}^\perp$ has constant generalized Kähler angle $(0,0,0,0,\pi/2,\pi/2,\pi/2)$.

The next theorem, which has been proved in [7], implies that every tube M^r around the above defined submanifold $S_\mathfrak{w}$ is an isoparametric hypersurface with constant principal curvatures. Therefore, $S_\mathfrak{w}$ and the tubes M^r, $r>0$, define a global isoparametric family of hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane.

Theorem 3.1 [7] *Let AN be a Damek-Ricci space with Lie algebra $\mathfrak{a}\oplus\mathfrak{n}$, where $\mathfrak{a}$ is one-dimensional and $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{z}$ is a generalized Heisenberg algebra with center $\mathfrak{z}$. Let $S_\mathfrak{w}$ be the connected subgroup of AN whose Lie algebra is $\mathfrak{s}_\mathfrak{w}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{z}$, where $\mathfrak{w}$ is any proper subspace of $\mathfrak{v}$.*

Then, the tubes M^r around the submanifold $S_\mathfrak{w}$ are isoparametric hypersurfaces of AN, and have constant principal curvatures if and only if $\mathfrak{w}^\perp=\mathfrak{v}\ominus\mathfrak{w}$ has constant generalized Kähler angle.

The proof of Theorem 3.1 in [7] is based on the calculation of the shape operator of the tubes M^r. To this aim, the technique used is similar to standard Jacobi field theory, but adapted in a suitable way to the study of submanifolds of Lie groups with left-invariant metric. In the proof, the notion of generalized Kähler angle plays a crucial role, because otherwise calculations would become unmanageable.

Let us come back to our construction in $\mathbb{O}H^2$. According to the discussion of the principal curvatures of the tubes M^r in [7, p. 12], and since $\mathfrak{w}^\perp$ has constant generalized Kähler angle $(0,0,0,0,\pi/2,\pi/2,\pi/2)$, the principal curvatures of the tube M^r of radius $r>0$ around $S_\mathfrak{w}$ are:

$$\lambda,\quad \lambda+\frac{1}{4\lambda},\quad \frac{1}{2}(3\lambda+\sqrt{1-3\lambda^2}),\quad \frac{1}{2}(3\lambda-\sqrt{1-3\lambda^2}),$$

where $\lambda=\frac{1}{2}\tanh\left(\frac{r}{2}\right)$. The corresponding multiplicities are 5, 4, 3, and 3, respectively.

Let us now show why no tube M^r is homogeneous. If for some r the hypersurface M^r were homogeneous, there would exist a group acting isometrically on $\mathbb{O}H^2$ with cohomogeneity one and with M^r as an orbit. The orbits of this action would then coincide with the leaves of the isoparametric family. Hence, the cohomogeneity one action would have one singular orbit of codimension 5, namely $S_\mathfrak{w}$. But this is impossible according to the classification of cohomogeneity one actions on $\mathbb{O}H^2$ [3, Th. 4.7, Th. 4.8].

Hence, the homogeneous minimal submanifold $S_\mathfrak{w}$ of codimension 5 and the tubes around it form an *isoparametric family of inhomogeneous hypersurfaces with constant principal curvatures in the Cayley hyperbolic plane*.

Finally, let us say a word about another interesting situation that arises from our method when one considers a subspace $\mathfrak{w}$ of $\mathfrak{v}$ with $\dim\mathfrak{w}=4$, instead of $\dim\mathfrak{w}=3$. Arguing as above, one can show that $\mathfrak{w}^\perp$ has constant generalized Kähler angle $(0,0,0,\pi/2,\pi/2,\pi/2,\pi/2)$. Again, using [7, p. 12], we obtain that the principal curvatures of the tube M^r of radius $r>0$ around $S_\mathfrak{w}$ are the same as in the case $\dim\mathfrak{w}=3$

above, but this time with respective multiplicities 4, 3, 4 and 4. In particular, these principal curvatures and multiplicities are independent of the subspace $\mathfrak{w}$ of dimension 4 chosen. By means of Theorem 3.1, every tube M^r is an isoparametric hypersurface with constant principal curvatures in $\mathbb{O}H^2$. In fact, these tubes are homogeneous hypersurfaces [1], so that they are the principal orbits of cohomogeneity one actions on $\mathbb{O}H^2$.

Contrary to what happened for $\dim \mathfrak{w} = 3$, now different choices of the subspace $\mathfrak{w}$ of $\mathfrak{v}$ with $\dim \mathfrak{w} = 4$ can lead to (uncountably many) orbit inequivalent cohomogeneity one actions [3, Th. 4.7]. This means that, for an arbitrary but fixed radius r, in spite of having the same constant principal curvatures and corresponding multiplicities, the tubes of radius r produced by different choices of $\mathfrak{w}$ can be noncongruent. Thus, for every $r > 0$, we obtain an *uncountable set of noncongruent homogeneous hypersurfaces with the same constant principal curvatures and multiplicities*.

The existence of noncongruent hypersurfaces with the same constant principal curvatures was known for spheres [9] (in which case such examples are inhomogeneous), and also for symmetric spaces of noncompact type and rank greater than two [2] (where the examples are homogeneous). Our result shows that this phenomenon also occurs for homogeneous hypersurfaces in symmetric spaces of rank one.

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