On the existence of inhomogenous isoparametric foliations of higher codimension on complex projective spaces

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Abstract

An isoparametric foliation is a certain kind of decomposition of a given Riemannian manifold into equidistant submanifolds, where those of the highest dimension have parallel mean curvature and the distribution defined by their normal bundles is integrable. Typical examples are given by the orbit foliations of polar actions, that is, of isometric actions that admit a submanifold intersecting all the orbits perpendicularly; such isoparametric foliations are called homogeneous.

In this paper we explain the main ideas of the classification of isoparametric foliations of codimension greater than one on complex projective spaces, as well as the surprising existence of inhomogeneous examples.

1 Isoparametric foliations: a quick introduction

The term *isoparametric* was first used by Levi-Civita [13] in 1937. Initially, it was only applied to hypersurfaces. Thus, a hypersurface of a given Riemannian manifold is called isoparametric if, locally, it and its nearby equidistant hypersurfaces have constant mean curvature. This notion was motivated by a problem in Geometric Optics and was first studied by Somigliana, Levi-Civita, Segre and Cartan; see [20] for a survey on the origins of the theory as well as for an extensive list of references. Cartan showed that, for spaces of constant curvature, a hypersurface is isoparametric if and only if it has constant principal curvatures, so the notion turns out to be rather rigid. In fact, it follows from Segre's and Cartan's classifications of isoparametric hypersurfaces in Euclidean and real hyperbolic spaces that all examples are *homogeneous*, that is, orbits of isometric actions. For example, in $\mathbb{R}^n$ the only isoparametric hypersurfaces are affine hyperplanes, spheres and cylinders. Cartan also addressed the problem in spheres, but he failed to obtain a classification. The amount of examples in this case is larger and the most surprising fact is the existence of many inhomogeneous isoparametric hypersurfaces. These were constructed by Ferus,

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Karcher and Münzner [8] using Clifford modules. There have been several major contributions to the classification problem in spheres over the last few years (see for example [2], [3], [10], [15]), but a complete classification is not yet known nowadays.
In the eighties, the notion of isoparametricity was generalized from hypersurfaces to submanifolds of arbitrary codimension by several authors (see [20]). However, it was Terng [18] who developed what we now know as the general theory of isoparametric submanifolds in real space forms. According to her, a submanifold of a space form is called isoparametric if its normal bundle is flat and if it has constant principal curvatures in the direction of any parallel normal vector field. Any isoparametric submanifold in the Euclidean space turns out to be the product of an isoparametric submanifold in a sphere times an affine subspace. Thus, it is equivalent to study these objects in Euclidean spaces or in spheres. The case of real hyperbolic spaces $\mathbb{R}H^n$ was tackled by Wu [22], who reduced the classification problem of isoparametric submanifolds in $\mathbb{R}H^n$ to the problem in spheres.
In real space forms, every isoparametric submanifold extends to a global isoparametric foliation, as follows from the works by Terng and Wu. Here and henceforth, the word foliation is used to refer to a *singular Riemannian foliation*. Let us explain this concept; see [21] for more details. Consider a decomposition $\mathcal{F}$ of a Riemannian manifold into connected injectively immersed submanifolds, called *leaves*, which may have different dimensions. We say that $\mathcal{F}$ is a singular Riemannian foliation if it is a transnormal system (i.e. every geodesic orthogonal to one leaf is also orthogonal to all the leaves that it intersects), and if $T_pL = \{X_p : X \in \Xi_{\mathcal{F}}\}$ for every leaf L in $\mathcal{F}$ and every $p \in L$, where T_pL is the tangent space to L at p, and Ξ is the module of smooth vector fields on the ambient manifold that are everywhere tangent to the leaves of $\mathcal{F}$. The leaves of maximal dimension are called *regular*, and the other ones are called *singular*. The codimension of the (singular Riemannian) foliation is the codimension of any regular leaf. Finally, we say that a foliation is isoparametric if its regular leaves are isoparametric submanifolds.
As explained above, for isoparametric foliations of codimension one on spheres, there are many inhomogeneous examples and the classification problem is still open. However, the situation for higher codimension is very different. Using the theory of Tits buildings, Thorbergsson [19] showed that all such examples are homogeneous. More precisely:

