Tese de Doutoramento

RULED HYPERSURFACES AND HOMOGENEOUS SUBMANIFOLDS IN SEMI-RIEMANNIAN MANIFOLDS

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Introduction

Intuitively, symmetry is the property that makes objects look similar when regarded from different perspectives. This is an important quality from the viewpoint of disciplines such us biology, chemistry and art. The concept of symmetry also has a place in the field of mathematics. In this context, it should be noted that symmetry is a property not only applicable to geometric figures, but also to more abstract objects. The study of such objects has led to important results in various areas of mathematics. For instance, Galois theory asserts that if a polynomial equation does not have adequate symmetries, then it is not solvable by radicals. Another example is the well-known Noether theorem, which claims that the symmetries of a physical system are translated into conservation laws.

The notion of symmetry can be defined in a rigorous way by means of group theory. Indeed, given a group G, we say that a certain object is G-symmetric if it is invariant under the action of G. Thus, the notion of symmetry is not an isolated concept, but it is linked to the action of a group, which actually specifies the kind of symmetry that such object has.

In the context of semi-Riemannian geometry, the natural group to consider is the *isometry group*, that is, the group consisting of the transformations of the space that preserve its metric. In the Riemannian setting, where the metric induces a distance, the isometry group turns out to be the group of transformations of the space preserving distance. The action of a subgroup of the isometry group on a given manifold is said to be an *isometric action*. A semi-Riemannian manifold is said to be homogeneous if its isometry group acts transitively on it. A submanifold is called *(extrinsically) homogeneous* if it arises as an orbit of the action of a subgroup of the isometry group.

The problem of classifying isometric actions on a given semi-Riemannian manifold turns out to be very involved. For this reason, it is common to restrict this problem and focus on classifying specific types of isometric actions which are more manageable. For example, transitive actions, in which the only orbit is the manifold itself, have been thoroughly studied in several contexts.

Another special type of isometric actions that has given rise to a fruitful area of research is that of *cohomogeneity one actions*. An isometric action is said to be of cohomogeneity one if it has an orbit of codimension one. Classifying cohomogeneity one actions on a given manifold is equivalent to studying homogeneous hypersurfaces on such manifold. There are not many known results concerning the classification of cohomogeneity one actions on Lorentzian manifolds. One of the main purposes of this thesis is to get a better understanding of this type of actions starting with the simplest Lorentzian manifold: the *Minkowski spacetime*.

However, in the Riemannian setting, several results related to the classification of cohomogeneity one actions have already been obtained. For instance, the problem of classifying cohomogeneity one actions on spaces of constant curvature, namely, Euclidean spaces $\mathbb{R}^n$, spheres S^n and real hyperbolic spaces $\mathbb{R}H^n$, has been achieved by several authors. More specifically, the classification of homogeneous hypersurfaces in Euclidean spaces can be derived from two works of Levi-Civita [49] and Segre [68] which deal with isoparametric hypersurfaces in such spaces. The classification in spheres can be deduced from a work about minimal submanifolds due to Hsiang and Lawson [44], whereas homogeneous hypersurfaces in real hyperbolic spaces have been studied by Cartan in [20].

The classification of homogeneous submanifolds turns out to be much more difficult when the ambient manifold is equipped with a Kähler structure, for example, when dealing with *complex space forms*, namely, complex Euclidean spaces $\mathbb{C}^n$, complex projective spaces $\mathbb{C}P^n$ and complex hyperbolic spaces $\mathbb{C}H^n$. However, the classification of homogeneous hypersurfaces in these spaces has been successfully achieved. In particular, the classification in the projective case has been obtained by Takagi in [70], whereas homogeneous hypersurfaces in complex hyperbolic spaces have been classified by Berndt and Tamaru in [15].

A large part of this thesis focuses on the study of submanifold geometry in the context of nonflat complex space forms, in which the underlying complex structure plays an important role. In general, in the Kähler setting, it is possible to define the notions of both totally real and complex submanifold, which depend on the complex structure of the ambient manifold. The notion of *CR submanifold* constitutes a generalization of these two concepts. In this work we study homogeneous CR submanifolds in complex hyperbolic spaces. There exist several known examples of homogeneous CR submanifolds in $\mathbb{C}H^n$ which motivate this problem, for instance the so-called *Berndt-Brück submanifolds* [11], and in particular, the *Lohnherr hypersurface* [50].

The Lohnherr hypersurface of $\mathbb{C}H^n$ satisfies interesting properties. For example, it can be characterized as the only homogeneous minimal hypersurface in the complex hyperbolic space [13]. It is also the only complete hypersurface of $\mathbb{C}H^n$ having constant principal curvatures that is *ruled* [51]. The notion of ruled real hypersurface in a complex space form is intimately related to the complex structure of the ambient space. Several results concerning ruled real hypersurfaces in nonflat complex space forms have been obtained. In this work, we present some classification results of ruled real hypersurfaces in nonflat complex space forms satisfying some important additional geometric properties.

We now present the main contributions and goals of this thesis.

Ruled real hypersurfaces in nonflat complex space forms

A *ruled real hypersurface* in a nonflat complex space form, namely $\mathbb{C}P^n$ or $\mathbb{C}H^n$, is a submanifold of real codimension one that is (locally) foliated by totally geodesic complex hypersurfaces of the ambient space, namely $\mathbb{C}P^{n-1}$ or $\mathbb{C}H^{n-1}$, respectively. Ruled hypersurfaces in nonflat complex space forms constitute a large class of real hypersurfaces, so it becomes an interesting problem to classify these objects under some additional geometric assumptions. For instance, Lohnherr and Reckziegel have classified ruled minimal hypersurfaces in nonflat complex space forms into three classes [51]: Kimura-type hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$, bisectors in $\mathbb{C}H^n$, or Lohnherr hypersurfaces in $\mathbb{C}H^n$.

In Chapter 2 we present the classification of ruled real hypersurfaces in nonflat complex space forms that satisfy some additional properties related to the constancy of its higher order mean curvatures. The higher order mean curvatures of a given hypersurface are defined to be the elementary symmetric polynomials in the principal curvatures of such hypersurface. Any ruled real hypersurface in a nonflat complex space form is known to have only two nonzero principal curvatures, say α and β, so there exist only two nontrivial elementary symmetric polynomials, which turn out to be the (first order) mean curvature $\alpha + \beta$ and the second order mean curvature $\alpha\beta$. In Section 2.3 we study ruled real hypersurfaces in nonflat complex space forms with constant mean curvature, finding that all the examples are minimal, and hence deriving their complete classification. Ruled real hypersurfaces with constant second order mean curvature in nonflat complex space forms have been characterized in [48].

The squared norm of the shape operator, $\alpha^2 + \beta^2$, which can be expressed in a simple way in terms of the mean curvatures of first and second orders, constitutes another classical geometric invariant of hypersurfaces. Thus, it seems natural to pose the question: what happens with ruled real hypersurfaces in nonflat complex space forms whose shape operators have constant norm? This is what we study in Section 2.4, obtaining a complete classification which includes a new inhomogeneous example.

Finally, motivated by a recent result due to Sasahara [67], where biharmonic ruled hypersurfaces in complex projective spaces are classified, we study these objects in the general context of nonflat complex space forms. This is settled in Section 2.5, where we prove that such hypersurfaces must be minimal, from where their classification follows.

Homogeneous CR submanifolds in complex hyperbolic spaces

A submanifold of a Hermitian manifold is said to be a *CR (Cauchy-Riemann or complex-real) submanifold* if its maximal holomorphic tangent subspaces define a distribution and its complementary distribution is totally real. In other words, a CR submanifold of a Hermitian manifold is a submanifold satisfying that the tangent space at each point can be decomposed into an orthogonal direct sum of a totally real subspace and a complex one. This notion has been introduced by Bejancu in [10], and generalizes the concepts of both totally real and complex submanifolds.

In this work we are mainly interested in the classification of homogeneous CR submanifolds in complex hyperbolic spaces. Our motivation comes from the fact that this kind of submanifolds include several special examples of interest in the context of Hermitian symmetric spaces, such as real hypersurfaces, Kähler or Lagrangian submanifolds, among others.

For instance, the classification of homogeneous real hypersurfaces, or equivalently, of cohomogeneity one actions, in complex hyperbolic spaces has been shown to be a very involved problem that has been successfully solved by Berndt and Tamaru in [15]. Homogeneous Kähler submanifolds in complex hyperbolic spaces have also been classified by Di Scala, Ishi and Loi in [26], finding that the only examples are totally geodesic complex hyperbolic subspaces.

Lagrangian submanifolds, that is, totally real submanifolds of maximal dimension, constitute a nice particular case of CR submanifolds. Classifying homogeneous Lagrangian submanifolds in complex hyperbolic spaces seems to be a very involved problem due to the noncompactness of the isometry group of the ambient space. However, some partial results have been achieved. For instance, Hashinaga and Kajigaya have obtained in [43] the classification of homogeneous Lagrangian submanifolds that arise as orbits of a subgroup of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$.

In view of the above results, we restrict our attention to classifying homogeneous CR submanifolds in complex hyperbolic spaces that arise as orbits of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$. We point out that our classification includes uncountably many congruence classes of examples, some of them of particular importance, such as some Berndt-Brück submanifolds or certain orbits of polar actions. This is accomplished in Chapter 3.

Cohomogeneity one actions on the 4-dimensional Minkowski spacetime

One of the main motivations of this thesis is to study isometric actions on Lorentzian manifolds. In the context of Lorentzian geometry, the *Minkowski spacetime* $\mathbb{L}^{n+1}$, that is, the Lorentzian analog to the Euclidean space, is the simplest example of manifold. From the physical viewpoint, the particular case of the four-dimensional Minkowski spacetime constitutes an important example of Lorentzian manifold since it is the space which models the Theory of Special Relativity.

Several results concerning the classification of isometric actions on Lorentzian manifolds have been achieved. For example, Adams and Stuck studied transitive isometric actions on Lorentzian manifolds in [2] and [3]. In this thesis we are mainly interested in the particular case of *cohomogeneity one actions*.

In the Riemannian setting it is customary to assume that the actions are *proper* due to the nice properties that their isotropy groups and orbits satisfy. For example, isotropy groups are compact, orbits are closed embedded submanifolds and the set of orbits is a Hausdorff space. Proper cohomogeneity one actions on Minkowski spacetimes have been investigated, for example, in [4]. However, in the Lorentzian case, there exist simple examples which motivate the study of nonproper actions. For instance, the natural action of the Lie group $SO^0(1, n)$ on the $(n + 1)$-dimensional

Minkowski spacetime is not proper: indeed, the past and future lightcones are orbits of this action, but they are not closed submanifolds. Thus, we will not assume the actions to be proper. Then, a (not necessarily proper) isometric action is said to be of cohomogeneity one if the minimum possible codimension of an orbit is one. A classification of cohomogeneity one actions on the Minkowski spacetimes of dimensions two and three has been achieved in [14]. In Chapter 4, we present the corresponding classification in the four-dimensional case. Moreover, we derive a splitting theorem and some structural results for cohomogeneity one actions on Minkowski spacetimes of arbitrary dimensions.

Structure of the thesis

This thesis is organized as follows.

Chapter 1 is devoted to introducing the basic notation and terminology needed for this thesis. More precisely, we introduce the notion of semi-Riemannian manifold (Section 1.1), some important tools in the setting of submanifold geometry (Section 1.2), the main concepts for studying isometric actions (Section 1.3) and the construction of the Iwasawa decomposition of a semisimple Lie group (Section 1.4). We finish this chapter by presenting the construction and description of nonflat complex space forms (Section 1.5), focusing on the algebraic description of complex hyperbolic spaces, as well as Minkowski spacetimes (Section 1.6).

The original contributions of this thesis are presented in Chapters 2, 3, and 4.

Chapter 2 is devoted to classifying ruled real hypersurfaces satisfying some additional geometric properties in complex projective and hyperbolic spaces. In order to do so, we will firstly recall some basic definitions and known results concerning ruled hypersurfaces in nonflat complex space forms (Section 2.1) and compute the Levi-Civita connection of an arbitrary ruled real hypersurface in this kind of spaces (Section 2.2). After that, we present our classification results. In particular, we study ruled real hypersurfaces in nonflat complex space forms that have constant mean curvature (Section 2.3), whose shape operators have constant norm (Section 2.4), and finally, those ones that are biharmonic (Section 2.5).

In Chapter 3 we investigate homogeneous CR submanifolds in complex hyperbolic spaces that arise as orbits of the solvable part AN of the Iwasawa decomposition of the isometry group of this symmetric space. We start by introducing the definition of CR submanifold of a Hermitian manifold (Section 3.1) and by characterizing homogeneous CR submanifolds in Hermitian symmetric spaces of noncompact type in terms of Lie algebras (Section 3.2). The rest of this chapter is devoted to presenting the classification of this type of submanifolds. More specifically, we firstly study the subgroups of AN that produce a CR orbit (Subsection 3.3.1), and after that, we decide whether the remaining orbits of such subgroups are CR submanifolds or not (Subsection 3.3.2). Finally, we study the congruence classes of the examples that we have previously obtained (Section 3.4).

Finally, Chapter 4 is devoted to studying cohomogeneity one actions on Minkowski spacetimes. We will firstly recall some known results needed for this study (Section 4.1) and give an alternative proof for the classical classification result of cohomogeneity one actions on Euclidean spaces of arbitrary dimension (Section 4.2). To finish, we derive some structural results on cohomogeneity one actions on Minkowski spacetimes of arbitrary dimension, and present the classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime up to orbit equivalence (Section 4.3).

Chapter 1

Preliminaries

This first chapter is devoted to introducing the basic concepts, notation and known results that we are going to use throughout this thesis. In Section 1.1 we recall the definition of semi-Riemannian manifold and fix our sign convention for the curvature tensor. Section 1.2 is devoted to reviewing the main concepts and equations of submanifold geometry needed for this thesis. In Section 1.3 we present the basic terminology related to isometric actions on a semi-Riemannian manifold. In Section 1.4 we briefly recall some fundamental facts concerning the theory of semisimple Lie groups. Moreover, we present the Iwasawa decomposition of the semisimple Lie algebras $\mathfrak{so}(1,n)$ and $\mathfrak{su}(1,n)$. Section 1.5 is devoted to introducing both the description and construction of the nonflat complex space forms, namely complex projective and hyperbolic spaces. Finally, in Section 1.6 we settle the main notation concerning the Minkowski spacetime $\mathbb{L}^{n+1}$.

1.1 Semi-Riemannian manifolds

Let M be a smooth manifold of dimension n. For each $p \in M$, T_pM will denote the tangent space of M at p. The tangent bundle to M is denoted by TM and $\Gamma(TM)$ is the module of smooth vector fields on M. If $\mathcal{D}$ is a distribution along M, then $\Gamma(\mathcal{D})$ will denote the module of sections of $\mathcal{D}$, that is, those vector fields $X \in \Gamma(TM)$ satisfying that $X_p \in \mathcal{D}_p$ for every $p \in M$.

Let T denote a symmetric bilinear tensor of type (0,2) in a given vector space V. T is said to be symmetric if $T(x,y) = T(y,x)$ for all x, $y \in V$, and nondegenerate if $T(x,y) = 0$ for each $y \in V$ implies that $x = 0$. As a symmetric nondegenerate tensor, T is linearly congruent to a diagonal matrix of the form $\operatorname{diag}(-1, \overset{r}{\ldots}, -1, 1, \overset{s}{\ldots}, 1)$. The signature of T is precisely the pair (r,s).

Let V denote a vector space equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. An element $v \in V$ is said to be spacelike, timelike or lightlike if $\langle v, v \rangle$ is positive, negative or zero, respectively. For each $v \in V$, we write $|v| = \sqrt{|\langle v, v \rangle|}$. If U and W are subspaces of V, we denote $U \ominus W = \{u \in U : \langle u, w \rangle = 0, \text{ for all } w \in W\}$. Notice that, in particular, if $\langle \cdot, \cdot \rangle$ is positive definite, this notation stands for the orthogonal complement of W in U.

A *semi-Riemannian manifold* is a pair $(M, \langle \cdot, \cdot \rangle)$, where M is a smooth manifold and $\langle \cdot, \cdot \rangle$ is a nondegenerate symmetric bilinear tensor field of type $(0,2)$ and constant signature. This fact means that, in particular, at each point $p \in M$, the tangent space T_pM is endowed with a nondegenerate symmetric bilinear tensor $\langle \cdot, \cdot \rangle_p$. M is said to be a *Riemannian manifold* if its signature is $(0, n)$. If the signature of M is $(1, n-1)$, it is called a *Lorentzian manifold*.

In the setting of semi-Riemannian geometry, one of the most important concepts is the curvature. The curvature information of a semi-Riemannian manifold is codified in its *curvature tensor* R, which is a tensor of type (1,3) that we define with the following sign convention:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ denotes the Levi-Civita connection of M, that is, the unique torsion-free metric connection on M.

A semi-Riemannian manifold M is said to be *flat* if its curvature tensor vanishes identically. It is said to have *constant curvature* c if its curvature tensor can be written as $R(X,Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$ for every X, Y, $Z \in \Gamma(TM)$. The only connected, complete, simply connected Riemannian manifolds having constant curvature are the so-called *(real) space forms*, that is, Euclidean spaces $\mathbb{R}^n$ $(c = 0)$, spheres S^n $(c > 0)$ and real hyperbolic spaces $\mathbb{R}H^n$ $(c < 0)$.

1.2 Submanifold geometry

This section is devoted to introducing the main definitions and fundamental formulas for studying submanifolds of a given semi-Riemannian manifold. For more information on this topic, we refer to [12, Chapters 1 and 10] for the Riemannian case and to [59, Chapter 4] for the case of arbitrary signature.

Let $(\bar{M}, \langle \cdot, \cdot \rangle)$ be a semi-Riemannian manifold and M an embedded submanifold of $\bar{M}$ in such a way that the restriction of $\langle \cdot, \cdot \rangle$ to M is nondegenerate. Then, M is called a semi-Riemannian submanifold of $\bar{M}$. We denote by νM and $\Gamma(\nu M)$ the normal bundle of M and the module of normal vector fields to M, respectively. At each point $p \in M$, the canonical orthogonal decomposition $T_p\bar{M} = T_pM \oplus \nu_p M$ holds. In this work, the symbol $\oplus$ will denote direct sum (not necessarily orthogonal direct sum). Moreover, if $X \in \Gamma(T\bar{M})$ is a vector field along M, then $X^\top$ and $X^\perp$ will denote the orthogonal projections of X onto TM and νM, respectively.

In this thesis, we are mostly interested in studying the extrinsic geometry of semi-Riemannian submanifolds, which refers to the geometry of such submanifolds in relation to the geometry of the ambient manifold.

We denote by $\bar{\nabla}$ and $\bar{R}$ the Levi-Civita connection and curvature tensor of $\bar{M}$, and by ∇ and R the corresponding objects for M, respectively. With the notation above, one can decompose $\bar{\nabla}_X Y$ into its tangent and normal parts. The tangent part $(\bar{\nabla}_X Y)^\top$ turns out to be the Levi-Civita connection of M whereas the normal part defines the *second fundamental form* $I\!I$ of M. Thus, we have an orthogonal decomposition

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X,Y),$$

for every X, Y, $Z \in \Gamma(TM)$, which is known as the *Gauss formula*. Let $\xi \in \Gamma(\nu M)$ a normal vector field to M. The *shape operator* of M with respect to ξ is the operator

$\mathcal{S}_\xi$ on M defined by $\langle \mathcal{S}_\xi X, Y\rangle = \langle I\!I(X,Y), \xi\rangle$ for any X, $Y \in \Gamma(TM)$. Moreover, if we denote by $\nabla^\perp$ the normal connection of M, that is, $\nabla^\perp_X \xi = (\nabla_X \xi)^\perp$ for every $X \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$, then we have the following orthogonal decomposition

$$\bar{\nabla}_X \xi = -\mathcal{S}_\xi X + \nabla^\perp_X \xi,$$

which is known as the *Weingarten formula*.

The *Gauss equation* gives the relation between the curvature tensors of $\bar{M}$ and M by means of the second fundamental form and, for X, Y, Z, $W \in \Gamma(TM)$, it reads

$$\langle \bar{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle - \langle I\!I(Y,Z), I\!I(X,W)\rangle + \langle I\!I(X,Z), I\!I(Y,W)\rangle.$$

In this work we will also use the *Codazzi equation*,

$$\langle \bar{R}(X,Y)Z, \xi\rangle = \langle (\nabla^\perp_X I\!I)(Y,Z) - (\nabla^\perp_Y I\!I)(X,Z), \xi\rangle,$$

where $\xi \in \Gamma(\nu M)$ and the covariant derivative of the second fundamental form is given by

$$(\nabla^\perp_X I\!I)(Y,Z) = \nabla^\perp_X I\!I(Y,Z) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

The last of the three fundamental equations of second order in submanifold geometry is the *Ricci equation*,

$$\langle R^\perp(X,Y)\xi, \eta\rangle = \langle R(X,Y)\xi, \eta\rangle + \langle [\mathcal{S}_\xi, \mathcal{S}_\eta]X, Y\rangle,$$

where X, $Y \in \Gamma(TM)$, ξ, $\eta \in \Gamma(\nu M)$, and $R^\perp$ denotes the curvature tensor of the normal bundle to M, defined by $R^\perp(X,Y)\xi = [\nabla^\perp_X, \nabla^\perp_Y]\xi - \nabla^\perp_{[X,Y]}\xi$.

A submanifold whose second fundamental form $I\!I$ vanishes identically is said to be *totally geodesic*. This is equivalent to saying that every geodesic in M is also a geodesic in $\bar{M}$.

The *mean curvature vector field* $\mathcal{H}$ of a semi-Riemannian submanifold is defined as the trace of the second fundamental form. In terms of a local orthonormal basis $\{E_i\}_i$, one may write $\mathcal{H} = \sum_i \langle E_i, E_i\rangle I\!I(E_i, E_i)$. The norm of the mean curvature vector field $|\mathcal{H}|$ is commonly called the *mean curvature function*. A submanifold is said to be *minimal* if its mean curvature function vanishes.

Two semi-Riemannian submanifolds M_1 and M_2 of $\bar{M}$ are said to be *congruent* if there exists an isometry of $\bar{M}$ which takes M_1 onto M_2.

Geometry of hypersurfaces

Assume now that M is a *hypersurface* of $\bar{M}$, that is, a submanifold of codimension one. Since we keep on assuming that M is a nondegenerate submanifold, it follows that, locally and up to sign, there exists a unique normal vector field $\xi \in \Gamma(\nu M)$ with $\epsilon := \langle \xi, \xi\rangle \in \{-1, 1\}$. We will write $\mathcal{S} = \mathcal{S}_\xi$ to denote the shape operator of M with respect to ξ. In this case, the Gauss and Weingarten formulas can be written as

$$\bar{\nabla}_X Y = \nabla_X Y + \epsilon\langle \mathcal{S}X, Y\rangle \xi, \qquad \bar{\nabla}_X \xi = -\mathcal{S}X.$$

Thus, Gauss and Codazzi equations reduce to

$$\begin{aligned}\langle \bar{R}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle - \epsilon\langle \mathcal{S}Y,Z\rangle\langle \mathcal{S}X,W\rangle + \epsilon\langle \mathcal{S}X,Z\rangle\langle \mathcal{S}Y,W\rangle,\\ \langle \bar{R}(X,Y)Z,\xi\rangle &= \langle (\nabla_X \mathcal{S})Y - (\nabla_Y \mathcal{S})X, Z\rangle,\end{aligned}$$

whereas the Ricci equation does not give further information for hypersurfaces.

When dealing with hypersurfaces of a semi-Riemannian manifold, the second fundamental form is a multiple of ξ, and hence, the mean curvature is proportional to ξ. Thus, we will usually talk about the *mean curvature* of the hypersurface, which is defined as the trace of its shape operator $\mathcal{S}$.

Let ξ be a unit normal vector field defined on an open subset $\mathcal{U}$ of the hypersurface M. Given $p \in \mathcal{U}$, we say that $\lambda \in \mathbb{R}$ is a *principal curvature at* p if there exists a tangent vector $X \in T_p\mathcal{U}$ such that $\mathcal{S}_\xi X = \lambda X$. In this case, X is said to be a *principal curvature vector at* p, and the eigenspace of λ, which we will denote by $T_\lambda(p)$, is commonly called the *principal curvature space* associated with λ. A continuous function $\lambda\colon \mathcal{U} \to \mathbb{R}$ is called a *principal curvature (function)* of M on $\mathcal{U}$ if $\lambda(p)$ is a principal curvature at p for any $p \in \mathcal{U}$. If each principal curvature space has constant dimension on $\mathcal{U}$, then there exists an orthonormal frame consisting of principal curvature vector fields. If $\bar{M}$ is a Riemannian manifold, the shape operator $\mathcal{S}$ is diagonalizable at every point since it is a self-adjoint operator and the metric is positive definite. In this case, the multiplicity of a principal curvature λ is defined to be the dimension of its associated principal curvature space, or equivalently, the multiplicity of λ as a eigenvalue of the shape operator.

A connected hypersurface is said to have *constant principal curvatures* if the eigenvalues of the shape operator are the same at every point. In this case, if the ambient manifold is Riemannian, the multiplicities of the principal curvatures are constant.

1.3 Isometric actions

In this section we briefly recall the main terminology and notation for the study of isometric actions on semi-Riemannian manifolds. For more information, we refer to [6, Chapter 3], [12, Chapter 2] and [59, Chapter 9].

Let $\bar{M}$ denote a semi-Riemannian manifold and G a Lie group. An *isometric action* of G on $\bar{M}$ is a smooth map

$$\varphi\colon G \times \bar{M} \to \bar{M}, \quad (g,p) \mapsto \varphi(g,p) = gp,$$

satisfying the following properties:

(i) $(g\tilde{g})p = g(\tilde{g}p)$, for all g, $\tilde{g} \in G$ and $p \in \bar{M}$;

(ii) $ep = p$ for any $p \in \bar{M}$, where e is the identity element of G;

(iii) the map $\varphi_g\colon \bar{M} \to \bar{M}$ defined by $\varphi_g(p) = gp$ is an isometry of $\bar{M}$ for each $g \in G$.

For each $p \in \bar{M}$, the *orbit* of the action of G through p is

$$G \cdot p = \{gp : g \in G\},$$

and the *isotropy group* or *stabilizer* at p is defined as

$$G_p = \{g \in G : gp = p\}.$$

If $G \cdot p = \bar{M}$ for some $p \in \bar{M}$, and hence for each $p \in \bar{M}$, then φ is said to be a *transitive* action. In this case, we say that $\bar{M}$ is a *homogeneous G-space*. The action is said to be *trivial* if each point in $\bar{M}$ is a fixed point, i.e. $G \cdot p = \{p\}$ for all $p \in \bar{M}$.

Let $I(\bar{M})$ denote the isometry group of $\bar{M}$, which is known to be a Lie group (see [57] and [59, Chapter 9, Theorem 32]). Then, we have a Lie group homomorphism $\rho\colon G \to I(\bar{M})$ given by $\rho(g) = \varphi_g$. If ρ is an injective map, the action is said to be *effective*, which means that the Lie group G is isomorphic to a subgroup of $I(\bar{M})$. The action is called *free* if for every $p \in \bar{M}$ and every g, $h \in G$, the equality $gp = hp$ implies $g = h$. If the action is free and transitive, we will say that G acts *simply transitively* on $\bar{M}$.

Two isometric actions $G \times \bar{M} \to \bar{M}$ and $G' \times \bar{M}' \to \bar{M}'$ are said to be *conjugate* or *equivalent* if there exists a Lie group isomorphism $\psi\colon G \to G'$ and an isometry $f\colon \bar{M} \to \bar{M}'$ in such a way that $f(gp) = \psi(g)f(p)$, for each $p \in \bar{M}$ and $g \in G$. We say that they are *orbit equivalent* if there exists an isometry $f\colon \bar{M} \to \bar{M}'$ which maps the orbits of the G-action on $\bar{M}$ onto the orbits of the G'-action on $\bar{M}'$. Clearly, two conjugate actions are orbit equivalent.

Each orbit $G \cdot p$ of an isometric action $G \times \bar{M} \to \bar{M}$ is a (generally immersed) submanifold of $\bar{M}$. One may study the intrinsic geometry of this orbit with the induced metric. However, we will be mostly interested in studying the geometry of the orbit $G \cdot p$ in relation to the geometry of $\bar{M}$, that is, its extrinsic geometry. An *(extrinsically) homogeneous submanifold* of $\bar{M}$ is an orbit of an isometric action on $\bar{M}$. With respect to the induced (possibly degenerate) metric, each orbit $G \cdot p$ is a homogeneous space $G \cdot p = G/G_p$ on which G acts transitively by isometries.

Any isometric action induces certain (pseudo-)orthogonal representations in a natural way. Recall that a representation of a Lie group G on a vector space V is a Lie group homomorphism $\rho\colon G \to GL(V)$, and that, if V is endowed with a nondegenerate inner product, ρ is said to be (pseudo-)orthogonal if $\rho(g)$ is a (pseudo-)orthogonal transformation of V for any $g \in G$.

Let $\varphi\colon G \times \bar{M} \to \bar{M}$ be an isometric action on $\bar{M}$ and consider $p \in \bar{M}$. Since the isotropy group G_p fixes p and leaves the orbit $G \cdot p$ invariant, the differential of each isometry $\varphi_g\colon \bar{M} \to \bar{M}$, $p \mapsto gp$, for $g \in G_p$, leaves both the tangent space $T_p(G \cdot p)$ and the normal space $\nu_p(G \cdot p)$ invariant. The action

$$G_p \times T_p(G \cdot p) \to T_p(G \cdot p), \quad (g, X) \mapsto (\varphi_g)_{*p} X,$$

is called the *isotropy representation* of the action φ at p, and

$$G_p \times \nu_p(G \cdot p) \to \nu_p(G \cdot p), \quad (g, \xi) \mapsto (\varphi_g)_{*p} \xi,$$

is said to be the *slice representation* of the action φ at p. If $G \cdot p$ is a semi-Riemannian submanifold, these representations are pseudo-orthogonal with respect to the induced inner products on $T_p(G \cdot p)$ and $\nu_p(G \cdot p)$.

Let $\bar{M}/G$ denote the set of orbits of the action of G on $\bar{M}$ and equip $\bar{M}/G$ with the quotient topology relative to the canonical projection $\bar{M} \to \bar{M}/G$, $p \mapsto G \cdot p$. In general, $\bar{M}/G$ is not a Hausdorff space. In order to avoid this unpleasant behavior, the particular type of proper isometric actions has been introduced. An isometric action of G on $\bar{M}$ is said to be *proper* if, for any two points $p, q \in \bar{M}$, there exist open neighborhoods $\mathcal{U}_p$ and $\mathcal{U}_q$ of p and q in $\bar{M}$, respectively, in such a way that $\{g \in G : g\mathcal{U}_p \cap \mathcal{U}_q \neq \emptyset\}$ is relatively compact in G. An equivalent definition is that the map

$$G \times \bar{M} \to \bar{M} \times \bar{M}, \quad (g, p) \mapsto (p, gp)$$

is a proper map, that is, the inverse image of each compact set in $\bar{M} \times \bar{M}$ is also compact in $G \times \bar{M}$. Every compact Lie group action is proper. If G is a Lie subgroup of $I(\bar{M})$ and $\bar{M}$ is a Riemannian manifold, then the G-action is proper if and only if G is closed in $I(\bar{M})$. Proper actions satisfy nice properties. For example, the quotient space $\bar{M}/G$ is Hausdorff, each isotropy group G_p is compact, and every orbit $G \cdot p$ is closed, and hence an embedded submanifold of $\bar{M}$ [56].

One can distinguish three different types of orbits of a proper action. Let $G \cdot p$ be an orbit, for some $p \in \bar{M}$. If for each $q \in \bar{M}$ the isotropy group G_p is conjugate in G to some subgroup of G_q, then $G \cdot p$ is called a *principal orbit*. The union of all the principal orbits is an open and dense subset of $\bar{M}$. Any principal orbit has maximal dimension. An orbit of maximal dimension which is not principal is said to be an *exceptional orbit*. A *singular orbit* is an orbit whose dimension is lower than the dimension of a principal orbit. The *cohomogeneity* of a proper action is defined to be the codimension of a principal orbit.

Restricting to the study of proper actions in the semi-Riemannian setting does not seem to be a natural assumption. For instance, the action of the connected component of the identity of $SO(1, n)$ (see Subsection 1.4.2) on the $(n+1)$-dimensional Minkowski spacetime $\mathbb{L}^{n+1}$ is a natural action which, nevertheless, is not proper. Indeed, this action has four types of orbits: a fixed point, the past and future lightcones, and the hyperbolic and De Sitter spaces (see Section 1.6). Since the lightcones are not closed in $\mathbb{L}^{n+1}$, one deduces that this action cannot be proper. In any case, we can consider a natural notion of cohomogeneity even in the semi-Riemannian setting. Thus, a not necessarily proper isometric action of G on a semi-Riemannian manifold $\bar{M}$ is said to be of *cohomogeneity* k if the minimum codimension of the orbits of such action is k.

Finally, let us comment on an important kind of isometric actions on the particular setting of Riemannian manifolds, namely the class of polar actions. A proper isometric action of a connected Lie group G on a Riemannian manifold $\bar{M}$ is called *polar* if there exists an immersed submanifold Σ of $\bar{M}$ satisfying:

(i) Σ intersects every G-orbit, and

(ii) for any $p \in \Sigma$, the tangent space of Σ at p, $T_p\Sigma$, and the tangent space of the orbit through p at p, $T_p(G \cdot p)$, are orthogonal.

Under these conditions, the submanifold Σ is called a *section* of the G-action, and it is always a totally geodesic submanifold of $\bar{M}$. Any polar action admits sections through any given point. Every cohomogeneity one action on a Riemannian manifold is polar.

1.4 Semisimple Lie algebras

The main purpose of this section is to briefly recall some basic definitions and properties concerning semisimple Lie groups and Lie algebras, with a focus on their Iwasawa decomposition. We refer the reader to [12], [45] and [46] to get further information on the Iwasawa decomposition of a semisimple Lie algebra. Subsections 1.4.1 and 1.4.2 below will be devoted to describing the Iwasawa decompositions of two well-known examples of semisimple Lie algebras that will play an important role in this thesis: $\mathfrak{so}(1,n)$ and $\mathfrak{su}(1,n)$.

We firstly fix some notation concerning Lie groups and Lie algebras. Given a Lie group G, we will denote its associated Lie algebra by the corresponding gothic letter $\mathfrak{g}$. Let Exp denote the Lie exponential map. For each $g \in G$, the conjugation map is $\mathrm{I}_g \colon G \to G$, $h \mapsto ghg^{-1}$. We will denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, that is, the linear bijective transformations $\varphi \colon \mathfrak{g} \to \mathfrak{g}$ satisfying $\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$, for any X, $Y \in \mathfrak{g}$. Then, the Lie group adjoint map $\mathrm{Ad} \colon G \to \mathrm{Aut}(\mathfrak{g})$ is defined by $\mathrm{Ad}(g) = (\mathrm{I}_g)_{*e}$, where $g \in G$ and e denotes the identity element of G. Furthermore, the differential of Ad at e defines the Lie algebra adjoint map $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$, $X \mapsto \mathrm{ad}(X) = [X, \cdot]$. Moreover, the following relation holds:

$$\mathrm{Ad}(\mathrm{Exp}(X))(Y) = e^{\mathrm{ad}(X)}(Y) = \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k(X)(Y)}{k!}.$$

The *Killing form* of $\mathfrak{g}$ is the bilinear map $\mathcal{B} \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $\mathcal{B}(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$, for each $X, Y \in \mathfrak{g}$. Note that, given any automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$, then $\mathcal{B}(\sigma(X), \sigma(Y)) = \mathcal{B}(X,Y)$.

From now on, we will suppose that $\mathfrak{g}$ is a semisimple Lie algebra, which means that the Killing form $\mathcal{B}$ is nondegenerate. A *Cartan involution* is an involutive homomorphism $\theta \colon \mathfrak{g} \to \mathfrak{g}$ in such a way that the map given by $\mathcal{B}_\theta(X,Y) = -\mathcal{B}(\theta X, Y)$ is a positive definite inner product on $\mathfrak{g}$. It follows immediately that this inner product satisfies $\mathcal{B}_\theta(\mathrm{ad}(X)Y, Z) = -\mathcal{B}_\theta(X, \mathrm{ad}(\theta X)Z)$, for any X, Y, $Z \in \mathfrak{g}$. Any semisimple Lie algebra admits a Cartan involution. As any involution, the Cartan involution θ has two eigenvalues, $+1$ and -1. Let us denote by $\mathfrak{k}$ and $\mathfrak{p}$ the eigenspaces associated with these eigenvalues, respectively. Then, the Lie algebra $\mathfrak{g}$ can be rewritten as the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is known as the Cartan decomposition of $\mathfrak{g}$. It is known that $\mathfrak{k}$ turns out to be the Lie algebra of a maximal compact Lie subgroup K of G, and that $\mathfrak{p}$ is its orthogonal complement in $\mathfrak{g}$ with respect to $\mathcal{B}$. Moreover, the restriction to $\mathfrak{k}$ of the Killing form is negative definite, the corresponding restriction to $\mathfrak{p}$ is positive definite, and the following relations hold:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{1.1}$$

Now, we consider a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ its dual space. Given $H \in \mathfrak{a}$ and $X, Y \in \mathfrak{g}$, we have that $\mathcal{B}_\theta(\operatorname{ad}(H)X, Y) = \mathcal{B}_\theta(X, \operatorname{ad}(H)Y)$, which implies that each operator $\operatorname{ad}(H) \in \operatorname{End}(\mathfrak{g})$ is self-adjoint with respect to the inner product $\mathcal{B}_\theta$. Moreover, if H_1, $H_2 \in \mathfrak{a}$, then $[\operatorname{ad}(H_1), \operatorname{ad}(H_2)] = \operatorname{ad}([H_1, H_2]) = 0$, since $\mathfrak{a}$ is abelian. Thus, $\{\operatorname{ad}(H) : H \in \mathfrak{a}\}$ is a commuting family of self-adjoint endomorphisms of $\mathfrak{g}$. In particular, they diagonalize simultaneously. The common eigenspaces are said to be the *restricted root spaces* and their associated nonzero eigenvalues are called the *restricted roots* of $\mathfrak{g}$. In other words, if for each covector $\lambda \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\},$$

then any $\mathfrak{g}_\lambda \neq 0$ is a restricted root space, and any $\lambda \neq 0$ such that $\mathfrak{g}_\lambda \neq 0$ is a restricted root. Notice that $\mathfrak{g}_0 \neq 0$ since $\mathfrak{a} \subset \mathfrak{g}_0$, and that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, where $\mathfrak{k}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let Σ be the set of restricted roots of $\mathfrak{g}$. Then, given $\lambda \in \Sigma$, we have that $\theta\mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$. In particular, $\lambda \in \Sigma$ if, and only if, $-\lambda \in \Sigma$. Moreover, we have the bracket relation $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$ for any λ, $\mu \in \mathfrak{a}^*$. One can consider the *restricted root space decomposition* of $\mathfrak{g}$, that is, the direct sum of vector subspaces defined by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right),$$

which is an orthogonal decomposition with respect to $\mathcal{B}_\theta$.