Theorem 1.1. [19] *Every irreducible isoparametric foliation of codimension higher than one on a sphere S^n is the orbit foliation of an s-representation.*

Here irreducible means that the corresponding foliation on the Euclidean space $\mathbb{R}^{n+1}$ determined by homotheties is not the product of two nontrivial foliations on linear subspaces of smaller dimensions. Recall as well that the term s-representation refers to the isotropy representation of a semisimple symmetric space. Specifically, if $M = G/K$ is a simply connected symmetric space, where G is the identity connected component of the isometry group of M and K is the isotropy group at some point $o \in M$, then K acts infinitesimally on the tangent space T_oM: $K \times T_oM \to T_oM$, $(k, v) \mapsto k_*v$. This action is an orthogonal representation with respect to the inner product on T_oM, and is called the *isotropy representation* of the symmetric space. Such actions are also called *s-representations* in case G is semisimple. The isotropy representations of two dual symmetric spaces are equivalent. Every s-representation restricts to an isometric action on the unit sphere of T_oM. Hence, Thorbergsson's theorem guarantees that the only irreducible isoparametric foliations of codimension at least two on spheres are the restrictions to the unit sphere of T_oM of the orbit foliations of the isotropy representations of irreducible symmetric spaces M of compact type (equivalently, of noncompact type). Finally, the well-known classification of symmetric spaces allows to obtain the explicit classification of irreducible isoparametric foliations of codimension at least two on spheres.

In spaces of nonconstant curvature the study of isoparametric submanifolds is much more involved. Even for isoparametric hypersurfaces, no complete classifications are known in symmetric spaces other than Euclidean and real hyperbolic spaces. The homogeneous hypersurfaces (orbits of cohomogeneity one actions) are quite well understood in symmetric spaces of compact type [12], but not yet in spaces of noncompact type [1]. Without the assumption of the homogeneity, very few results and examples were known; see [5] for a recent construction of many inhomogeneous isoparametric hypersurfaces in the noncompact rank one symmetric spaces.

In [9], Heintze, Liu and Olmos proposed a definition of isoparametric submanifold in an arbitrary Riemannian manifold which subsumes the notions of isoparametric hypersurface in a Riemannian manifold, and of isoparametric submanifold of a real space form. Thus, we will say that a submanifold M of a Riemannian manifold is an *isoparametric submanifold* if the following conditions are satisfied:

(a) The normal bundle of M is flat.

(b) Every parallel submanifold M' of M has constant mean curvature with respect to every parallel normal vector field of M'.

(c) M admits sections, that is, for each $p \in M$ there exists a totally geodesic submanifold Σ_p that meets M at p orthogonally and whose dimension is the codimension of M.

The locally defined parallel submanifolds of an isoparametric submanifold are isoparametric as well, and thus define locally a regular isoparametric foliation. Similarly as for space forms, we will use the term *isoparametric foliation* to refer to a singular Riemannian foliation whose regular leaves are isoparametric submanifolds. The main set of examples of isoparametric submanifolds are provided by the principal orbits of the so-called *polar actions*. Let us recall that a proper isometric action is polar if there exists a submanifold intersecting all the orbits of the action and always perpendicularly. In this case, there is a globally defined isoparametric foliation (i.e. the orbit foliation of the action), which is something that cannot be guaranteed in general. These isoparametric foliations obtained by (polar) isometric actions are said to be *homogeneous*.

In the preprint [6] we have obtained a complete classification of irreducible isoparametric foliations of codimension at least two on complex projective spaces $\mathbb{C}P^n$. The main implication of this classification is the existence of inhomogeneous examples, which contrasts with the situation in spheres and Thorbergsson's theorem. Very recently as well, Lytchak [14] proposed a new approach for the study of the so-called *polar foliations* (that is, singular Riemannian foliations where the distribution of normal spaces to the regular leaves is integrable) on symmetric spaces of compact type. Together with a previous result by Christ [4], the result of Lytchak implies the homogeneity (and hence the classification, thanks to [12]) of every irreducible isoparametric foliation of codimension at least three on a simply connected irreducible symmetric space of compact type and rank at least two.