We now choose a criterion of positivity on the set of restricted roots by defining a root to be positive if it lies at the same side of a hyperplane in $\mathfrak{a}^*$ which does not contain any root. Let Σ^+ be the set of positive roots. We define

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda,$$

which turns out to be a nilpotent Lie subalgebra of $\mathfrak{g}$ by virtue of the properties of the root space decomposition of $\mathfrak{g}$. Moreover, $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$ since $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$ is nilpotent. The *Iwasawa decomposition theorem* states that, at the level of Lie algebras,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

is a direct sum of vector spaces (but not an orthogonal direct sum), and at the level of Lie groups, that there exists an analytic diffeomorphism $K \times A \times N \to G$, $(k, a, n) \mapsto kan$, where A and N denote the connected and simply connected subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Since $\mathfrak{a}$ normalizes $\mathfrak{n}$, the semidirect product AN is the connected Lie subgroup of G whose Lie algebra is $\mathfrak{a} \oplus \mathfrak{n}$. Furthermore, since A is abelian and N is nilpotent, and both are simply connected, then each of them is diffeomorphic to a Euclidean space [46, Theorem 1.127]. Then, AN is also diffeomorphic to a Euclidean space.

1.4.1 The Iwasawa decomposition of $\mathfrak{so}(1,n)$

We now give an explicit description of the Iwasawa decomposition of the Lie algebra of the semisimple Lie group

$$SO(1,n) = \{A \in GL(n+1,\mathbb{R}) : AI_{1,n}A^t = I_{1,n},\ \det(A) = 1\},$$

where $I_{1,n}$ denotes the diagonal matrix $I_{1,n} = \mathrm{diag}(-1,1,\dots,1)$ and A^t is the transpose matrix of A. Its associated semisimple Lie algebra is

$$\begin{aligned}\mathfrak{so}(1,n) &= \{X \in \mathfrak{gl}(n+1,\mathbb{R}) : XI_{1,n} + I_{1,n}X^t = 0,\ \mathrm{tr}(X) = 0\}\\ &= \{(X,v) : X \in \mathfrak{so}(n),\ v \in \mathbb{R}^n\},\end{aligned}$$

where

$$(X,v) \equiv \begin{pmatrix} 0 & v^t \\ v & X \end{pmatrix}.$$

Let (X,v), $(Y,w) \in \mathfrak{so}(1,n)$. With the notation above, the Killing form of $\mathfrak{so}(1,n)$ is $\mathcal{B}((X,v),(Y,w)) = (n-1)\,\mathrm{tr}((X,v)(Y,w))$, the Cartan involution is given by $\theta(X,v) = -(X,v)^t = (X,-v)$, and hence, we have an inner product $\mathcal{B}_\theta$ in $\mathfrak{so}(1,n)$ given by $\mathcal{B}_\theta((X,v),(Y,w)) = -\mathcal{B}(\theta(X,v),(Y,w)) = (n-1)(\mathrm{tr}(X^tY)+2v^tw)$. With respect to the Cartan involution, we have the Cartan decomposition $\mathfrak{so}(1,n) = \mathfrak{so}(n)\oplus\mathfrak{p}$, where we identify $\mathfrak{so}(n)$ with the subgroup $\{(X,0) : X \in \mathfrak{so}(n)\}$ of $\mathfrak{so}(1,n)$ and $\mathfrak{p} = \{(0,v) : v \in \mathbb{R}^n\} \cong \mathbb{R}^n$.

Consider $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$ and define $\mathfrak{a} = \mathbb{R}(0,e_1)$, which turns out to be a maximal abelian subspace of $\mathfrak{p}$. The root space decomposition is very simple in this case: there exist only two roots, $\pm\alpha$, and hence $\mathfrak{so}(1,n) = \mathfrak{g}_{-\alpha}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_\alpha$, where

$$\mathfrak{g}_\alpha = \left\{\begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix} : v \in \mathbb{R}^{n-1}\right\} \cong \mathbb{R}^{n-1}, \qquad \mathfrak{g}_{-\alpha} = \theta\mathfrak{g}_\alpha,$$

$$\mathfrak{k}_0 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & X \end{pmatrix} : X \in \mathfrak{so}(n-1)\right\} \cong \mathfrak{so}(n-1), \qquad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}.$$

Defining $\mathfrak{n} = \mathfrak{g}_\alpha$, we obtain the Iwasawa decomposition $\mathfrak{so}(1,n) = \mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$. The Lie subalgebra $\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$ is known to be a maximal proper subalgebra of $\mathfrak{so}(1,n)$, called a *parabolic subalgebra*. In this work, an element in $\mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$ will be written as

$$Y + a + v \equiv \begin{pmatrix} 0 & a & v^t \\ a & 0 & v^t \\ v & -v & Y \end{pmatrix}, \qquad \text{where } a \in \mathbb{R},\ v \in \mathbb{R}^{n-1} \text{ and } Y \in \mathfrak{so}(n-1).$$

Reductive subalgebras, that is, subalgebras of the form $\mathfrak{so}(1,k)\oplus\mathfrak{so}(n-k)$, for some $k \in \{0,1,\dots,n-1\}$, are also maximal proper subalgebras of $\mathfrak{so}(1,n)$. Any maximal subalgebra of $\mathfrak{so}(1,n)$ is either reductive or parabolic (see, for example, [62, Chapter 6, Theorem 1.9] or [16]).

1.4.2 The Iwasawa decomposition of $\mathfrak{su}(1,n)$

In this section we describe the Iwasawa decomposition of the Lie algebra of the semisimple Lie group

$$SU(1,n) = \{A \in GL(n+1,\mathbb{C}) : AI_{1,n}A^* = I_{1,n},\ \det(A) = 1\},$$

where, again, $I_{1,n}$ denotes the diagonal matrix $I_{1,n} = \operatorname{diag}(-1,1,\dots,1)$ and A^* is the conjugate transpose matrix of A. Its corresponding semisimple Lie algebra is

$$\begin{aligned}\mathfrak{su}(1,n) &= \{X \in \mathfrak{gl}(n+1,\mathbb{C}) : XI_{1,n} + I_{1,n}X^* = 0,\ \operatorname{tr}(X) = 0\}\\ &= \{(\lambda, v, X) : \lambda \in \mathbb{R},\ v \in \mathbb{C}^n,\ X \in \mathfrak{u}(n),\ i\lambda + \operatorname{tr}(X) = 0\},\end{aligned}$$

where

$$(\lambda, v, X) \equiv \begin{pmatrix} i\lambda & v^* \\ v & X \end{pmatrix}.$$

Let now (λ, v, X), $(\mu, w, Y) \in \mathfrak{su}(1,n)$. With the notation above, the Killing form of $\mathfrak{su}(1,n)$ is $\mathcal{B}((\lambda, v, X),(\mu, w, Y)) = 2(n+1)\operatorname{tr}((\lambda, v, X)(\mu, w, Y))$. The Cartan involution is given, in this case, by $\theta(\lambda, v, X) = -(\lambda, v, X)^* = (\lambda, -v, X)$. Thus, we have an inner product $\mathcal{B}_\theta$ in $\mathfrak{su}(1,n)$ induced by the Killing form, which is given by $\mathcal{B}_\theta((\lambda, v, X),(\mu, w, Y)) = -\mathcal{B}(\theta(\lambda, v, X),(\mu, w, Y)) = -2(n+1)\operatorname{tr}((\lambda, -v, X)(\mu, w, Y))$. With respect to the Cartan involution, one can consider the Cartan decomposition of $\mathfrak{su}(1,n)$, that is, $\mathfrak{su}(1,n) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\begin{aligned}&\mathfrak{k} = \{(\lambda, 0, X) : \lambda \in \mathbb{R},\ X \in \mathfrak{u}(n),\ i\lambda + \operatorname{tr}(X) = 0\} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n)),\\ &\mathfrak{p} = \{(0, v, 0) : v \in \mathbb{C}^n\} \cong \mathbb{C}^n.\end{aligned}$$

Let $e_1 = (1,0,\dots,0) \in \mathbb{C}^n$ and define $\mathfrak{a} = \mathbb{R}(0, e_1, 0)$, which turns out to be a maximal abelian subspace of $\mathfrak{p}$. In this case, the set of roots consists of four elements, $\{\pm\alpha, \pm 2\alpha\}$, so the root space decomposition of $\mathfrak{su}(1,n)$ reads $\mathfrak{su}(1,n) = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_{-\lambda} = \theta\mathfrak{g}_\lambda$, for $\lambda \in \{\alpha, 2\alpha\}$, and

$$\begin{aligned}\mathfrak{g}_\alpha &= \left\{ \begin{pmatrix} 0 & 0 & v^* \\ 0 & 0 & v^* \\ v & -v & 0 \end{pmatrix} : v \in \mathbb{C}^{n-1} \right\} \cong \mathbb{C}^{n-1},\\ \mathfrak{g}_{2\alpha} &= \left\{ \begin{pmatrix} i\mu & -i\mu & 0 \\ i\mu & -i\mu & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mu \in \mathbb{R} \right\} \cong \mathbb{R},\\ \mathfrak{g}_0 &= \left\{ \begin{pmatrix} i\mu & x & 0 \\ x & i\mu & 0 \\ 0 & 0 & X \end{pmatrix} : x, \mu \in \mathbb{R},\ X \in \mathfrak{u}(n-1),\ 2i\mu + \operatorname{tr}(X) = 0 \right\}.\end{aligned}$$

If we define $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, we obtain the Iwasawa decomposition $\mathfrak{su}(1,n) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The Lie subalgebra $\mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{su}(1,n)$ is a solvable Lie algebra which will play an important role in several chapters of this thesis.

1.5 Complex space forms

This section is devoted to presenting the construction and main properties of the two families of Hermitian symmetric spaces of rank one: complex projective and hyperbolic spaces. We refer to [35] and [60] for more information on this topic.

We start by recalling some terminology concerning complex, Hermitian and Kähler manifolds. See [74] for more details. Let V be a vector space equipped with an inner product $\langle\cdot,\cdot\rangle$. An orthogonal transformation J of V satisfying $J^2 = -\operatorname{Id}$ is said to be a *complex structure* on V. In particular, if J is a complex structure on V, the following properties are satisfied for every u, $v \in V$:

(i) $\langle Ju, Jv\rangle = \langle u, v\rangle$, that is, $\langle\cdot,\cdot\rangle$ is a Hermitian inner product;

(ii) $\langle Ju, v\rangle = -\langle u, Jv\rangle$, that is, $J \in \mathfrak{so}(V)$.

A *complex manifold* is a smooth manifold $\bar{M}$ that admits charts with image onto open subsets of $\mathbb{C}^n$ in such a way that the coordinate transitions are holomorphic. This induces an almost complex structure J on $\bar{M}$, that is, an endomorphism of the tangent bundle of $\bar{M}$ satisfying $J^2 = -\operatorname{Id}$. In particular, complex manifolds have even (real) dimension. $\bar{M}$ is said to be a *Hermitian manifold* if it is Riemannian and complex, and J restricts to a complex structure on each tangent space $T_p\bar{M}$, with $p \in \bar{M}$. A *Kähler manifold* is a Hermitian manifold $\bar{M}$ such that $\bar{\nabla}J = 0$, where $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$. The endomorphism J is called the *Kähler structure* or the *complex structure* of $\bar{M}$.

It is known that Kähler manifolds of constant curvature and dimension greater than two are necessarily flat. A suitable concept is then introduced in this context. Let $\bar{M}$ be a Kähler manifold with complex structure J and curvature tensor $\bar{R}$. The *holomorphic sectional curvature* $\bar{K}_{hol}$ of $\bar{M}$ is the restriction of the sectional curvature $\bar{K}$ to J-invariant two-dimensional subspaces of the tangent space. These subspaces are generated by pairs of the form $\{v, Jv\}$, with $v \in T_p\bar{M} - \{0\}$ for $p \in \bar{M}$, so the holomorphic sectional curvature can be understood as the function which maps each unit tangent vector $v \in T\bar{M}$ to the real number $\bar{K}_{hol}(v) = \bar{K}(v, Jv) = \langle \bar{R}(v, Jv)Jv, v\rangle$.

A Kähler manifold $\bar{M}$ is said to have *constant holomorphic sectional curvature* if $\bar{K}_{hol}(v)$ is equal to a constant value c for each unit tangent vector v to $\bar{M}$. If $\bar{M}$ has constant holomorphic sectional curvature c, its curvature tensor can be written as

$$\bar{R}(X,Y)Z = \frac{c}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ). \quad (1.2)$$

Any complete, simply connected Kähler manifold of constant holomorphic sectional curvature c is isometric to one of the following spaces: a complex Euclidean space $\mathbb{C}^n$, if $c = 0$; a complex projective space $\mathbb{C}P^n$, if $c > 0$; or a complex hyperbolic space $\mathbb{C}H^n$, if $c < 0$. These spaces are the so-called *complex space forms*. A complex Euclidean space is nothing but an even dimensional Euclidean space equipped with a complex structure. A description of complex projective and hyperbolic spaces is settled below.

1.5.1 Symmetric spaces

Both complex projective and hyperbolic spaces are particular examples of symmetric spaces, so this section is devoted to presenting a brief review on this kind of Riemannian manifolds. We refer to [45], [52] and [53] to get further information on this topic.

Let $\bar{M}$ be a connected Riemannian manifold and let $o \in \bar{M}$. Consider $r > 0$ sufficiently small so that normal coordinates are defined on the open geodesic ball $B_r(o) = \{p \in \bar{M} : d(o,p) < r\}$. One may consider the local geodesic symmetry at o, that is, the smooth map $s_o\colon B_r(o) \to B_r(o)$ given by $\exp_o(v) \mapsto \exp_o(-v)$, for $v \in T_o\bar{M}$. In this thesis, exp denotes the exponential map of a (semi-)Riemannian manifold. The Riemannian manifold $\bar{M}$ is said to be *locally symmetric* if at each point there exists a geodesic ball in such a way that the corresponding local geodesic symmetry is an isometry. Locally symmetric spaces are characterized by the fact that $\bar{\nabla}\bar{R} = 0$. A connected Riemannian manifold $\bar{M}$ is called a *(Riemannian) symmetric space* if each local geodesic symmetry s_o can be extended to a global isometry $s_o\colon \bar{M} \to \bar{M}$. Equivalently, $\bar{M}$ is a symmetric space if, for each point $o \in \bar{M}$, there exists an involutive isometry of $\bar{M}$ such that o is an isolated fixed point of such isometry. This involutive isometry turns out to be, precisely, s_o.

One may deduce some properties from the definition of symmetric space. For example, any symmetric space is complete, which follows from the fact that geodesics can be extended by means of geodesic reflections. Moreover, every symmetric space is a homogeneous space, that is, for any p, $q \in \bar{M}$, there exists an isometry φ of $\bar{M}$ such that $\varphi(p) = q$ (indeed, it is enough to consider the geodesic reflection $\varphi = s_m$, where m denotes the midpoint of a geodesic joining p and q).

We now give a more algebraic description of Riemannian symmetric spaces. In order to do so, let us denote by $G = I^0(\bar{M})$ the connected component of the identity of the isometry group $I(\bar{M})$, and let $\mathfrak{g}$ be the Lie algebra of G. We can consider the action $G \times \bar{M} \to \bar{M}$ given by $(g,p) \mapsto g(p)$, which turns out to be transitive since $\bar{M}$ is homogeneous. In what follows, we fix a point $o \in \bar{M}$ and define K as the isotropy group of G at o, which is compact. Then, $\bar{M}$ is diffeomorphic to the quotient space G/K by means of the map $\Phi\colon G/K \to \bar{M}$ given by $gK \mapsto g(o)$. Let $\langle \cdot, \cdot \rangle$ denote the metric in G/K obtained after pulling back the metric of $\bar{M}$. Then, Φ becomes an isometry and the metric $\langle \cdot, \cdot \rangle$ is G-invariant, which means that the map $gK \mapsto hgK$ is an isometry for any $h \in G$. The *isotropy representation* of the symmetric space $\bar{M} = G/K$ at o is the orthogonal representation defined by $K \times T_o\bar{M} \to T_o\bar{M}$, $(k,v) \mapsto k_*v$, where k_* denotes the differential of $k \in K$.

Consider now the map $\sigma\colon G \to G$ given by $g \mapsto s_o g s_o$, which is an involutive automorphism of G. Moreover, $G^0_\sigma \subset K \subset G_\sigma$, where $G_\sigma = \{g \in G : \sigma(g) = g\}$ and G^0_σ is the connected component of the identity of G_σ. Consider the differential $\theta = \sigma_*\colon \mathfrak{g} \to \mathfrak{g}$, which is a Lie algebra homomorphism called the Cartan involution of the symmetric space. The Lie algebra of K turns out to be $\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}$, and its complementary in $\mathfrak{g}$, namely $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$, can be identified with $T_o\bar{M}$ by means of Φ_*. Then, $\mathfrak{p}$ can be equipped with an inner product that turns out to be $\mathrm{Ad}(K)$-invariant. In fact, the isotropy representation of G/K is equivalent to the adjoint representation of K on $\mathfrak{p}$, $K\times\mathfrak{p}\to\mathfrak{p}$, $(k,X)\mapsto \mathrm{Ad}(k)X$. As in Section 1.4, the so-called Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is an orthogonal decomposition with respect to the Killing form $\mathcal{B}$ of $\mathfrak{g}$ and relations (1.1) hold.

In the description above we have taken $G=I^0(\bar{M})$. However, when dealing with concrete examples, it is customary to use a coset description G/K of $\bar{M}$ where G is a finite covering of $I^0(\bar{M})$. More precisely, one has to require the pair (G,K) to be an (almost effective) *symmetric pair*, which means that G is a connected Lie group, K is a compact subgroup of G, there is an involutive automorphism σ of G such that $G_\sigma^0\subset K\subset G_\sigma$, and G acts almost effectively on G/K. The condition that G acts almost effectively on G/K means that the isotropy groups of such transitive action are finite, and this is equivalent, in terms of the Cartan decomposition, to saying that $\mathfrak{g}$ and $\mathfrak{k}$ do not share any nonzero ideals. In what follows, whenever we refer to a symmetric space $\bar{M}\cong G/K$, we will implicitly assume that (G,K) is a symmetric pair.

Let $\bar{M}=G/K$ be a symmetric space and let us denote by $\widetilde{M}$ its universal covering space, which is also a symmetric space. Consider the isotropy representation restricted to the connected component of the identity of K, say K^0. If such representation is irreducible, we say that $\bar{M}$ is an *irreducible symmetric space*. Equivalently, the universal covering space $\widetilde{M}$ does not split as a nontrivial product of symmetric spaces (unless $\bar{M}$ is an Euclidean space). If a symmetric space $\bar{M}$ is not irreducible, then it is called *reducible*. The De Rham theorem ensures that the universal covering space $\widetilde{M}$ can be decomposed as $\widetilde{M}=\widetilde{M}_0\times\widetilde{M}_1\times\cdots\times\widetilde{M}_k$, where $\widetilde{M}_0$ is isometric to a Euclidean space and, for each $i\in\{1,\dots,k\}$, $\widetilde{M}_i$ is a simply connected irreducible symmetric space.

A symmetric space $\bar{M}=G/K$ is called *semisimple* if the Euclidean factor of its universal covering space has dimension zero. In such case, the Lie algebra of the isometry group of $\widetilde{M}$ is semisimple, so one may apply the theory of semisimple Lie algebras given in Section 1.4. A semisimple symmetric space $\bar{M}=G/K$ is said to be of *compact type* if all the De Rham factors of the universal covering space $\widetilde{M}$ are compact, and it is said to be of *noncompact type* if all the De Rham factors of $\widetilde{M}$ are non-Euclidean and noncompact. The Lie algebra $\mathfrak{g}$ of the isometry group of a symmetric space of compact (resp. noncompact) type is compact (resp. noncompact). By definition, an irreducible symmetric space must be of one of the following three types: of Euclidean type (that is, a flat Euclidean space $\mathbb{R}^n$), of compact type, or of noncompact type. Moreover, if $\mathcal{B}$ is the Killing form of $\mathfrak{g}$, then the symmetric space is of compact, noncompact, or Euclidean type if, and only if, $\mathcal{B}|_{\mathfrak{p}\times\mathfrak{p}}$ is negative definite, positive definite, or identically zero, respectively.

In particular, if $\bar{M}\cong G/K$ is a symmetric space of noncompact type, $\mathfrak{g}$ is a semisimple Lie algebra and the restriction of the Killing form $\mathcal{B}$ to $\mathfrak{p}$ is positive definite. One can consider the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$, and hence the corresponding Iwasawa decomposition at the level of Lie groups, $G=KAN$. Moreover, every symmetric space of noncompact type can be regarded as a solvable Lie group endowed with a left-invariant metric. Indeed, consider the smooth function $\phi\colon G\to\bar{M}$ given by $h\mapsto h(o)$. By the Iwasawa decoposition, the restriction $\phi|_{AN}\colon AN\to\bar{M}$ turns out to be a diffeomorphism, so $\mathfrak{a}\oplus\mathfrak{n}$ can be identified with the tangent space $T_o\bar{M}$ using

ϕ_*. The metric g of $\bar{M}$ can be pulled back to obtain a Riemannian metric $(\phi|_{AN})^*g$ on AN. Thus, $(AN, (\phi|_{AN})^*g)$ and $(\bar{M}, g)$ are isometric Riemannian manifolds. Let L_h denote the left translation in G by the element $h \in G$. The metric g on $\bar{M}$ is invariant under isometries, and hence under elements of G. Then it follows that, for any $h \in G$,

$$\mathrm{L}_h^*(\phi^* g) = \mathrm{L}_h^* \phi^*(h^{-1})^* g = (h^{-1} \circ \phi \circ \mathrm{L}_h)^* g = \phi^* g,$$

since $(h^{-1} \circ \phi \circ \mathrm{L}_h)(h') = h^{-1}(hh'(o)) = h'(o) = \phi(h')$ for all $h' \in G$. Thus, every symmetric space of noncompact type $\bar{M}$ can be seen as the solvable Lie group AN endowed with a left-invariant metric $(\phi|_{AN})^*g$.

An important particular class of symmetric spaces is that of Hermitian symmetric spaces. A symmetric space $\bar{M}$ is said to be *Hermitian* if it is a Hermitian manifold, and for each $p \in \bar{M}$, the geodesic symmetry s_p is a holomorphic transformation. Any Hermitian symmetric space is Kähler. Irreducible nonflat Hermitian symmetric spaces G/K are characterized by the property that K is not semisimple (indeed K has one-dimensional center).

The *rank* of a symmetric space $\bar{M}$ is defined to be the dimension of a maximal flat, totally geodesic submanifold of $\bar{M}$, or equivalently, as the dimension of a maximal abelian subspace of $\mathfrak{p}$. In this thesis, we are mostly interested in rank one irreducible semisimple Hermitian symmetric spaces. These are, precisely, the nonflat complex space forms, namely the complex projective spaces $\mathbb{C}P^n$ and the complex hyperbolic spaces $\mathbb{C}H^n$, which we will describe in the next subsections.

1.5.2 The complex projective space

As a smooth manifold, the complex projective space of complex dimension n (real dimension $2n$), commonly denoted by $\mathbb{C}P^n$, is the set of complex lines of $\mathbb{C}^{n+1}$ through the origin, or equivalently, it is the quotient manifold of a sphere $S^{2n+1}(r) \subset \mathbb{C}^{n+1}$ of radius r by the equivalence relation given by $z \sim \lambda z$, where $z \in \mathbb{C}^{n+1}$ and $\lambda \in S^1 \subset \mathbb{C}$. We denote by π the *Hopf map*, that is, the quotient projection of the sphere onto the complex projective space, $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$, which is a smooth surjective submersion. The metric that we consider in $\mathbb{C}P^n$ is the one which makes the Hopf map a Riemannian submersion.

We now give a more detailed description of the complex projective space. In order to do so, we consider a complex structure J on $\mathbb{R}^{2n+2}$ which allows us to identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, where the multiplication by the imaginary unit i is induced by J. Consider the scalar product on $\mathbb{C}^{n+1}$ given by

$$\langle z, w\rangle = \operatorname{Re}\left(\sum_{k=0}^{n} z_k \bar{w}_k\right),$$

for $z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1}$, which yields the standard Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n+2}$.

The $(2n+1)$-dimensional sphere of radius r is $S^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = r^2\}$ and its tangent space at $z \in S^{2n+1}(r)$ is $T_z S^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z, w\rangle = 0\}$. The restriction of the above inner product yields a Riemannian metric of constant curvature $1/r^2$ on $S^{2n+1}(r)$. A unit normal vector field along $S^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r}z$.

Consider the equivalence relation on $S^{2n+1}(r)$ generated by $z \sim \lambda z$, with $\lambda \in S^1 \subset \mathbb{C}$, which defines a principal fiber bundle over $\mathbb{C}P^n$ with total space $S^{2n+1}(r)$, fiber S^1 and projection map $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$. Define $V = J\xi$. Then, V is a unit tangent vector to $S^{2n+1}(r)$ and we can write

$$TS^{2n+1}(r) = \mathbb{R}V \oplus V^\perp,$$

where $V^\perp$ denotes the orthogonal complement to V. Notice that if $z \in S^{2n+1}(r)$, $\mathbb{R}V_z$ is precisely the kernel of π_{*z}, where π_* is the differential of π. Then, π_* maps $V_z^\perp$ isomorphically onto $T_{\pi(z)}\mathbb{C}P^n$, and for each $X \in T_{\pi(z)}\mathbb{C}P^n$ one can define the horizontal lift X_z^L of X to z as the unique tangent vector in $V_z^\perp$ such that $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it}z$ is exactly the geodesic in $S^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, and so $X^L_{\varphi_t(z)} = (\varphi_t)_{*z} X_z^L$.

The complex structure J on $\mathbb{C}P^n$ is defined by $JX = \pi_*(JX^L)$ for $X \in T\mathbb{C}P^n$, whereas the metric on $\mathbb{C}P^n$ is given by $\langle X, Y\rangle = \langle X^L, Y^L\rangle$, for every X, $Y \in T_p\mathbb{C}P^n$, $p \in \mathbb{C}P^n$. This metric, called the *Fubini-Study metric*, makes $\pi\colon S^{2n+1}(r) \to \mathbb{C}P^n$ a Riemannian submersion, and moreover, it satisfies $\langle JX, JY\rangle = \langle X, Y\rangle$ for any tangent vectors X and Y. By virtue of the formulas for Riemannian submersions [58], the Levi-Civita connection of $\mathbb{C}P^n$ is given by

$$\bar{\nabla}_X Y = \pi_*\left(\widetilde{\nabla}_{X^L} Y^L\right),$$

for all tangent vector fields X, Y on $\mathbb{C}P^n$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $S^{2n+1}(r)$. Using this formula, one can prove that J is Kähler.

The theory of Riemannian submersions [58] allows us to compute the holomorphic sectional curvature of $\mathbb{C}P^n$, which turns out to be $\bar{K}_{hol}(X) = 4/r^2$ for any unit tangent vector $X \in T\mathbb{C}P^n$. Therefore, $\mathbb{C}P^n$ is a space of constant holomorphic sectional curvature $c = 4/r^2$.

The unitary group $U(n+1) = \{A \in GL(n+1, \mathbb{C}) : AA^* = \mathrm{Id}\}$, where A^* denotes the conjugate transpose matrix of A, preserves the standard metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+1}$. Since it preserves complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them, $U(n+1)$ acts transitively by isometries on $\mathbb{C}P^n$ by $A(p) = \pi(Az)$, where $p = \pi(z) \in \mathbb{C}P^n$ and $A \in U(n+1)$. However, this action is not effective since all the transformations of the form $z\,\mathrm{Id}$, with $|z| = 1$, act trivially on $\mathbb{C}P^n$. The subgroup $SU(n+1)$ consisting of those matrices of $U(n+1)$ whose determinant is one keeps acting transitively on $\mathbb{C}P^n$, but with finite kernel constituted by the matrices of the form $z\,\mathrm{Id}$, where z is an $(n+1)$-th root of the unit.

Hence, $\mathbb{C}P^n$ is a homogeneous Riemannian manifold. The isotropy group at, for example, the point $p = \pi(r, 0, \dots, 0) \in \mathbb{C}P^n$ is $S(U(1)U(n))$, which is isomorphic to $U(n)$. Then, the complex projective space turns out to be the Hermitian symmetric space of rank one given by $\mathbb{C}P^n = SU(n+1)/S(U(1)U(n))$. The fact that its rank is one can be deduced, for example, from a classification of totally geodesic submanifolds, which implies that any totally geodesic, flat submanifold of maximal dimension in

$\mathbb{C}P^n$ is a geodesic [73]. More precisely, any totally geodesic submanifold of $\mathbb{C}P^n$ is holomorphically congruent to an open part of a real projective space $\mathbb{R}P^k$ for some $k \in \{1, \dots, n\}$, or to a complex projective space $\mathbb{C}P^k$ for some $k \in \{0, 1, \dots, n\}$. Moreover, any two totally geodesic submanifolds of $\mathbb{C}P^n$ are locally holomorphically congruent to each other if, and only if, they are locally isometric.

1.5.3 The complex hyperbolic space

The construction of the complex hyperbolic space is formally quite similar to the one of the complex projective space. However, their geometries turn out to be very different.

As in the previous section, consider the complex structure J on $\mathbb{R}^{2n+2}$, which allows us to identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$. Now we take the scalar product on $\mathbb{C}^{n+1}$ defined by

$$\langle z, w \rangle = \operatorname{Re}\left(-z_0\bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k\right),$$

for $z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n) \in \mathbb{C}^{n+1}$, which yields a standard semi-Riemannian metric of signature $(2, 2n)$.

The *anti-De Sitter space* of radius r, which can be regarded as the Lorentzian analog to the real hyperbolic space, is defined as

$$H_1^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z \rangle = -r^2\}$$

Its tangent space at $z \in H_1^{2n+1}(r)$ is $T_z H_1^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z, w \rangle = 0\}$. The restriction of the previous inner product yields a Lorentzian metric of constant sectional curvature $-1/r^2$ on $H_1^{2n+1}(r)$. A unit normal vector field ξ along $H_1^{2n+1}(r)$ is given by $\xi_z = \frac{1}{r} z$, which, in this case, satisfies $\langle \xi, \xi \rangle = -1$.

The complex hyperbolic space, as a smooth manifold, is defined as the space of timelike lines through the origin of $\mathbb{C}^{n+1}$, or equivalently, as the quotient manifold $\mathbb{C}H^n = H_1^{2n+1}(r)/\sim$, where $\sim$ is the equivalence relation generated by $z \sim \lambda z$, with $\lambda \in S^1 \subset \mathbb{C}$. The canonical projection $\pi \colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ is called the *Hopf map* of $\mathbb{C}H^n$. As a Riemannian manifold, the metric of $\mathbb{C}H^n$ is induced by the metric of the anti-De Sitter space through the map π.

Define $V = J\xi$. Then V turns out be a unit tangent vector field to $H_1^{2n+1}(r)$, where now unit means $\langle V, V \rangle = -1$. Thus, both ξ and V are timelike vector fields. One can write

$$TH_1^{2n+1}(r) = \mathbb{R}V \oplus V^\perp,$$

where $V^\perp$ denotes the orthogonal complement to V with respect to the Lorentzian metric on $H_1^{2n+1}(r)$. If $z \in H_1^{2n+1}(r)$, then $\mathbb{R}V_z$ is in fact the kernel of π_{*z}. Thus, π_{*z} maps $V_z^\perp$ isomorphically onto $T_{\pi(z)}\mathbb{C}H^n$, and for each $X \in T_{\pi(z)}\mathbb{C}H^n$ one can define the horizontal lift X_z^L of X to z as the unique tangent vector in $V_z^\perp$ satisfying $\pi_* X_z^L = X$. The map $t \mapsto \varphi_t(z) = e^{it} z$ is exactly the geodesic on $H_1^{2n+1}(r)$ starting at z with initial speed $Jz = iz = rV_z$. We have $\pi \circ \varphi_t = \pi$, so $X_{\varphi_t(z)}^L = (\varphi_t)_{*z} X_z^L$.

The complex structure J on $\mathbb{C}H^n$ is then defined by $JX = \pi_*(JX^L)$, for each $X \in T\mathbb{C}H^n$, and the metric on $\mathbb{C}H^n$ is given by $\langle X, Y \rangle = \langle X^L, Y^L \rangle$, for every X, $Y \in T_p\mathbb{C}H^n$, $p \in \mathbb{C}H^n$.

It is important to point out that the metric of $H_1^{2n+1}(r)$ is positive definite on $V_z^\perp$ and thus the metric on $\mathbb{C}H^n$ is positive definite. Hence, the complex hyperbolic space becomes a Riemannian manifold. This metric, commonly called the *Bergman metric* of $\mathbb{C}H^n$, makes the Hopf map $\pi\colon H_1^{2n+1}(r) \to \mathbb{C}H^n$ a semi-Riemannian submersion. Moreover, it is a Hermitian metric, that is, $\langle JX, JY \rangle = \langle X, Y \rangle$ for any tangent vectors X and Y. By virtue of the formulas for semi-Riemannian submersions [59], the Levi-Civita connection of $\mathbb{C}H^n$ is given by

$$\bar{\nabla}_X Y = \pi_* \left(\widetilde{\nabla}_{X^L} Y^L \right),$$

for tangent vector fields X, Y on $\mathbb{C}H^n$, where $\widetilde{\nabla}$ denotes here the Levi-Civita connection of $H^{2n+1}(r)$. Using this formula, one can show that J is Kähler.

Again, the theory of semi-Riemannian submersions allows us to compute the holomorphic sectional curvature of $\mathbb{C}H^n$, which turns out to be $\bar{K}_{hol}(X) = -4/r^2$ for any $X \in T\mathbb{C}H^n$. Thus, $\mathbb{C}H^n$ is a space of constant holomorphic sectional curvature $c = -4/r^2$.

The indefinite unitary group $U(1,n) = \{A \in GL(n, \mathbb{C}) : AI_{1,n}A^* = I_{1,n}\}$, where $I_{1,n}$ is the diagonal matrix $\operatorname{diag}(-1, 1, \ldots, 1)$, leaves invariant the metric of $\mathbb{R}^{2n+2} \equiv \mathbb{C}^{n+1}$ with signature $(2, 2n)$ that we have considered above. Moreover, it preserves timelike complex lines through the origin of $\mathbb{C}^{n+1}$ and acts transitively on them. Hence, one deduces that $U(1,n)$ acts transitively by isometries on $\mathbb{C}H^n$. As in the projective case, we can restrict to $SU(1,n)$, that is, the subgroup consisting of those matrices of $U(1,n)$ with determinant one, which still acts transitively on $\mathbb{C}H^n$. Then, $\mathbb{C}H^n$ is a homogeneous Riemannian manifold. Furthermore, the complex hyperbolic space is a Hermitian symmetric space $\mathbb{C}H^n = SU(1,n)/S(U(1)U(n))$.

The fact that the complex hyperbolic space has rank one as a symmetric space comes from a known result, which also completely determines both the extrinsic and intrinsic geometry of totally geodesic submanifolds of $\mathbb{C}H^n$. More precisely, every totally geodesic submanifold of $\mathbb{C}H^n$ is holomorphically congruent to an open part of a real hyperbolic space $\mathbb{R}H^k$ for some $k \in \{1, \ldots, n\}$, or to a complex hyperbolic space $\mathbb{C}H^k$ for some $k \in \{0, 1, \ldots, n\}$. Moreover, any two totally geodesic submanifolds of $\mathbb{C}H^n$ are locally holomorphically congruent to each other if, and only if, they are locally isometric.

1.5.4 The solvable Lie group model of $\mathbb{C}H^n$

As we have pointed out in Subsection 1.5.1, any symmetric space of noncompact type can be regarded as a solvable Lie group and its metric is left-invariant with respect to the Lie group structure. In this section we focus on the particular case of the complex hyperbolic space, and we give a description of $\mathbb{C}H^n$ as a symmetric space and as a solvable Lie group. In order to do so, we will use the notation introduced in

Section 1.4 and Subsections 1.4.2, 1.5.1 and 1.5.3. We refer to [31] and [35] for more information on this topic.