The results in [6] and in [14] are, as far as we know, the first ones involving classifications of isoparametric foliations of higher codimension on spaces of nonconstant curvature. However, an important difference between both contexts is the question of the homogeneity: while for symmetric spaces of rank greater than one the classified examples are homogeneous, rank one symmetric spaces of nonconstant curvature allow a greater diversity of examples, including inhomogeneous foliations of large codimension. The aim of the rest of this paper is to explain the results in [6] and, in particular, the existence of inhomogeneous isoparametric foliations.

2 Isoparametric foliations on complex projective spaces

The starting point for the study of isoparametric submanifolds in complex projective spaces is their good behaviour with respect to the Hopf map $\pi\colon S^{2n+1} \to \mathbb{C}P^n$. The idea of using the Hopf fibration for the study of geometric objects in $\mathbb{C}P^n$ is not new; it has been used, for example, by Takagi [17] for the classification of homogeneous hypersurfaces in $\mathbb{C}P^n$, by Xiao [23] in his study of isoparametric hypersurfaces in $\mathbb{C}P^n$, or by Podestà and Thorbergsson [16] for the classification of polar actions on $\mathbb{C}P^n$. Using results in [9], one can show the following:

Proposition 2.1. [6] *Let M be a submanifold of $\mathbb{C}P^n$ of positive dimension. Then M is isoparametric if and only if its lift $\pi^{-1}M$ to the sphere S^{2n+1} is isoparametric.*

Using this result, together with the fact that isoparametric submanifolds in spheres can be extended to globally defined analytic isoparametric foliations [18], one can also show that:

Proposition 2.2. [6] *Each isoparametric submanifold of $\mathbb{C}P^n$ is an open part of a unique complete isoparametric submanifold, and this one is a regular leaf of a unique isoparametric foliation on $\mathbb{C}P^n$.*

In view of the previous two propositions, we conclude that, in order to study isoparametric submanifolds of $\mathbb{C}P^n$, it is enough to analyse the projections via the Hopf map π of the isoparametric foliations existing on odd-dimensional spheres S^{2n+1}. We will focus on irreducible isoparametric foliations on $\mathbb{C}P^n$, that is, those foliations for which there is no totally geodesic $\mathbb{C}P^k$, $k \in \{0, \dots, n-1\}$, being a union of leaves of the foliation. Note that an isoparametric foliation on $\mathbb{C}P^n$ is irreducible if and only if its lift $\pi^{-1}M$ is irreducible as an isoparametric foliation on S^{2n+1}.
Since by Thorbergsson's result (Theorem 1.1) we know all irreducible isoparametric foliations of codimension higher than one on spheres, then it seems natural to propose the following approach to classify irreducible isoparametric foliations of codimension higher than one on complex projective spaces. For each isoparametric foliation $\mathcal{F}$ on S^{2n+1}, decide if it "can be projected" to an isoparametric foliation on $\mathbb{C}P^n$; more precisely, determine if $\mathcal{F}$ is the pullback under the Hopf map of an isoparametric foliation on $\mathbb{C}P^n$ or, equivalently, if the leaves of $\mathcal{F}$ are foliated by the Hopf S^1-fibers. If we do this for all irreducible isoparametric foliations of codimension at least two on S^{2n+1}, we obtain the classification of irreducible isoparametric foliations of codimension at least two on $\mathbb{C}P^n$.