Recall from Subsection 1.5.3 that, as a symmetric space, the complex hyperbolic space can be identified with the quotient space G/K, where, up to a finite quotient, $G = SU(1,n)$ is the connected component of the identity element of the isometry group of $\mathbb{C}H^n$ and $K = G_o = S(U(1)U(n))$ is the stabilizer of a fixed element $o \in \mathbb{C}H^n$. The Lie algebras of these two Lie groups are $\mathfrak{g} = \mathfrak{su}(1,n)$ and $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(n))$, respectively. Consider the Iwasawa decompositions of G and $\mathfrak{g}$, which have been described in Section 1.4.2. It follows from the Iwasawa decomposition at the Lie group level that the solvable part of the Iwasawa decomposition of G, AN, acts simply and transitively on $G/K \cong \mathbb{C}H^n$.

Consider the smooth map $\phi\colon G \to \mathbb{C}H^n$ given by $h \mapsto h(o)$. Recall from Subsection 1.5.1 that, since AN acts simply and transitively on $\mathbb{C}H^n$, the restriction $\phi|_{AN}\colon AN \to \mathbb{C}H^n$ is a diffeomorphism, so $\mathfrak{a} \oplus \mathfrak{n}$ can be identified with the tangent space $T_o\mathbb{C}H^n$ using ϕ_*. The Bergman metric g of $\mathbb{C}H^n$ can be pulled back to obtain a Riemannian metric $(\phi|_{AN})^*g$ on AN, which makes $(AN, (\phi|_{AN})^*g)$ and $(\mathbb{C}H^n, g)$ isometric Riemannian manifolds. As we have proved in Subsection 1.5.1, $\mathbb{C}H^n$ can be regarded as the solvable Lie group AN endowed with the left-invariant metric $(\phi|_{AN})^*g$ which, from now on, will be denoted by $\langle \cdot, \cdot \rangle$. If X, $Y \in \mathfrak{a} \oplus \mathfrak{n}$, the relation between $\langle \cdot, \cdot \rangle$ and $\mathcal{B}_\theta(\cdot, \cdot)$ is given, up to homothety of the metric of $\mathbb{C}H^n$, by

$$\langle X, Y \rangle = \mathcal{B}_\theta(X_\mathfrak{a}, Y_\mathfrak{a}) + \frac{1}{2}\mathcal{B}_\theta(X_\mathfrak{n}, Y_\mathfrak{n}),$$

where the subscripts mean, in this case, the orthogonal projections, with respect to $\mathcal{B}_\theta$, onto $\mathfrak{a}$ and $\mathfrak{n}$, respectively.

Moreover, the Lie group AN can be equipped with a Kähler structure induced by the Kähler structure of $\mathbb{C}H^n$ via $\phi|_{AN}$. One obtains then a complex structure on AN, and hence also on $\mathfrak{a} \oplus \mathfrak{n}$, which will be denoted by J. The complex structure J on $\mathfrak{a} \oplus \mathfrak{n}$ satisfies that $\mathfrak{g}_\alpha$ is a J-invariant subspace and that $J\mathfrak{a} = \mathfrak{g}_{2\alpha}$, where $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are the positive root spaces whose sum is precisely $\mathfrak{n}$ (see Subsection 1.4.2).

Thus, we have obtained a model for the complex hyperbolic space as a solvable Lie group AN endowed with a left-invariant Riemannian metric whose Lie algebra $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ can be identified with the tangent space $T_o\mathbb{C}H^n$, and such that $\mathfrak{g}_\alpha$ can be seen as the complex vector space $\mathbb{C}^{n-1}$.

Let $B \in \mathfrak{a}$ a unit vector and define $Z = JB \in \mathfrak{g}_{2\alpha}$. In particular, $\langle B, B \rangle = \mathcal{B}_\theta(B, B) = 1$ and $2\langle Z, Z \rangle = \mathcal{B}_\theta(Z, Z) = 2$. If U, $V \in \mathfrak{g}_\alpha$, the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$ is given by the following relations:

$$[B, Z] = \sqrt{-c}Z, \qquad [B, U] = \frac{\sqrt{-c}}{2}U, \qquad [Z, U] = 0, \qquad [U, V] = \sqrt{-c}\langle JU, V \rangle Z,$$

where c denotes the constant holomorphic sectional curvature of $\mathbb{C}H^n$. Moreover, the expression for the Levi-Civita connection $\bar{\nabla}$ of $(AN, \langle\cdot,\cdot\rangle)$ is (cf. [17])

$$\frac{1}{\sqrt{-c}}\bar{\nabla}_{aB+U+xZ}(bB+V+yZ) = \left(\frac{1}{2}\langle U,V\rangle + xy\right)B \\ -\frac{1}{2}(bU+yJU+xJV) + \left(\frac{1}{2}\langle JU,V\rangle - bx\right)Z,$$

where a,b,x,y are real numbers.

For each restricted root $\lambda \in \Sigma = \{\pm\alpha, \pm 2\alpha\}$, we define $\mathfrak{p}_\lambda = (1-\theta)\mathfrak{g}_\lambda$, the projection onto $\mathfrak{p}$ of the restricted root space associated with λ. Then, $\mathfrak{p} = \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$. Let i denote the complex structure of $\mathfrak{p}$. Then, for each $U \in \mathfrak{g}_\alpha$, $2iB = (1-\theta)Z$ and $i(1-\theta)U = (1-\theta)JU$.

Notice that the orthogonal projection map $\frac{1}{2}(1-\theta)\colon \mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha} \to \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$ defines an equivalence between the adjoint K_0-representation on $\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ and the adjoint K_0-representation on $\mathfrak{p} = \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}$, where K_0 denotes the connected Lie subgroup of K whose Lie algebra is $\mathfrak{k}_0$, which is isomorphic to $U(n-1)$. Furthermore, such equivalence is an isometry between $(\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha},\ \langle\cdot,\cdot\rangle)$ and $(\mathfrak{p},\ \mathcal{B}_\theta|_{\mathfrak{p}\times\mathfrak{p}})$, and $\frac{1}{2}(1-\theta)\colon \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ is a complex linear map.

Geometric interpretation

To finish this section, we briefly present some ideas about the geometric interpretation of the groups K, A and N which arise in the Iwasawa decomposition of the isometry group of the complex hyperbolic space. For more information, we refer to [35] and [38].

Let $\bar{M}$ be a complete, simply connected Riemannian manifold of nonpositive curvature and denote by $\bar{d}$ its Riemannian distance. Two complete unit speed geodesics γ and σ in $\bar{M}$ are said to be *asymptotic* if there exists a positive constant C in such a way that $\bar{d}(\gamma(t),\sigma(t)) \leq C$, for every $t \geq 0$. In the case of symmetric spaces of rank one, and in particular for the complex hyperbolic space, if γ and σ are asymptotic, then we have $\lim_{t\to\infty}\bar{d}(\gamma(t),\sigma(t)) = 0$. This definition yields an equivalence relation among the complete geodesics of $\bar{M}$. Each one of the equivalence classes is said to be a *point at infinity* of $\bar{M}$, and the set of the points at infinity of $\bar{M}$ is the so-called *ideal boundary* of $\bar{M}$, commonly denoted by $\bar{M}(\infty)$.

If one considers the particular case $\bar{M} = \mathbb{C}H^n$, it is possible to equip $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ with the so-called *cone topology*, which makes $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ homeomorphic to the closed unit ball of $\mathbb{R}^{2n}$ in such a way that the ideal boundary of $\mathbb{C}H^n$ corresponds to the unit sphere of $\mathbb{R}^{2n}$. In this model, two geodesics are asymptotic if they converge to the same point of the unit sphere. Moreover, for each $p \in \mathbb{C}H^n$ and each $x \in \mathbb{C}H^n(\infty)$, there exists a unique geodesic $\gamma_{px}\colon \mathbb{R} \to \mathbb{C}H^n$ satisfying

$$|\dot{\gamma}_{px}| = 1, \qquad \gamma_{px}(0) = p, \qquad \lim_{t\to\infty}\gamma_{px}(t) = x.$$

The Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is a one-dimensional abelian subspace of $\mathfrak{p}$. In $\mathfrak{p} \simeq T_o\mathbb{C}H^n$, the Riemannian exponential map exp and the Lie group exponential map Exp coincide, that is, $\operatorname{Exp}(tX)\cdot o = \exp_o(tX)$, for any $X \in \mathfrak{p}$ and $t \in \mathbb{R}$. It follows then that the orbit $A\cdot o$ is the trace of a geodesic through o whose tangent space at o is given by $\mathfrak{a} \subset \mathfrak{p} \simeq T_o\mathbb{C}H^n$. This totally geodesic one-dimensional submanifold determines two points at infinity, depending on the orientation we choose to parametrize it as a geodesic curve. Choose one of them, say x. Then, the submanifold A of AN corresponds to $\gamma_{ox}(\mathbb{R})$ under the isometry $\phi|_{AN}\colon AN \to \mathbb{C}H^n$. In other words, $\gamma_{ox}(\mathbb{R})$ is the orbit of A through o, while the remaining orbits of A are equidistant curves to $A \cdot o$. The geodesic $A \cdot o$ intersects orthogonally all K-orbits, which are the fixed point $o \in \mathbb{C}H^n$ and the geodesic spheres centered at o.

Finally, we comment on the action of the nilpotent part of the Iwasawa decomposition, N. Since N has dimension $2n-1$ and AN acts simply and transitively on $\mathbb{C}H^n$, it follows that N acts isometrically with cohomogeneity one on $\mathbb{C}H^n$. Then it turns out that the orbits of such action are hypersurfaces in $\mathbb{C}H^n$ that are orthogonal at every point to the integral curves of the left-invariant vector field $B \in \mathfrak{a}$. These integral curves are, according to the notation above, geodesics with common point at infinity x.

More precisely, the orbits of the action of N are the *horospheres* of $\mathbb{C}H^n$ determined by the point at infinity x. To define this concept, consider a complete unit speed geodesic curve γ in $\mathbb{C}H^n$. The *Busemann function* with respect to γ is the real function given as follows:

$$f\colon \mathbb{C}H^n \to \mathbb{R}, \qquad f_\gamma(p) = \lim_{t\to\infty}(\bar{d}(\gamma(t),p) - t).$$

The horospheres are defined as the level sets of a Busemann function, which are parallel real hypersurfaces of $\mathbb{C}H^n$ defining a regular Riemannian foliation, all of whose leaves have the same limit set of points at infinity, namely $\{x\}$. Thus, the N-orbits turn out to be the horospheres adherent to x.

As we have said above, once we choose the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, the orbit of A through o can be parametrized in exactly two ways as unit speed geodesics, which determine two points at infinity, say x and $-x$. The fact that the horospheres of the action of N have x, and not $-x$, as limit set comes from the choice of the criterion of positivity in the set of restricted roots $\{\pm\alpha, \pm 2\alpha\}$ (see Section 1.4.2). Thus, there exist two equivalent ways of defining a particular Iwasawa decomposition of a semisimple Lie algebra $\mathfrak{g}$. The algebraic one, described in Section 1.4, depends on the choice of a Cartan decomposition, a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a criterion of positivity in the set of roots. The geometric one, described in the present section, depends on the choice of a point o in the associated symmetric space and a point at infinity x.

1.6 The Minkowski spacetime

In the setting of Lorentzian geometry, the simplest manifold is the Minkowski spacetime $\mathbb{L}^{n+1}$, that is, the Lorentzian analog to the Euclidean space. In this section, we present the main properties of this space to be used in this work. We refer the reader to [59].

As a notational convention, in this thesis we will denote in bold font vectors of $\mathbb{L}^{n+1}$ in order to distinguish them from vectors lying in other vector spaces. Throughout this section, we will denote by $\mathbb{L}^{n+1}$ the $(n+1)$-dimensional *Minkowski spacetime*, that is, the smooth manifold $\mathbb{R}^{n+1}$ equipped with the Minkowski metric, which is the flat Lorentzian metric defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_0 v_0 + \sum_{i=1}^{n} u_i v_i,$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_n)$, $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ and $u_i, v_i \in \mathbb{R}$ for $i \in \{0, \ldots, n\}$. In this thesis, we are mainly interested in the case $n = 3$, that is, in the four-dimensional Minkowski spacetime, which is used to model Special Relativity.

Since the Minkowski spacetime is a Lorentzian manifold, a tangent vector $\mathbf{v} \in T\mathbb{L}^{n+1}$ can be *spacelike*, *timelike* or *lightlike* depending on whether $\langle \mathbf{v}, \mathbf{v} \rangle$ is positive, negative or zero, respectively (see Section 1.1). Moreover, a vector $\mathbf{v} \in T\mathbb{L}^{n+1}$ is said to be *causal* if $\langle \mathbf{v}, \mathbf{v} \rangle \leq 0$.

There exist several important subspaces of the Minkowski spacetime. The *hyperbolic space* of radius r is the set $H(r) = \{\mathbf{v} \in T_{\mathbf{p}}\mathbb{L}^{n+1} : \langle \mathbf{v}, \mathbf{v} \rangle = -r^2\}$, while the *De Sitter space* of radius r is $S(r) = \{\mathbf{v} \in T_{\mathbf{p}}\mathbb{L}^{n+1} : \langle \mathbf{v}, \mathbf{v} \rangle = r^2\}$, for $\mathbf{p} \in \mathbb{L}^{n+1}$. Moreover, the *lightlike cone* at $\mathbf{p}$ is the set $\mathcal{C}_{\mathbf{p}} = \{\mathbf{v} \in T_{\mathbf{p}}\mathbb{L}^{n+1} : \langle \mathbf{v}, \mathbf{v} \rangle = 0\}$, whereas the *timelike cone* is defined by $\{\mathbf{v} \in T_{\mathbf{p}}\mathbb{L}^{n+1} : \langle \mathbf{v}, \mathbf{v} \rangle < 0\}$.

One may introduce the notion of time-orientation, which is intimately related to the concept of timelike cone, as follows. At each point $\mathbf{p} \in \mathbb{L}^{n+1}$, one can consider the timelike cone in $T_{\mathbf{p}}\mathbb{L}^{n+1}$, which has two connected components. Once we have chosen one of these connected components, we say that $T_{\mathbf{p}}\mathbb{L}^{n+1}$ has been time-oriented. The connected component that we have chosen is called *future* of $\mathbf{p}$, whereas the remaining one is called *past* of $\mathbf{p}$. Moreover, the Minkowski spacetime is *time-orientable* since it is possible to make this choice at each point of $\mathbb{L}^{n+1}$ in a continuous way.

Consider the Lie group $O(1,n) = \{A \in GL(n+1, \mathbb{R}) : AI_{1,n}A^t = I_{1,n}\}$, usually referred to as Lorentz group, where $I_{1,n}$ denotes the diagonal matrix $I_{1,n} = \operatorname{diag}(-1, 1, \ldots, 1)$ and A^t is the transpose matrix of A. The isometry group of the Minkowski spacetime is known to be the semidirect product $I(\mathbb{L}^{n+1}) = O(1,n) \times_{\Phi} \mathbb{L}^{n+1}$, called Poincaré group, where Φ is defined by

$$\Phi \colon O(1,n) \to \operatorname{Aut}(\mathbb{L}^{n+1}), \qquad \Phi(a)(\mathbf{v}) = a\mathbf{v}.$$

Thus, the natural operation of this group is given by $(a, \mathbf{v})(b, \mathbf{w}) = (ab, \mathbf{v} + a\mathbf{w})$, for $(a, \mathbf{v})$, $(b, \mathbf{w}) \in I(\mathbb{L}^{n+1})$. Moreover, the inverse of an element $(a, \mathbf{v}) \in I(\mathbb{L}^{n+1})$ can be calculated as $(a, \mathbf{v})^{-1} = (a^{-1}, -a^{-1}\mathbf{v})$. Any connected Lie subgroup of the isometry group $I(\mathbb{L}^{n+1})$ acts on $\mathbb{L}^{n+1}$ in the obvious way by $(a, \mathbf{v}) \cdot \mathbf{p} = a\mathbf{p} + \mathbf{v}$.

The connected component of the identity of the isometry group of $\mathbb{L}^{n+1}$ is known to be the Lie group $I^0(\mathbb{L}^{n+1}) = SO^0(1,n) \times_{\Phi} \mathbb{L}^{n+1}$, where $SO^0(1,n)$ denotes the subgroup of $O(1,n)$ consisting of those matrices of determinant one which preserve both orientation and time-orientation. Its corresponding Lie algebra is the semidirect sum $\mathfrak{i}(\mathbb{L}^{n+1}) = \mathfrak{so}(1,n) \oplus_{\phi} \mathbb{L}^{n+1}$, where ϕ is given as follows:

$$\phi \colon \mathfrak{so}(1,n) \to \operatorname{Der}(\mathbb{L}^{n+1}), \qquad \phi(X)(\mathbf{v}) = X\mathbf{v}.$$

The Lie bracket is given by the expression

$$[X+\mathbf{v}, Y+\mathbf{w}] = (XY-YX)+(X\mathbf{w}-Y\mathbf{v}),$$

for $X+\mathbf{v}$, $Y+\mathbf{w} \in \mathfrak{i}(\mathbb{L}^{n+1})$. Moreover, the adjoint representation is, in this case,

$$\mathrm{Ad}(a,\mathbf{v})(X+\mathbf{w}) = aXa^{-1} - aXa^{-1}\mathbf{v} + a\mathbf{w},$$

where $(a,\mathbf{v}) \in I^0(\mathbb{L}^{n+1})$ and $X+\mathbf{w} \in \mathfrak{i}(\mathbb{L}^{n+1})$.

One may consider the rotational part of the Lie algebra $\mathfrak{i}(\mathbb{L}^{n+1})$, that is, $\mathfrak{so}(1,n)$, whose Iwasawa decomposition has been described in Section 1.4.1. The so-called parabolic subalgebra of $\mathfrak{so}(1,n)$, that is, $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, is known to be a maximal proper subalgebra of $\mathfrak{so}(1,n)$ which will play an important role throughout Chapter 4. Notice that, if we define $\mathbf{e} = (1,1,0,\dots,0) \in \mathbb{L}^{n+1}$, then it follows that this Lie algebra can be regarded as $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n} = \{X \in \mathfrak{so}(1,n) : X\mathbf{e} \in \mathbb{R}\mathbf{e}\}$. Moreover, with the notation established in Section 1.4.1, if $\mathbf{p} = (p_0, p_1, p) \in \mathbb{L}^{n+1}$, with p_0, $p_1 \in \mathbb{R}$ and $p \in \mathbb{R}^{n-1}$, we have

$$(Y+a+v)\cdot\mathbf{p} = \big(ap_1 + v^t p,\ ap_0 + v^t p,\ (p_0-p_1)v + Yp\big).$$

Chapter 2

Ruled real hypersurfaces in nonflat complex space forms

This chapter is devoted to classifying ruled real hypersurfaces satisfying some additional geometric properties in nonflat complex space forms. These properties include having constant mean curvature, having shape operator of constant norm, or being biharmonic. The results of this chapter have given rise to the articles [36], [37] and [66].

In Section 2.1, we will briefly recall some basic definitions and known results concerning ruled hypersurfaces in complex space forms. Moreover, we will give the construction and description of ruled minimal hypersurfaces in this type of ambient manifolds. In Section 2.2, we will compute the Levi-Civita connection of an arbitrary ruled real hypersurface in a nonflat complex space form. After that, we will impose several particular geometric assumptions to get simpler expressions of the Levi-Civita connection and to finally obtain classification results. Specifically, Section 2.3 is devoted to classifying ruled real hypersurfaces in nonflat complex space forms having constant mean curvature, and in Section 2.4 we study ruled real hypersurfaces whose shape operators have constant norm. Finally, in Section 2.5 we focus on biharmonic ruled real hypersurfaces.

2.1 Ruled hypersurfaces

To start with, we briefly review the terminology concerning ruled real hypersurfaces in nonflat complex space forms needed for this work. We refer the reader to [22, Chapter 8], [51] and [54].

Throughout this chapter, $\bar{M} = \bar{M}^n(c)$ will denote a nonflat complex space form with complex structure J and nonzero constant holomorphic sectional curvature c. Let $M \subset \bar{M}$ be a real hypersurface, that is, a submanifold of real codimension one. Then, locally, there exists a unique unit normal vector field to M, up to sign, say ξ. Denote by $\mathcal{S} = \mathcal{S}_\xi$ the shape operator of M with respect to ξ. The tangent vector field $J\xi$ is the so-called *Hopf vector field* of M.

We now introduce the functions $g, h \colon M \to \mathbb{N}$, where $g(p)$ is the number of distinct principal curvatures of M at $p \in M$, and $h(p)$ is the number of eigenspaces of $\mathcal{S}$ onto which $J\xi$ has nontrivial projection. If $h = 1$, then M is called a *Hopf hypersurface*, which is equivalent to saying that the Hopf vector field is an eigenvector of the shape operator $\mathcal{S}$ of M.

In the setting of non-Hopf hypersurfaces, ruled ones constitute a nice particular class. A *ruled hypersurface* M of $\bar{M}$ is a real hypersurface satisfying $\mathcal{S}(J\xi)^\perp \subset \mathbb{R}J\xi$, where $(J\xi)^\perp$ denotes the distribution on M given by the tangent vectors to M that are orthogonal to $J\xi$. However, there exist equivalent definitions for the notion of a ruled hypersurface in a nonflat complex space form which turn out to be geometrically more clear. For example, from [22, Proposition 8.27], a ruled real hypersurface in a nonflat complex space form $\bar{M}$ can be characterized as a real hypersurface satisfying that the maximal holomorphic distribution $(J\xi)^\perp$ is integrable and its leaves are totally geodesic submanifolds of $\bar{M}$. This result means that any ruled hypersurface $M \subset \bar{M}$ is foliated by totally geodesic leaves, commonly called *rulings*, which are in fact open subsets of totally geodesic $\mathbb{C}P^{n-1}$ or $\mathbb{C}H^{n-1}$, depending on whether the ambient space $\bar{M}$ is $\mathbb{C}P^n$ or $\mathbb{C}H^n$, respectively.

Let us introduce some notation to deal with ruled real hypersurfaces. For an arbitrary real hypersurface $M \subset \bar{M}$, we define the functions $\lambda = \langle \mathcal{S}J\xi, J\xi\rangle$ and $\mu = |\mathcal{S}J\xi - \lambda J\xi|$, which give a measure of how far $J\xi$ is from being a principal vector. Notice that, in particular, if M is a Hopf hypersurface, then λ turns out to be the principal curvature associated with $J\xi$, and μ vanishes identically on M. However, in the non-Hopf setting (for example, if M is ruled), μ is nonvanishing on an open subset of M. In this case, on such open subset we can consider a smooth unit vector field U_1 in such a way that $\mathcal{S}J\xi - \lambda J\xi = \mu U_1$. Note that the pair $\{J\xi, U_1\}$ is orthonormal.

The notion of a ruled hypersurface in a nonflat complex space form can be characterized in terms of the functions λ and μ, and the vector field U_1 (see, for example, [1]). More specifically, a real hypersurface $M \subset \bar{M}$ is ruled if, and only if, the following conditions hold:

(a) the function μ does not vanish on any open subset of M, and

(b) the unit vector field U_1 defined above satisfies $\mathcal{S}J\xi = \lambda J\xi + \mu U_1$, $\mathcal{S}U_1 = \mu J\xi$ and $\mathcal{S}X = 0$ for any $X \in TM \ominus \operatorname{span}\{J\xi, U_1\}$.

Note that condition (b) can be reformulated by saying that the only nonzero elements of the matrix expression of $\mathcal{S}$ with respect to an orthonormal basis $\{J\xi, U_1, \dots, U_{2n-2}\}$ lie in its 2×2 upper-left submatrix, which has the form

$$\mathcal{S}|_{\operatorname{span}\{J\xi,U_1\}} = \begin{pmatrix} \lambda & \mu \\ \mu & 0 \end{pmatrix}.$$

Moreover, M is minimal if, and only if, $\lambda = 0$.

This algebraic description of the shape operator of M will allow us to easily deduce some properties of ruled real hypersurfaces in nonflat complex space forms concerning, for example, their principal curvatures. Such description will be used in Section 2.2 in order to compute the Levi-Civita connection of M.

Any ruled real hypersurface in a nonflat complex space form $\bar{M}$ can be constructed as follows (see, for example, [1], [22], [51]). We consider a unit speed curve $\gamma\colon I \subset \mathbb{R} \to \bar{M}$, which will be called *generating curve*, and at each point $\gamma(s)$, we attach totally geodesic complex hyperplanes $\mathbb{C}P^{n-1}$ or $\mathbb{C}H^{n-1}$ (depending on the ambient space) orthogonally to the plane spanned by $\{\dot\gamma(s), J\dot\gamma(s)\}$, where $\dot\gamma$ denotes the velocity of γ. The union of these hyperplanes yields a ruled real hypersurface of $\bar{M}$. Locally, ruled real hypersurfaces are embedded, but globally they may have singularities, so in general one may have to restrict to a small neighborhood of γ in order to avoid them. Every integral curve of the Hopf vector field $J\xi$ on a ruled real hypersurface M is a generating curve of M and, conversely, any generating curve of a ruled real hypersurface M is an integral curve of $J\xi$, up to reparametrization.

2.1.1 Geometry of a ruled real hypersurface in terms of a generating curve

Lohnherr and Reckziegel [51] investigated the extrinsic geometry of a ruled real hypersurface M in terms of the geometry of a generating curve γ. In this subsection we summarize some of their results, which will be needed in Section 2.4. It will be enough, for our purposes, to restrict to complex hyperbolic spaces, although similar results hold for the projective case.

Let then $\gamma\colon I \to \mathbb{C}H^n$ be a unit speed curve, where $I \subset \mathbb{R}$ denotes an open interval with $0 \in I$. We decompose $\bar{\nabla}_{\dot\gamma}\dot\gamma$ into its $(J\dot\gamma)$ and $(\mathbb{C}\dot\gamma)^\perp$ components as

$$\bar{\nabla}_{\dot\gamma}\dot\gamma = -\lambda_\gamma J\dot\gamma + N_\gamma,$$

where λ_γ is a smooth function along γ, and N_γ is a smooth vector field in $(\mathbb{C}\dot\gamma)^\perp$ along γ. We also denote $\mu_\gamma = |N_\gamma|$.

Fix a linear isomorphism between $\mathbb{C}^{n-1}$ and $(\mathbb{C}\dot\gamma(0))^\perp$. Then, for each unit vector $v \in \mathbb{C}^{n-1} \cong (\mathbb{C}\dot\gamma(0))^\perp$ and $s \in \mathbb{R}$, let us denote by $Z_{sv}(t)$ the parallel transport of sv along γ with respect to the connection $\widetilde{\nabla}$ of the bundle $(\mathbb{C}\dot\gamma)^\perp$ given by $\widetilde{\nabla}_X Y := (\bar{\nabla}_X Y)_{(\mathbb{C}\dot\gamma)^\perp}$, where the subscript means orthogonal projection. Now we can consider the (maybe singular) parametrization $f\colon I \times \mathbb{C}^{n-1} \to \mathbb{C}H^n$ given by

$$f(t, sv) = \exp_{\gamma(t)} Z_{sv}(t),$$

which formalizes the construction of a (maybe singular) ruled real hypersurface $M := f(I \times \mathbb{C}^{n-1})$. The singular points of M, that is, the points $f(t, sv)$ such that $f_{*(t,sv)}$ is not an immersion, turn out to be precisely the points for which $f_{*(t,sv)}(\partial_t) = 0$, which happens exactly when the function

$$\rho(t, sv) := |f_{*(t,sv)}(\partial_t)| = \left(x(t, sv)^2 + y(t, sv)^2\right)^{1/2} \tag{2.1}$$

vanishes, where

$$x(t, sv) := \cosh\left(\frac{\sqrt{|c|}}{2}s\right) - \frac{2}{\sqrt{|c|}}\langle Z_v(t), N_\gamma(t)\rangle \sinh\left(\frac{\sqrt{|c|}}{2}s\right), \tag{2.2}$$

$$y(t, sv) := -\frac{2}{\sqrt{|c|}}\langle Z_v(t), JN_\gamma(t)\rangle \sinh\left(\frac{\sqrt{|c|}}{2}s\right).$$

It easily follows that the ruling $M_t = \{f(t, sv) : sv \in \mathbb{C}^{n-1}\}$ through $\gamma(t)$ contains singular points if, and only if, $\mu_\gamma(t) > \sqrt{-c}/2$. Let $M_{reg} = \{f(t, sv) : \rho(t, sv) \neq 0\}$

be the open subset of M of regular points. It turns out that M_{reg} is connected. Moreover, M_{reg} is a smooth ruled real hypersurface with unit normal vector field $\xi_{f(t,sv)} = -Jf_{*(t,sv)}(\partial_t)/\rho(t,sv)$ and Hopf vector field $J\xi_{f(t,sv)} = f_{*(t,sv)}(\partial_t)/\rho(t,sv)$. In particular, the facts stated at the beginning of this section hold for M_{reg}. It turns out that the functions μ and λ describing the shape operator of M_{reg} are given by

$$\begin{aligned}\mu(t,sv) &= \left(\frac{1}{\rho(t,sv)^2}\left(\mu_\gamma(t)^2 + \frac{c}{4}\right) - \frac{c}{4}\right)^{1/2}, \\ \lambda(t,sv) &= \frac{1}{\rho(t,sv)}\left(\lambda_\gamma(t) + \frac{1}{\rho(t,sv)^2}(y\partial_t x - x\partial_t y)(t,sv)\right).\end{aligned} \tag{2.3}$$

From the first expression, it follows that either $\mu < \sqrt{-c}/2$, $\mu = \sqrt{-c}/2$ or $\mu > \sqrt{-c}/2$ on each ruling. Indeed, we have either $\mu(M_t \cap M_{reg}) = [0, \sqrt{-c}/2)$, $\mu(M_t \cap M_{reg}) = \{\sqrt{-c}/2\}$, or $\mu(M_t \cap M_{reg}) = (\sqrt{-c}/2, +\infty)$, respectively.

As we have already mentioned, similar formulas as above (with spherical trigonometric functions instead of hyperbolic functions) hold when the ambient space is a complex projective space, although some behaviors are different. For example, in the projective case, each ruling has singularities (i.e. $M_t \cap M_{reg} \neq M_t$, for each $t \in I$), and $\mu(M_t \cap M_{reg})$ is always $[0, +\infty)$. For more details, we refer to [51].

2.1.2 Ruled minimal hypersurfaces. The examples

Ruled minimal hypersurfaces constitute an important subclass of ruled real hypersurfaces in complex projective and hyperbolic spaces. The classification of ruled minimal hypersurfaces in nonflat complex space forms has been achieved by Lohnherr and Reckziegel in [51] (see also [1]), where they proved that any ruled minimal hypersurface in $\bar{M}$ must be an open part of one of the following hypersurfaces:

(i) a Kimura-type hypersurface in a complex projective or hyperbolic space, or

(ii) a bisector in a complex hyperbolic space, or

(iii) a Lohnherr hypersurface in a complex hyperbolic space.

One can easily construct ruled minimal hypersurfaces in nonflat complex space forms as above by requiring the generating curve γ to be a circle contained in a totally geodesic real projective plane $\mathbb{R}P^2$ or in a totally geodesic real hyperbolic plane $\mathbb{R}H^2$, depending on whether $\bar{M}$ is a complex projective or hyperbolic space, respectively. By definition (see [12, Subsection 10.4.2]), a *circle* is a smooth curve $\gamma\colon I \to \bar{M}$ parametrized by arc length with constant curvature $\kappa = |\bar{\nabla}_{\dot\gamma}\dot\gamma|$ satisfying the relation $\bar{\nabla}_{\dot\gamma}\bar{\nabla}_{\dot\gamma}\dot\gamma = -\kappa^2\dot\gamma$. Indeed, every generating curve of an arbitrary minimal ruled hypersurface in a nonflat complex space form is a circle which lies in some totally geodesic real projective or hyperbolic plane of constant sectional curvature $c/4$, $\mathbb{R}P^2$ or $\mathbb{R}H^2$, respectively (see, for example, [1]).

There exist three types of circles in $\mathbb{R}H^2$ depending on the value of the curvature κ with respect to the critical value $\sqrt{|c|}/2$. If $\kappa = \sqrt{|c|}/2$, the circle is a horocycle, that is, a one-dimensional horosphere in $\mathbb{R}H^2$. If $\kappa > \sqrt{|c|}/2$, the circle is a one-dimensional geodesic sphere in $\mathbb{R}H^2$ (we call it also a closed circle), whereas if $\kappa < \sqrt{|c|}/2$, we obtain an equidistant curve to a geodesic in $\mathbb{R}H^2$, which we will call unbounded circle. Each of the three types of circles corresponds exactly to a type of ruled minimal hypersurface in $\mathbb{C}H^n$ according to Table 2.1. Recall that by c we denote the constant holomorphic sectional curvature of the ambient complex space form $\bar{M}$.

Circle type	κ	Hypersurface type
Closed circle	$\kappa > \sqrt{\vert c\vert}/2$	Kimura-type hypersurface
Horocycle	$\kappa = \sqrt{\vert c\vert}/2$	Lohnherr hypersurface
Unbounded circle	$\kappa < \sqrt{\vert c\vert}/2$	Bisector

Table 2.1: Circle types and ruled minimal hypersurfaces in $\mathbb{C}H^n$

In the projective case, there exists a unique type of circle in $\mathbb{R}P^2$ (namely one-dimensional geodesic spheres, which we call also closed circles), which corresponds with the unique type of minimal ruled hypersurfaces in $\mathbb{C}P^n$.

We now give a more detailed description of these three examples of minimal ruled hypersurfaces in nonflat complex space forms.

Kimura-type hypersurfaces

A *Kimura-type hypersurface* [47] in a nonflat complex space form $\bar{M}$ is the ruled minimal hypersurface constructed by attaching totally geodesic complex hyperplanes $\mathbb{C}P^{n-1}$ or $\mathbb{C}H^{n-1}$ orthogonally to the points of a closed circle (of curvature $\kappa > \sqrt{|c|}/2$ in the hyperbolic case) contained in a totally geodesic real projective or hyperbolic plane, $\mathbb{R}P^2$ or $\mathbb{R}H^2$, respectively.

If $n = 2$, this hypersurface is called a *Clifford cone* and it can also be constructed as follows. The Lie group $U(1)\times U(1)$ acts polarly with cohomogeneity two on $\bar{M}^2(c)$. Such action has three fixed points in $\mathbb{C}P^2$ whereas it has only one fixed point in $\mathbb{C}H^2$. Let p denote one of these fixed points and let $S^3_p(r)$ be a geodesic sphere centered at p. Then, the Clifford cone with vertex p is the singular hypersurface consisting of all geodesic rays starting from p and hitting the only two-dimensional orbit of the action of $U(1) \times U(1)$ that is minimal as a submanifold of $S^3_p(r)$ (see [28]). If $n > 2$, one can construct a Kimura-type hypersurface using a Clifford cone as follows. We consider a Clifford cone inside a totally geodesic complex projective or hyperbolic plane, $\mathbb{C}P^2 \subset \mathbb{C}P^n$ or $\mathbb{C}H^2 \subset \mathbb{C}H^n$, and we attach totally geodesic $\mathbb{C}P^{n-2}$ or $\mathbb{C}H^{n-2}$, respectively, perpendicularly to the complex projective or hyperbolic plane along the points of the Clifford cone.

Bisectors

A *bisector* in a complex hyperbolic space is a ruled minimal hypersurface constructed by attaching totally geodesic hyperplanes $\mathbb{C}H^{n-1}$ perpendicularly to the points of an unbounded circle of curvature $\kappa < \sqrt{|c|}/2$ contained in a totally geodesic real hyperbolic plane $\mathbb{R}H^2$.

One can give an alternative definition of a bisector as a geometric locus. Indeed, given two points p, $q \in \mathbb{C}H^n$, the bisector they determine is the set of points which are equidistant to p and q. In the setting of real space forms, bisectors are totally geodesic hypersurfaces. However, in the context of complex space forms, totally geodesic hypersurfaces cannot occur (see [72]), so bisectors constitute one of the examples of real hypersurfaces in such spaces that are closer to being totally geodesic.

The Lohnherr hypersurface

The *Lohnherr hypersurface* of $\mathbb{C}H^n$ (also called a fan [40]) is the ruled minimal hypersurface constructed by attaching totally geodesic hyperplanes $\mathbb{C}H^{n-1}$ orthogonally to a horocycle contained in a totally geodesic real hyperbolic plane $\mathbb{R}H^2$.

It is known that the Lohnherr hypersurface can be characterized as the unique complete ruled hypersurface of $\mathbb{C}H^n$ (or even of a nonflat complex space form) having constant principal curvatures [51]; specifically, its principal curvatures are $\pm\sqrt{-c}/2$, both with multiplicity one, and 0. Furthermore, the Lohnherr hypersurface is also the only minimal hypersurface of $\mathbb{C}H^n$ which is homogeneous [13].

Taking into account its homogeneity, one can give an alternative construction of the Lohnherr hypersurface using the Lie group structure of the isometry group $I(\mathbb{C}H^n)$. Following Subsections 1.4.2 and 1.5.4, let $G = SU(1,n)$, which is a finite covering of the connected component of the identity of the isometry group of the complex hyperbolic space, and consider its Iwasawa decomposition $G = KAN$, where $K = S(U(1)U(n))$ is the isotropy group of a fixed point $o \in \mathbb{C}H^n$. Consider also the Iwasawa decomposition at the level of Lie algebras, that is, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ according to the root space decomposition theorem. We now choose a linear hyperplane $\mathfrak{w} \subset \mathfrak{g}_\alpha$ and define $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, which turns out to be a Lie subalgebra of the solvable part of the Iwasawa decomposition of $\mathfrak{g}$, $\mathfrak{a} \oplus \mathfrak{n}$. Let us denote by S the connected Lie subgroup of G whose Lie algebra is $\mathfrak{s}$. The orbit of this subgroup S through o is a Lohnherr hypersurface of $\mathbb{C}H^n$ that will be denoted by $W^{2n-1} := S \cdot o$.