However, it turns out that it is not trivial to carry out this procedure. The reason is that, somewhat surprisingly, there is not a one-to-one correspondence between congruence classes of isoparametric foliations on S^{2n+1} that can be projected and congruence classes of isoparametric foliations on $\mathbb{C}P^n$. If two foliations on $\mathbb{C}P^n$ are congruent, then their pullbacks are congruent as well, but it can happen that two noncongruent foliations on $\mathbb{C}P^n$ pullback to congruent foliations on S^{2n+1}. This phenomenon (whose analogue does not occur for homogeneous hypersurfaces or polar actions) was discovered by Xiao [23] for the orbit foliations of codimension one induced by the real Grassmannians $SO(n+3)/S(O(2)\times O(n+1))$ with odd n. In [6] we show that this behaviour happens for foliations of higher codimension as well.
But, as shown in [6], there is another phenomenon which is even more interesting. There are homogeneous isoparametric foliations on S^{2n+1} that can be projected to *inhomogeneous* isoparametric foliations on $\mathbb{C}P^n$. This happens in codimension one (as noticed by Xiao [23]), but also in higher codimension, which gives rise to the inhomogeneous examples announced above.
In order to analyse the possible projections of a fixed isoparametric foliation $\mathcal{F}$ on an odd-dimensional sphere S^{2n+1}, we can follow two equivalent procedures. The first one would consist in determining all possible orthogonal transformations $A \in O(2n+2)$ such that the foliation $A(\mathcal{F})$ is the pullback of a foliation on $\mathbb{C}P^n$, then study the congruence in $\mathbb{C}P^n$ of the projections $\pi(A(\mathcal{F}))$ for all possible A, and finally decide the homogeneity of these projections.
However, we will follow a second procedure, which we formalize now. We fix a congruence class of isoparametric foliations on S^{2n+1}, and take any fixed representative of this class, say $\mathcal{F}$. The first step is to find the set $\mathcal{J}_{\mathcal{F}}$ of complex structures on $\mathbb{R}^{2n+2}$ that preserve $\mathcal{F}$. Here and henceforth, by *complex structure* we mean an orthogonal linear transformation $J \in O(2n+2)$ such that $J^2 = -\mathrm{Id}$. We will say that a complex structure *preserves* the foliation $\mathcal{F}$ if $\mathcal{F}$ is the pullback of some foliation on $\mathbb{C}P^n$ under the Hopf map determined by J or, equivalently, if for every $x \in S^{2n+1}$ the Hopf circle $\{\cos(t)x + \sin(t)Jx \in \mathbb{R}^{2n+2} : t \in \mathbb{R}\}$ through x determined by J is contained in the leaf of $\mathcal{F}$ through x. Secondly, we have to determine the quotient set $\mathcal{J}_{\mathcal{F}}/\!\sim$, where $\sim$ is the equivalence relation "yield congruent foliations on $\mathbb{C}P^n$". Note that $\mathcal{J}_{\mathcal{F}}/\sim$ is isomorphic to the set of congruence classes of isoparametric foliations on $\mathbb{C}P^n$ that pullback under a fixed Hopf map to a foliation congruent to $\mathcal{F}$. Finally, we have to decide which congruence classes correspond to homogeneous foliations on $\mathbb{C}P^n$.
This procedure has been applied in [6] to analyse the possible projections of:

- the orbit foliations of the isotropy representations of semisimple symmetric spaces, and
- the isoparametric foliations of codimension one on spheres S^{2n+1} constructed by Ferus, Karcher and Münzner [8], except when $n = 15$.

As a result, and thanks to the classification theorems of isoparametric foliations on spheres, we obtained a classification of irreducible isoparametric foliations on $\mathbb{C}P^n$ of arbitrary codimension q, except for the case $(n, q) = (15, 1)$.
Here, we will focus on the first type of foliations. It will be convenient to introduce some notation. Let G/K be an irreducible simply connected compact symmetric space of dimension $2n+2$ and rank at least two. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of

the Lie algebra of G, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. It is well-known that we can identify $\mathfrak{p}$ with the tangent space $T_o(G/K)$ and regard the isotropy representation of G/K at o as the adjoint representation $\mathrm{Ad}\colon K \to O(\mathfrak{p})$, where $O(\mathfrak{p})$ is the orthogonal group of the Euclidean vector space $\mathfrak{p}$ (endowed with the negative of the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{p}$). Finally, we denote by $\mathcal{F}_{G/K}$ the orbit foliation of the adjoint action $\mathrm{Ad}\colon K \to O(\mathfrak{p})$ restricted to the unit sphere S^{2n+1} of $\mathfrak{p}$. This is an isoparametric foliation whose codimension on S^{2n+1} agrees with the rank of G/K minus 1.
Now we explain the main ideas of how to carry out the three steps in the procedure described above, namely: determination of the complex structures, congruence of the projected foliations, and homogeneity.

2.1 Complex structures preserving $\mathcal{F}_{G/K}$

The first step consists in finding all complex structures on $\mathfrak{p}$ that preserve a given foliation $\mathcal{F}_{G/K}$. The key observation is the fact that the group $\mathrm{Ad}(K)|_{\mathfrak{p}}$ is the largest connected subgroup of $O(\mathfrak{p})$ acting on $\mathfrak{p}$ with the same orbits as the isotropy representation of G/K. This maximality property was proved by Eschenburg and Heintze [7]. Now, let J be a complex structure on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$. Then $\mathcal{T}^1 = \{\cos(t)\mathrm{Id} + \sin(t)J : t \in \mathbb{R}\}$ is a 1-dimensional group leaving invariant the leaves of $\mathcal{F}_{G/K}$. Let K' be the subgroup of $O(\mathfrak{p})$ generated by $\mathrm{Ad}(K)$ and $\mathcal{T}^1$, which is connected and leaves each leaf of $\mathcal{F}_{G/K}$ invariant. Using the maximality property we get that $K' \subset \mathrm{Ad}(K)|_{\mathfrak{p}}$, so $\mathcal{T}^1$ is a subgroup of $\mathrm{Ad}(K)|_{\mathfrak{p}}$. If we differentiate, we get that $J \in \mathrm{ad}(\mathfrak{k})|_{\mathfrak{p}}$, where ad is the adjoint action at the Lie algebra level. Thus, a complex structure J preserves $\mathcal{F}_{G/K}$ if and only if it can be written in the form $J = \mathrm{ad}(X)|_{\mathfrak{p}}$ for some $X \in \mathfrak{k}$. In other words, the set of complex structures on $\mathfrak{p}$ preserving $\mathcal{F}_{G/K}$ can be parametrized by the set

$$\mathcal{J}_{\mathcal{F}_{G/K}} = \{X \in \mathfrak{k} : \mathrm{ad}(X)|_{\mathfrak{p}}^2 = -\mathrm{Id}\},$$

since every transformation $\mathrm{ad}(X)|_{\mathfrak{p}}$, with $X \in \mathfrak{k}$, is skew-symmetric.
Now let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. A general fact about compact groups guarantees that if $X \in \mathfrak{k}$, then there is a $k \in K$ such that $\mathrm{Ad}(k)X \in \mathfrak{t}$. In this situation, one can then show that $X \in \mathcal{J}_{\mathcal{F}_{G/K}}$ if and only if $\mathrm{Ad}(k)X \in \mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$. This means that we will know $\mathcal{J}_{\mathcal{F}_{G/K}}$ once we determine the subset $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$. This restriction to a maximal abelian subalgebra of $\mathfrak{k}$ suggests the utilization of the theory of roots of the compact Lie algebra $\mathfrak{g}$ and, more specifically, the Borel-de Siebenthal theory. Here we give only some basic terminology needed to state the main consequences of the use of this theory in our problem; we refer to our article [6] for more details, and to [11, §VI.8-10] for an introduction to the Borel-de Siebenthal theory.
We say that the symmetric space G/K is inner if the rank of $\mathfrak{g}$ equals the rank of $\mathfrak{k}$. This means that a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is also a maximal abelian subalgebra of $\mathfrak{g}$. Let $\Delta_{\mathfrak{g}}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$ be the corresponding root space decomposition of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. One can show that, for each $\alpha \in \Delta_{\mathfrak{g}}$, the root space $\mathfrak{g}_\alpha$ is either contained in $\mathfrak{k}^{\mathbb{C}}$ or in $\mathfrak{p}^{\mathbb{C}}$. In the first case we say that the root α is *compact*, whereas we call it *noncompact* in the second case.
We can now state an algebraic method that can be used to completely determine the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$ for each symmetric space G/K.