2.2 The Levi-Civita connection of a ruled hypersurface in a complex space form

The main purpose of this section is to compute the Levi-Civita connection of an arbitrary ruled real hypersurface in a nonflat complex space form. Let then M be a ruled real hypersurface in a nonflat complex space form $\bar{M}$ with (locally defined) unit normal vector field ξ. We start by studying the principal curvatures of such a hypersurface. Recall that $h\colon M \to \mathbb{N}$ is the function which gives, for each point $p \in M$, the number of eigenspaces of the shape operator of M at p onto which the Hopf vector field $J\xi$ has nontrivial projection.

Proposition 2.2.1. *Let $p \in M$ such that $h(p) \neq 1$. Then $h(p) = 2$, M has exactly two nonzero principal curvatures α, β at p, both of multiplicity one, and 0 is always a principal curvature of M. Moreover, $J\xi_p = au + bv$ for $u \in T_\alpha(p)$, $v \in T_\beta(p)$ and a, $b \in \mathbb{R}$ such that $a^2 + b^2 = 1$ and*

$$a^2 = \frac{\alpha}{\alpha - \beta}, \qquad b^2 = \frac{\beta}{\beta - \alpha}. \tag{2.4}$$

Proof. Fix $p \in M$. From the definition of ruled real hypersurface in a nonflat complex space form above we know that $\mathcal{S}(J\xi)^\perp \subset \mathbb{R}J\xi$, and hence, the shape operator $\mathcal{S}$ of M at p satisfies

$$\mathcal{S}J\xi_p = \lambda J\xi_p + \mu z, \qquad \mathcal{S}z = \mu J\xi_p, \qquad \mathcal{S}w = 0,$$

for certain unit vector $z \in T_pM$ orthogonal to $J\xi_p$, and for all $w \in T_pM$ perpendicular to $J\xi_p$ and z. Let α and β be the eigenvalues of $\mathcal{S}$ restricted to the invariant subspace $\mathbb{R}J\xi_p \oplus \mathbb{R}z$, and u, v some corresponding orthogonal eigenvectors of unit length. Thus, one can write $J\xi_p = au + bv$ for some a, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Moreover, $a \neq 0 \neq b$ since $h(p) \neq 1$ by assumption, and hence, we must have $h(p) = 2$. Then

$$\lambda = \langle \mathcal{S}J\xi_p, J\xi_p \rangle = \langle a\alpha u + b\beta v, au + bv \rangle = a^2\alpha + b^2\beta.$$

Moreover, we have $\lambda = \alpha + \beta$ due to the invariance of the trace of $\mathcal{S}$. Both equations imply $\alpha \neq \beta$, since otherwise this would give us that $\alpha = \lambda = 2\alpha$, producing $\alpha = \beta = 0$, which contradicts $h(p) \neq 1$. Finally, combining again both equations with $a^2 + b^2 = 1$ we obtain the formulas for a^2 and b^2 in the statement. $\square$

Notice that, from the matrix form of the shape operator $\mathcal{S}$ given in the previous section, without restriction of generality, the principal curvatures α and β of M can be expressed in terms of the functions λ and μ defined in Section 2.1 as follows:

$$\alpha = \frac{\lambda - \sqrt{\lambda^2 + 4\mu^2}}{2} \qquad \text{and} \qquad \beta = \frac{\lambda + \sqrt{\lambda^2 + 4\mu^2}}{2}.$$

It is known that ruled hypersurfaces in a nonflat complex space form cannot be Hopf. This implies that no open subset of a ruled hypersurface M in $\bar{M}^n(c)$, $c \neq 0$, is Hopf. Thus, by virtue of Proposition 2.2.1, $h = 2$ on an open and dense subset $\mathcal{U}$ of M.

Again by Proposition 2.2.1 we know that, at each point, $\mathcal{U}$ has exactly two distinct nonzero principal curvatures. Altogether, $\mathcal{U}$ has exactly two nonzero principal curvature functions α and β, both of multiplicity one at every point. We also have $J\xi = aU + bV$ for some unit vector fields $U \in \Gamma(T_\alpha)$ and $V \in \Gamma(T_\beta)$ and smooth functions a, $b \colon \mathcal{U} \to \mathbb{R}$ with $a^2 + b^2 = 1$, and again by Proposition 2.2.1, satisfying (2.4). In particular, since $h = 2$ on $\mathcal{U}$, this equation implies $\alpha \neq 0 \neq \beta$ at every point of $\mathcal{U}$. From now on we will work on the open and dense subset $\mathcal{U}$ of M.

The following result has been proved under slightly different assumptions in [27], [28] and [29]. Although the proof in our setting is similar, we include it for completeness.

Proposition 2.2.2. *With the notation above, there exists $A \in \Gamma(T_0)$ such that*

$$J\xi = aU + bV, \qquad JU = -bA - a\xi,$$
$$JA = bU - aV, \qquad JV = aA - b\xi.$$

Proof. The Hopf vector field $J\xi$ is a unit tangent vector field to M that has nontrivial projection onto the principal curvature spaces T_α and T_β. Then, one can write $J\xi = aU + bV$, where $U \in \Gamma(T_\alpha)$ and $V \in \Gamma(T_\beta)$ are unit vector fields, and a, b are smooth functions on $\mathcal{U}$ such that $a^2 + b^2 = 1$ and $a, b > 0$.

Since $-\xi = J^2\xi = aJU + bJV$ and $a \neq 0$, taking inner product with V one gets $\langle JU, V\rangle = 0$. Moreover, $\langle JU, \xi\rangle = -\langle U, J\xi\rangle = -\langle U, aU + bV\rangle = -a$. Then, there exists a unit vector field $A \in \Gamma(T_0)$ such that $JU \in \operatorname{span}\{A, \xi\}$, and since U has length 1, we get $\langle JU, A\rangle = \pm b$. By changing the sign of A if necessary, we can assume that $JU = -bA - a\xi$. Similarly, one can show that $JV = aA - b\xi$. Finally, these expressions imply $\langle JA, U\rangle = b$, $\langle JA, V\rangle = -a$ and $\langle JA, \xi\rangle = 0$, from where $JA = bU - aV$. $\square$

With these ingredients, we compute the Levi-Civita connection of M.

Proposition 2.2.3. *Let M be a ruled hypersurface in a nonflat complex space form. Then, its Levi-Civita connection satisfies the following equations:*

$$\langle \nabla_U U, V\rangle = \frac{V(\alpha)}{\alpha - \beta}, \qquad \langle \nabla_V V, U\rangle = -\frac{U(\beta)}{\alpha - \beta},$$
$$\langle \nabla_U U, A\rangle = \frac{4A(\alpha) - 3abc}{4\alpha}, \qquad \langle \nabla_V V, A\rangle = \frac{4A(\beta) + 3abc}{4\beta},$$
$$\langle \nabla_U V, A\rangle = \frac{3c}{4(\alpha - \beta)} + \alpha - \frac{aA(\alpha)}{b\alpha}, \qquad \langle \nabla_V U, A\rangle = \frac{3c}{4(\alpha - \beta)} - \beta - \frac{bA(\beta)}{a\beta},$$
$$\langle \nabla_A U, V\rangle = \frac{ac\beta - 4a\alpha\beta^2 - 4b\alpha A(\beta)}{4a\beta(\alpha - \beta)}, \qquad \nabla_A A = 0.$$

Moreover, for any unit vector field orthogonal to A in the 0-principal curvature distribution, $W \in \Gamma(T_0 \ominus \mathbb{R}A)$, the following relations hold:

$$\langle \nabla_U U, W\rangle = \frac{W(\alpha)}{\alpha}, \qquad \langle \nabla_V V, W\rangle = \frac{W(\beta)}{\beta}, \qquad \langle \nabla_W U, V\rangle = \frac{b\alpha W(\beta)}{a\beta(\beta - \alpha)},$$
$$\langle \nabla_U V, W\rangle = -\frac{aW(\alpha)}{b\alpha}, \qquad \langle \nabla_V U, W\rangle = -\frac{bW(\beta)}{a\beta}.$$

In addition,

$$U(\beta) = -\frac{\beta^2 U(\alpha) + 2ab\alpha(\alpha - \beta)V(\beta)}{3\alpha\beta}, \qquad V(\alpha) = \frac{2ab\beta(\alpha - \beta)U(\alpha) - \alpha^2 V(\beta)}{3\alpha\beta},$$
$$A(\alpha) = \frac{b(\alpha - \beta)(a\beta(4\alpha\beta - c) + 2b\alpha A(\beta))}{2\beta^2}, \qquad W(\alpha) = -\frac{\alpha}{\beta}W(\beta).$$

Proof. Since U and V are orthogonal eigenvectors of the shape operator $\mathcal{S}$ associated with the eigenvalues α and β, respectively, we have

$$\begin{aligned}\langle(\nabla_U\mathcal{S})V,U\rangle &= \langle\nabla_U(\mathcal{S}V)-\mathcal{S}\nabla_U V,U\rangle = \langle\nabla_U(\beta V),U\rangle-\alpha\langle\nabla_U V,U\rangle\\ &= \langle U(\beta)V+\beta\nabla_U V,U\rangle-\alpha\langle\nabla_U V,U\rangle = -(\alpha-\beta)\langle\nabla_U V,U\rangle.\end{aligned}$$

As U has constant length, $\langle\nabla_V U,U\rangle=0$, and thus, proceeding as before,

$$\begin{aligned}\langle(\nabla_V\mathcal{S})U,U\rangle &= \langle\nabla_V(\mathcal{S}U)-\mathcal{S}\nabla_V U,U\rangle = \langle\nabla_V(\alpha U),U\rangle-\alpha\langle\nabla_V U,U\rangle\\ &= \langle V(\alpha)U+\alpha\nabla_V U,U\rangle-\alpha\langle\nabla_V U,U\rangle = V(\alpha).\end{aligned}$$

Using the expression for the curvature tensor of a complex space form (1.2) and the relations given in Proposition 2.2.2, we obtain that $\langle\bar{R}(U,V)U,\xi\rangle=0$. Then, using the previous relations and applying the Codazzi equation to the triple (U,V,U), we get

$$0=V(\alpha)+(\alpha-\beta)\langle\nabla_U V,U\rangle, \tag{2.5}$$

which gives the first relation in the statement. Analogously, the Codazzi equation applied to the triple (V,U,V) yields

$$0=U(\beta)-(\alpha-\beta)\langle\nabla_V U,V\rangle, \tag{2.6}$$

which is equivalent to the second relation in the statement.

Since $\bar{\nabla}J=0$, using the definition of the shape operator and the relations $J\xi = aU+bV$ and $\langle\nabla_U U,U\rangle=0$, we obtain

$$\begin{aligned}U(a) &= U\langle J\xi,U\rangle = \langle\bar{\nabla}_U J\xi,U\rangle+\langle J\xi,\bar{\nabla}_U U\rangle = \langle J\bar{\nabla}_U\xi,U\rangle+\langle J\xi,\nabla_U U\rangle\\ &= \langle\mathcal{S}U,JU\rangle+\langle aU+bV,\nabla_U U\rangle = b\langle V,\nabla_U U\rangle = -b\langle\nabla_U V,U\rangle.\end{aligned}$$

By multiplying this expression by $2a$ and taking into account that

$$2aU(a)=U(a^2)=U\left(\frac{\alpha}{\alpha-\beta}\right)=\frac{\alpha U(\beta)-\beta U(\alpha)}{(\alpha-\beta)^2},$$

we get, using (2.5),

$$0=\beta U(\alpha)-\alpha U(\beta)+2ab(\alpha-\beta)V(\alpha). \tag{2.7}$$

Analogously, expanding the relation $V(a)=V\langle J\xi,U\rangle$, we deduce, inserting (2.6), that

$$0=\beta V(\alpha)-\alpha V(\beta)-2ab(\beta-\alpha)U(\beta). \tag{2.8}$$

Equations (2.7) and (2.8) constitute a linear system in the unknowns $U(\beta)$ and $V(\alpha)$. After some calculations using (2.4), we get that the determinant of the matrix of this system vanishes if and only if $\alpha\beta$ does, which cannot occur in $\mathcal{U}$. Then, there exists a unique solution given by

$$U(\beta)=-\frac{2ab\alpha(\alpha-\beta)V(\beta)+\beta^2U(\alpha)}{3\alpha\beta},\qquad V(\alpha)=\frac{2ab\beta(\alpha-\beta)U(\alpha)-\alpha^2V(\beta)}{3\alpha\beta}. \tag{2.9}$$

Now, proceeding as above, the Codazzi equation applied to the triples (U, A, U), (V, A, V), (A, U, A) and (A, V, A) yields

$$\begin{aligned} \langle \nabla_U U, A \rangle &= \frac{4A(\alpha) - 3abc}{4\alpha}, & \langle \nabla_V V, A \rangle &= \frac{3abc + 4A(\beta)}{4\beta}, \\ \langle \nabla_A A, U \rangle &= 0, & \langle \nabla_A A, V \rangle &= 0. \end{aligned} \tag{2.10}$$

Since $\bar{\nabla} J = 0$ and $J\xi = aU + bV$, expanding the relations $0 = U\langle J\xi, A\rangle$ and $0 = V\langle J\xi, A\rangle$, inserting the expressions for $\langle \nabla_U A, U\rangle$ and $\langle \nabla_V A, V\rangle$ that follow from (2.10), and using (2.4), we have

$$\langle \nabla_U V, A \rangle = \frac{3c}{4(\alpha-\beta)} + \alpha - \frac{aA(\alpha)}{b\alpha}, \quad \langle \nabla_V A, U \rangle = \frac{bA(\beta)}{a\beta} + \beta - \frac{3c}{4(\alpha-\beta)}. \tag{2.11}$$

Now, the Codazzi equation applied to the triple (V, A, U) yields, after inserting the expression for $\langle \nabla_V A, U\rangle$ given in (2.11),

$$\begin{aligned} 0 &= \langle \bar{R}(V, A)U, \xi \rangle - \langle (\nabla_V \mathcal{S})A, U \rangle + \langle (\nabla_A \mathcal{S})V, U \rangle \\ &= \frac{c(2\alpha+\beta)}{4(\alpha-\beta)} + \alpha \langle \nabla_V A, U \rangle - (\alpha-\beta)\langle \nabla_A V, U \rangle \\ &= -\frac{c}{4} + \alpha\beta + (\beta-\alpha)\langle \nabla_A V, U \rangle + \frac{b\alpha A(\beta)}{a\beta}, \end{aligned}$$

from where

$$\langle \nabla_A V, U \rangle = \frac{4b\alpha A(\beta) + 4a\alpha\beta^2 - ac\beta}{4a\beta(\alpha-\beta)}. \tag{2.12}$$

Similarly, applying the Codazzi equation to the triple (U, V, A) and using the expressions for $\langle \nabla_U V, A\rangle$ and $\langle \nabla_V U, A\rangle$ given by (2.11), we obtain

$$\begin{aligned} 0 &= \langle \bar{R}(U, V)A, \xi \rangle - \langle (\nabla_U \mathcal{S})V, A \rangle + \langle (\nabla_V \mathcal{S})U, A \rangle \\ &= -\frac{c}{4} - \beta \langle \nabla_U A, V \rangle + \alpha \langle \nabla_V A, U \rangle = \frac{c}{2} - 2\alpha\beta + \frac{a\beta A(\alpha)}{b\alpha} - \frac{b\alpha A(\beta)}{a\beta}, \end{aligned}$$

from where, using (2.4), we get the following relation between $A(\alpha)$ and $A(\beta)$:

$$A(\alpha) = \frac{b(\alpha-\beta)(a\beta(4\alpha\beta - c) + 2b\alpha A(\beta))}{2\beta^2}. \tag{2.13}$$

Finally, let $W \in \Gamma(T_0 \ominus \mathbb{R}A)$ be an arbitrary unit vector field orthogonal to A in the 0-principal curvature distribution. Using the expressions for $J\xi$, JU and JA given in Proposition 2.2.2, the Codazzi equation applied to the triple (A, U, W) yields $\langle \nabla_A U, W\rangle = 0$, from where we deduce, using (2.10), that $\nabla_A U$ is proportional to V. In particular, since $T_0 \ominus \mathbb{R}A$ is a complex distribution, $\langle \nabla_A U, JW\rangle = 0$. Expanding the relation $0 = A\langle JU, W\rangle$, and taking the previous fact into account, as well as the expression for JU given in Proposition 2.2.2, one gets

$$b\langle \nabla_A A, W \rangle = \langle \nabla_A U, JW \rangle = 0.$$

Then, $\langle \nabla_A A, W\rangle = 0$, and in fact, using (2.10), we obtain that $\nabla_A A = 0$.

Proceeding as above, the Codazzi equation applied to the triples (U, W, U) and (V, W, V) yields

$$\langle \nabla_U U, W\rangle = \frac{W(\alpha)}{\alpha}, \qquad \langle \nabla_V V, W\rangle = \frac{W(\beta)}{\beta}. \tag{2.14}$$

Expanding the relations $0 = U\langle J\xi, W\rangle$ and $0 = V\langle J\xi, W\rangle$ and inserting the expressions for $\langle \nabla_U U, W\rangle$ and $\langle \nabla_V V, W\rangle$ given by (2.14), we have

$$\langle \nabla_U V, W\rangle = -\frac{aW(\alpha)}{b\alpha}, \qquad \langle \nabla_V U, W\rangle = -\frac{bW(\beta)}{a\beta}. \tag{2.15}$$

After some calculations using (2.14) and (2.15), the Codazzi equation applied to the triple (V, W, U) yields

$$\langle \nabla_W V, U\rangle = \frac{b\alpha W(\beta)}{a\beta(\alpha-\beta)}, \tag{2.16}$$

and analogously, the Codazzi equation applied to the triple (U, V, W) reads

$$0 = \frac{a\beta W(\alpha)}{b\alpha} - \frac{b\alpha W(\beta)}{a\beta},$$

from where we get, after using (2.4), the last formula in the statement. □

2.3 Ruled hypersurfaces with constant mean curvature

The motivation for studying ruled hypersurfaces with constant mean curvature in a given Riemannian manifold comes from a classical theorem due to Catalan [21], which claims that the only ruled minimal surfaces in the Euclidean space $\mathbb{R}^3$ are planes and helicoids. This result has been extended in several directions in the context of spaces of constant curvature. In particular, Barbosa and Delgado proved in [9] that there are no ruled hypersurfaces with nonzero constant mean curvature in nonflat real space forms other than the 3-sphere. The aim of this section is to present the complex analog of this result, that is, the classification of ruled real hypersurfaces with constant mean curvature in nonflat complex space forms. More specifically, we prove the following result.