Theorem 2.1. [6] *There exists a complex structure on* $\mathfrak{p}$ *preserving* $\mathcal{F}_{G/K}$ *if and only if* G/K *is an inner symmetric space.*
In this situation, let $T \in \mathfrak{t}$*. Then* $\mathrm{ad}(T)|_{\mathfrak{p}}$ *is a complex structure on* $\mathfrak{p}$ *preserving* $\mathcal{F}_{G/K}$ *if and only if* $\alpha(T) \in \{-1, 1\}$ *for all noncompact roots* α*.*

2.2 Congruence of the projected foliations

We are now interested in classifying the complex structures parametrized by the set $\mathcal{J}_{\mathcal{F}_{G/K}}$ in terms of the congruence of the projected isoparametric foliations. More concretely, we have to study the equivalence relation $\sim$ on $\mathcal{J}_{\mathcal{F}_{G/K}}$ defined as follows: given X_1, $X_2 \in \mathcal{J}_{\mathcal{F}_{G/K}}$, we say that $X_1 \sim X_2$ if $\pi_1(\mathcal{J}_{\mathcal{F}_{G/K}})$ and $\pi_2(\mathcal{J}_{\mathcal{F}_{G/K}})$ are congruent foliations on $\mathbb{C}P^n$, where π_1, $\pi_2 \colon S^{2n+1} \to \mathbb{C}P^n$ are the Hopf fibrations determined by the complex structures $\mathrm{ad}(X_1)|_{\mathfrak{p}}$ and $\mathrm{ad}(X_2)|_{\mathfrak{p}}$, respectively.
The main object of study in this subproblem turns out to be the group $\mathrm{Aut}(\mathcal{F}_{G/K})$ of orthogonal transformations of $\mathfrak{p}$ that map leaves of $\mathcal{F}_{G/K}$ to leaves of $\mathcal{F}_{G/K}$. Roughly, the reason for this is that, if J_1 and J_2 are two complex structures preserving $\mathcal{F}_{G/K}$ and with corresponding Hopf maps π_1 and π_2, then $\pi_1(\mathcal{F}_{G/K})$ is congruent to $\pi_2(\mathcal{F}_{G/K})$ if and only if there exists $A \in \mathrm{Aut}(\mathcal{F}_{G/K})$ such that $AJ_1A^{-1} = \pm J_2$.
The determination of the group $\mathrm{Aut}(\mathcal{F}_{G/K})$ can be a very difficult task for an arbitrary singular Riemannian foliation on a sphere. Nonetheless, it can be done for the homogeneous isoparametric foliations $\mathcal{F}_{G/K}$, and also for most of the inhomogeneous foliations of codimension one constructed by Ferus, Karcher and Münzner (although in this case the task is more difficult and involves working with Clifford modules). For the case we are here interested in, it happens that $\mathrm{Aut}(\mathcal{F}_{G/K})$ is canonically isomorphic to the group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$ of automorphisms of the Lie algebra $\mathfrak{g}$ that restrict to automorphisms of $\mathfrak{k}$; the isomorphism is simply given by the restriction to $\mathfrak{p}$ of the elements of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{k})$. In particular, the adjoint transformations in $\mathrm{Ad}(K)|_{\mathfrak{p}}$ belong to $\mathrm{Aut}(\mathcal{F}_{G/K})$. This readily implies that every $\sim$-equivalence class interesects any maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. Thus, we can restrict the study of $\sim$ to the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$, which turns out to be much more manageable in view of the following result.

Theorem 2.2. [6] *Let* $\Pi_{\mathfrak{k}}$ *be a set of simple roots for* $\mathfrak{k}$*, and* $\bar{C}$ *the closed Weyl chamber in* $\mathfrak{t}$ *defined by the inequalities* $\alpha \geq 0$*, for all* $\alpha \in \Pi_{\mathfrak{k}}$*. Let* T_1*,* $T_2 \in \mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$*.*
Then $T_1 \sim T_2$ *if and only if there is an autormophism* φ *of the extended Vogan diagram of the pair* $(\mathfrak{g}, \mathfrak{k})$ *such that* $\varphi(T_1) = T_2$*.*

This result requires some explanations. We recall first that the *extended Dynkin diagram* of $\mathfrak{g}$ consists in the Dynkin diagram of $\mathfrak{g}$ together with an extra node corresponding to the lowest root of the root system of $\mathfrak{g}$; this node is linked to the other nodes according to the usual rules of Dynkin diagrams. Now, we define the *extended Vogan diagram* of the pair $(\mathfrak{g}, \mathfrak{k})$ as the extended Dynkin diagram of $\mathfrak{g}$, where the nodes corresponding to noncompact roots are painted while the ones corresponding to compact roots remain unpainted. Due to the Borel-de Siebenthal theorem, one can always assume that there are at most two painted nodes: one simple root of $\mathfrak{g}$ and, maybe, the lowest root. An automorphism of the extended Vogan diagram is a permutation of its nodes preserving the types of edges between nodes and the colours of the nodes. One can show that every automorphism of the extended Vogan diagram determines (in a unique way) an autormorphism $\varphi \colon \mathfrak{t} \to \mathfrak{t}$ of the root system of $\mathfrak{g}$ that restricts to an automorphism of the root system of $\mathfrak{k}$. For the

sake of brevity and since it should not lead to confusion, we did not distinguish between both automorphisms in the statement of Theorem 2.2.
Since for each inner irreducible symmetric space G/K the corresponding extended Vogan diagram is known (see Table 2 in [6] for the pictures), one can completely understand the equivalence relation $\sim$ on the set $\mathcal{J}_{\mathcal{F}_{G/K}} \cap \mathfrak{t}$ and, thus, on $\mathcal{J}_{\mathcal{F}_{G/K}}$. Going through all possible cases of inner symmetric spaces, one can obtain the desired classification of complex structures preserving the irreducible homogeneous isoparametric foliations on S^{2n+1}. As we mentioned earlier, it follows from this classification that the cardinality of $\mathcal{J}_{\mathcal{F}_{G/K}}/\sim$ is at least one for every inner symmetric space G/K, but it is strictly higher than one in several cases. For instance, its value is $1+[\nu/2]+[p+1-\nu]$ for the complex Grassmannian $SU(p+1)/S(U(\nu)U(p+1-\nu))$ (where $[\cdot]$ denotes the integer part of a real number), whereas such cardinality is 2 for the symmetric space $E_6/SU(6)\cdot SU(2)$.

2.3 Homogeneity

Although the arguments above complete the classification of irreducible isoparametric foliations of codimension at least two on complex projective spaces, it is important to know which of the examples in the classification are homogeneous. The first observation is that, if $\mathcal{G}$ is a homogeneous isoparametric foliation on $\mathbb{C}P^n$, then its pullback $\pi^{-1}\mathcal{G} \subset S^{2n+1}$ under the Hopf map is homogeneous as well. This implies that $\pi^{-1}\mathcal{G}$ is induced by some s-representation. In other words, $\pi^{-1}\mathcal{G} = \mathcal{F}_{G/K}$ for some semisimple symmetric space G/K, which must necessarily be inner because of Theorem 2.1. However, one can prove even more:

Theorem 2.3. [6] *Let G/K be an irreducible inner symmetric space of rank at least two, $J = \mathrm{ad}(X)|_{\mathfrak{p}}$ a complex structure preserving $\mathcal{F}_{G/K}$, with $X \in \mathfrak{k}$, and π the Hopf map determined by J. Then $\pi(\mathcal{F}_{G/K})$ is homogeneous if and only if G/K is a Hermitian symmetric space and X belongs to the one-dimensional center of $\mathfrak{k}$.*

Since there are inner non-Hermitian symmetric spaces of rank higher than two, Theorems 2.1 and 2.3 readily imply the existence of inhomogeneous irreducible isoparametric foliations of codimension higher than one on complex projective spaces. However, not for all dimensions does the complex projective space $\mathbb{C}P^n$ admit inhomogeneous irreducible isoparametric foliations. We conclude this article with a result that answers this question in a surprising way.

Theorem 2.4. [6] *Every irreducible isoparametric foliation on $\mathbb{C}P^n$ is homogeneous if and only if $n+1$ is a prime number.*

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