Theorem 2.3.1. *Let M be a ruled real hypersurface with constant mean curvature in a nonflat complex space form. Then, M is minimal.*

This theorem, together with the result in [1] and [51] described in Subsection 2.1.2, yields the classification of ruled real hypersurfaces with constant mean curvature in complex projective and hyperbolic spaces. In the flat setting, the corresponding classification follows form the results in [8] and [9], which deal with ruled hypersurfaces with constant mean curvature in the real space forms.

In order to prove Theorem 2.3.1, let $\mathcal{H} = \alpha + \beta$ be the mean curvature of M on the open subset $\mathcal{U}$ defined in Subsection 2.2. By hypothesis, $\mathcal{H}$ is a constant function, say k, so $X(k) = 0$ for each $X \in T\mathcal{U}$ or, equivalently,

$$X(\alpha) = -X(\beta) \quad \text{for each } X \in T\mathcal{U}. \tag{2.17}$$

Considering this fact, one can rewrite some of the relations given in Proposition 2.2.3 in an easier way.

Proposition 2.3.2. *The Levi-Civita connection of the open subset $\mathcal{U}$ satisfies the following equations:*

$$\nabla_U U = -\frac{ab(c+8\alpha(k-\alpha))}{4\alpha} A, \qquad \nabla_U V = \frac{c+4k\alpha}{4(2\alpha-k)} A,$$
$$\nabla_V V = \frac{ab(c+8\alpha(k-\alpha))}{4(k-\alpha)} A, \qquad \nabla_V U = \frac{c+4k(k-\alpha)}{4(2\alpha-k)} A.$$

Moreover,

$$U(\alpha) = V(\alpha) = W(\alpha) = 0, \qquad \text{and} \qquad A(\alpha) = \frac{ab}{2}(c+4\alpha(\alpha-k)), \tag{2.18}$$

where $W \in \Gamma(T_0 \ominus \mathbb{R}A)$ denotes an arbitrary unit vector field orthogonal to A in the 0-principal curvature distribution.

Proof. First of all, we prove that $U(\alpha) = V(\alpha) = 0$. In order to do so, we rewrite the expressions for $U(\beta)$ and $V(\alpha)$ given in Proposition 2.2.3 using (2.17). Some straightforward calculations show that these equations are equivalent to

$$\beta(3\alpha-\beta)U(\alpha) + 2ab\alpha(\alpha-\beta)V(\alpha) = 0,$$
$$2ab\beta(\alpha-\beta)U(\alpha) - \alpha(3\beta-\alpha)V(\alpha) = 0,$$

which constitute a homogeneous linear system in the unknowns $U(\alpha)$ and $V(\alpha)$. Using (2.4), one deduces that the determinant of the matrix of this system is $3\alpha\beta(\alpha-\beta)^2$, which cannot vanish because $\alpha\beta \neq 0$ and $\alpha \neq \beta$ on the open subset $\mathcal{U}$. Thus, $U(\alpha) = V(\alpha) = 0$.

Using again (2.17), some easy calculations involving (2.4) show that the expression for $A(\alpha)$ given in Proposition 2.2.3 can be rewritten as $A(\alpha) = ab(c - 4\alpha\beta)/2$, which is equivalent to the last formula in the statement since $\beta = k - \alpha$.

Let $W \in \Gamma(T_0 \ominus \mathbb{R}A)$. By virtue of Proposition 2.2.3, $W(\alpha) = -\alpha W(\beta)/\beta$ and, using (2.17), $W(\alpha) = \alpha W(\alpha)/\beta$. Since $\alpha \neq \beta$ on the open set $\mathcal{U}$, it follows that $W(\alpha) = 0$.

Inserting the expressions for $U(\alpha), V(\alpha), W(\alpha)$ and $A(\alpha)$ that we have just obtained into the relations given in Proposition 2.2.3, and using (2.17) and the fact that $\beta = k - \alpha$, one gets, after some calculations involving (2.4), the formulas for $\nabla_U U$, $\nabla_U V$, $\nabla_V U$ and $\nabla_V V$ in the statement. □

Now we can conclude the proof of Theorem 2.3.1.

Proof of Theorem 2.3.1. According to the discussion before Proposition 2.2.2, there is an open and dense subset $\mathcal{U}$ of M where $h = 2$. Propositions 2.2.2 and 2.3.2 hold on this open subset. By (2.18), the definition of the Lie bracket of M yields

$$[U, V](\alpha) = U(V(\alpha)) - V(U(\alpha)) = 0. \tag{2.19}$$

On the other hand, using the fact that the Levi-Civita connection of M is torsion-free, and inserting the expressions for $\nabla_U V$, $\nabla_V U$ and $A(\alpha)$ given in Proposition 2.3.2, we obtain

$$\begin{aligned}[U, V](\alpha) = (\nabla_U V)(\alpha) - (\nabla_V U)(\alpha) &= \left(\frac{c + 4k\alpha}{4(2\alpha - k)}\right) A(\alpha) + \left(\frac{c + 4k(k - \alpha)}{4(k - 2\alpha)}\right) A(\alpha) \\ = kA(\alpha) &= \frac{kab}{2}(c + 4\alpha(\alpha - k)).\end{aligned} \tag{2.20}$$

Assume that the mean curvature $\mathcal{H} = k$ of M is nonzero. Then, since on $\mathcal{U}$ we have $a \neq 0 \neq b$, (2.19) and (2.20) imply that $c + 4\alpha(\alpha - k) = 0$, from where we deduce that α is constant on $\mathcal{U}$, and by the density of $\mathcal{U}$ on M, also on M. Therefore, M has constant principal curvatures. Then, the work of Lohnherr and Reckziegel [51, Remark 5] (or the classification of real hypersurfaces with constant principal curvatures and $h = 2$ [27], together with their explicit principal curvatures [13]) implies that M must be congruent to the Lohnherr hypersurface in $\mathbb{C}H^n$, which is minimal. Hence, $\mathcal{H} = 0$, which gives us the desired contradiction. $\square$

2.4 Ruled hypersurfaces with shape operator of constant norm

From a general viewpoint, the higher order mean curvatures of a hypersurface are defined as the elementary symmetric polynomials in the principal curvatures of such hypersurface. As we have pointed out in the previous sections, any ruled real hypersurface in a nonflat complex space form has exactly two nonzero distinct principal curvatures, both of multiplicity one, α and β. Hence, for this kind of hypersurfaces there exist only two nontrivial elementary symmetric polynomials: the mean curvature, $\mathcal{H} = \alpha + \beta$, and the second order mean curvature, $\alpha\beta$.

The norm of the shape operator constitutes yet another geometric invariant for hypersurfaces. As well as for the mean curvature, the constancy of the norm of the shape operator is a classical property in Differential Geometry, insofar as it arises naturally in important problems, such as the longstanding Chern's conjecture: a closed minimal hypersurface of a round sphere having shape operator of constant norm must be isoparametric, that is, have constant principal curvatures [7, 23]. In the particular case of ruled real hypersurfaces in nonflat complex space forms, the squared norm of the shape operator reads $|\mathcal{S}|^2 = \alpha^2 + \beta^2$. Moreover, it can be written in a simple way in terms of the mean curvature and the second order mean curvature as $|\mathcal{S}|^2 = \mathcal{H}^2 - 2\alpha\beta$.

In Section 2.3 we have classified ruled real hypersurfaces in nonflat complex space forms having constant mean curvature, and a characterization of ruled real hypersurfaces with constant second order mean curvature (or, equivalently, with constant scalar curvature) has been achieved in [48] in terms of lightlike curves in an indefinite complex projective space. Then, in view of the simple relation among the three geometric invariants above, and in order to contribute to the general problem of identifying and characterizing the "simplest" ruled hypersurfaces, cf. [22, p. 446], we wonder what happens with ruled real hypersurfaces whose shape operators have constant norm in nonflat complex space forms. This is what we study in the present section. In particular, we prove the following theorem.

Theorem 2.4.1. *Let M be a ruled real hypersurface in a nonflat complex space form. Then, the shape operator $\mathcal{S}$ of M has constant norm if, and only if, M is an open part of:*

1. *A Lohnherr hypersurface of $\mathbb{C}H^n$, or*
2. *The ruled real hypersurface which is constructed by attaching totally geodesic complex hyperbolic spaces $\mathbb{C}H^{n-1}$ perpendicularly to a circle of curvature $\kappa = \sqrt{-c/2}$ in a totally geodesic complex hyperbolic line $\mathbb{C}H^1$.*

In order to prove this result, we firstly present Theorem 2.4.2, in which we show that there are no such hypersurfaces in complex projective spaces, whereas any possible example in a complex hyperbolic space must have certain geometric property.

Theorem 2.4.2. *Let M be a ruled real hypersurface in a nonflat complex space form whose shape operator has constant norm. Then, M is a strongly 2-Hopf real hypersurface in $\mathbb{C}H^n$ with $|\mathcal{S}|^2 = -c/2$. In particular, there are no ruled hypersurfaces in complex projective spaces whose shape operator has constant norm.*

The notion of strongly 2-Hopf hypersurface has been introduced in [28] in relation to the study of cohomogeneity one real hypersurfaces in complex space forms. A real hypersurface M in a nonflat complex space form is said to be *strongly 2-Hopf* if the following conditions hold:

(i) The smallest $\mathcal{S}$-invariant distribution $\mathcal{D}$ of M that contains the Hopf vector field $J\xi$ has rank 2.

(ii) $\mathcal{D}$ is integrable.

(iii) The spectrum of $\mathcal{S}|_{\mathcal{D}}$ is constant along the integral submanifolds of $\mathcal{D}$.

Notice that, in particular, the first condition is equivalent to $h = 2$.

To prove Theorem 2.4.2, let $|\mathcal{S}|^2 = \alpha^2 + \beta^2$ be the squared norm of the shape operator of M along $\mathcal{U}$. Since by hypothesis $|\mathcal{S}|^2$ is a constant function, say k, then $X(k) = 0$ for each $X \in T\mathcal{U}$. Thus, $\alpha X(\alpha) + \beta X(\beta) = 0$, from where

$$X(\beta) = -\frac{\alpha X(\alpha)}{\beta}, \quad \text{for each } X \in T\mathcal{U}. \tag{2.21}$$

Considering this fact, one can rewrite some of the relations given in Proposition 2.2.3 in an easier way.

Proposition 2.4.3. *Suppose that $\alpha \neq -\beta$ on an open subset of $\mathcal{U}$. Then, with the previous notations, the Levi-Civita connection of such open subset satisfies the following equations:*

$$\nabla_U U = -\frac{ab(8\alpha\beta^2 + c(3\alpha+\beta))}{4\alpha(\alpha+\beta)}A, \qquad \nabla_U V = \frac{c(3\alpha+\beta) + 4\alpha(\alpha^2+\beta^2)}{4(\alpha^2-\beta^2)}A,$$
$$\nabla_V V = \frac{ab(8\alpha^2\beta + c(\alpha+3\beta))}{4\beta(\alpha+\beta)}A, \qquad \nabla_V U = \frac{c(\alpha+3\beta) + 4\beta(\alpha^2+\beta^2)}{4(\alpha^2-\beta^2)}A.$$

Moreover:

$$U(\alpha) = V(\alpha) = W(\alpha) = 0, \qquad \textit{and} \qquad A(\alpha) = \frac{ab\beta(c - 4\alpha\beta)}{2(\alpha+\beta)}, \tag{2.22}$$

for any $W \in \Gamma(T_0 \ominus \mathbb{R}A)$.

Proof. First of all, in order to prove that $U(\alpha) = V(\alpha) = 0$, we rewrite the expressions for $U(\beta)$ and $V(\alpha)$ given in Proposition 2.2.3 using the relation (2.21). Some calculations using (2.4) show that such equations are equivalent to:

$$\beta(3\alpha^2 - \beta^2)\, U(\alpha) + 2ab\alpha^2(\alpha-\beta)\, V(\alpha) = 0,$$
$$-2ab\beta^2(\alpha-\beta)\, U(\alpha) + \alpha(3\beta^2 - \alpha^2)\, V(\alpha) = 0,$$

which constitute a homogeneous linear system in the unknowns $U(\alpha)$ and $V(\alpha)$. The determinant of the matrix of such system can be easily deduced to be, using (2.4), $-3\alpha\beta(\alpha^2-\beta^2)^2$, which cannot vanish since $\alpha\beta \neq 0$ and $\alpha \neq \pm\beta$ on an open subset of $\mathcal{U}$. Then, we conclude that $U(\alpha) = V(\alpha) = 0$.

Again, using (2.21), we can rewrite the expression for $A(\alpha)$ given in Proposition 2.2.3. Some calculations using (2.4) lead us to conclude that $2(\alpha+\beta)A(\alpha) = ab\beta(c - 4\alpha\beta)$, which is equivalent to the last formula in the statement.

Given $W \in \Gamma(T_0 \ominus \mathbb{R}A)$, $W(\alpha) = -\alpha W(\beta)/\beta$ by Proposition 2.2.3 and $W(\beta) = -\alpha W(\alpha)/\beta$ by (2.21). Then $W(\alpha)(1 - \alpha^2/\beta^2) = 0$, from where we deduce, since $\alpha \neq \pm\beta$, that $W(\alpha) = 0$.

Inserting the expressions for $U(\alpha)$, $V(\alpha)$, $W(\alpha)$ and $A(\alpha)$ that we have just obtained into the relations given in Proposition 2.2.3, one gets, after some calculations involving (2.4), the formulas for $\nabla_U U$, $\nabla_U V$, $\nabla_V U$, $\nabla_V V$ in the statement. □

We can now prove Theorem 2.4.2.

Proof of Theorem 2.4.2. First of all notice that, if $\alpha = -\beta$ on $\mathcal{U}$, then $0 = X(k) = X(2\alpha^2) = 4\alpha X(\alpha)$, which implies that $X(\alpha) = 0$ for any $X \in T\mathcal{U}$. Since $X \in TM$ is arbitrary, we deduce that both α and β have to be constant on $\mathcal{U}$, and by the density of $\mathcal{U}$, also on M.

Suppose now that there exists a point $p \in \mathcal{U}$ such that $\alpha(p) \neq -\beta(p)$. Then, in an open neighborhood of p, $\alpha \neq -\beta$. Taking the expressions given by (2.22) into account, the definition of the Lie bracket of M yields

$$[U, V](\alpha) = U(V(\alpha)) - V(U(\alpha)) = 0. \tag{2.23}$$

On the other hand, using the fact that the Levi-Civita connection of M is torsion-free, inserting the expressions for $\nabla_U V$ and $\nabla_V U$ given in Proposition 2.4.3, we obtain

$$[U, V] = \nabla_U V - \nabla_V U = \frac{c + 2(\alpha^2 + \beta^2)}{2(\alpha + \beta)} A. \tag{2.24}$$

Then, either $A(\alpha) = 0$ or $c + 2(\alpha^2 + \beta^2) = c + 2k = 0$. If $A(\alpha) = 0$ on an open subset, since $U(\alpha) = V(\alpha) = W(\alpha) = 0$ for any $W \in \Gamma(T_0 \ominus \mathbb{R}A)$, both α and β must be constant, and thus, M has constant principal curvatures on such open subset. Suppose now that $2k + c = 0$, or equivalently, that $k = -c/2$ on an open subset of $\mathcal{U}$.

In the projective case, since $c > 0$, the equation $\alpha^2 + \beta^2 = -c/2$ has no solution, and on the other hand, there is no ruled hypersurface in the complex projective case having constant principal curvatures [51, Remark 5]. Therefore, there is no example of ruled hypersurface in $\mathbb{C}P^n$ whose shape operator has constant norm.

In the hyperbolic case, let $\mathcal{D} := \operatorname{span}\{U, V\}$ be the smallest $\mathcal{S}$-invariant distribution of $\mathcal{U}$ that contains $J\xi$, which clearly has rank 2. On the one hand, if an open subset of M has constant principal curvatures, then it is an open part of a Lohnherr hypersurface [51, Remark 5], which is known to be strongly 2-Hopf and to satisfy $|\mathcal{S}|^2 = -c/2$. This can be checked directly from Proposition 2.4.3: taking into account that it has constant principal curvatures $\alpha = \sqrt{-c}/2$, $\beta = -\sqrt{-c}/2$ and 0, it follows that $[U, V] = \nabla_U V - \nabla_V U = 0$; hence $\mathcal{D}$ is integrable, and moreover $\mathcal{D}(\alpha) = \mathcal{D}(\beta) = 0$. On the other hand, if an open subset of M satisfies $k = -c/2$, it follows from (2.24) that $\mathcal{D}$ is integrable, and by virtue of (2.22) and the assumption that $\alpha^2 + \beta^2$ is constant, again $\mathcal{D}(\alpha) = \mathcal{D}(\beta) = 0$. Thus, in any case, M is a strongly 2-Hopf hypersurface with $|\mathcal{S}|^2 = -c/2$, which concludes the proof. □

Notice that, in particular, Lohnherr hypersurfaces, as homogeneous ruled hypersurfaces, are examples of ruled real hypersurfaces with shape operator of constant norm. The problem of deciding if the Lohnherr hypersurface is the only ruled one having shape operator of constant norm is what we address in Theorem 2.4.1. In order to prove this result, we will combine the information derived so far with some other geometric arguments to determine a particularly nice integral curve γ of the Hopf vector field $J\xi$. This γ will be a generating curve to which we must attach the totally geodesic complex hyperbolic hyperplanes $\mathbb{C}H^{n-1}$ to recover the inhomogeneous ruled hypersurface M described in item (2) of Theorem 2.4.1.

Proof of Theorem 2.4.1. Let M be a ruled real hypersurface in a nonflat complex space form whose shape operator has constant norm. Recall from Section 2.1 that the shape operator of M satisfies $\mathcal{S}J\xi = \lambda J\xi + \mu U_1$, $\mathcal{S}U_1 = \mu J\xi$ and $\mathcal{S}X = 0$ for any $X \in TM \ominus \operatorname{span}\{J\xi, U_1\}$, where $\{J\xi, U_1\}$ is an orthonormal set. In terms of the notation of Section 2.2, we can assume that $U_1 = JA$. Moreover, with respect to a set of principal curvature vector fields whose first two elements are U and V, the shape operator adopts a diagonal matrix form whose only nonzero entries are α and β. In terms of the functions λ and μ, the squared norm of the shape operator of M can be written as $|\mathcal{S}|^2 = \lambda^2 + 2\mu^2$, where $\mu \geq 0$.

Let us observe that we have the relation

$$\bar{\nabla}_{J\xi} J\xi = \mu A + \lambda\xi. \tag{2.25}$$

Indeed, using the fact that the ambient space is a Kähler manifold, the definition of the shape operator, and tacking into account that $\mathcal{S}(J\xi) = \lambda J\xi + \mu JA$, we obtain

$$\bar{\nabla}_{J\xi} J\xi = J\bar{\nabla}_{J\xi}\xi = -J\mathcal{S}(J\xi) = \mu A + \lambda\xi.$$

Therefore, since integral curves of $J\xi$ are precisely generating curves, for any generating curve γ of M the functions λ_γ and μ_γ defined at the beginning of Subsection 2.1.1 agree with the functions λ and μ along the curve γ, respectively.

By virtue of Theorem 2.4.2, ruled real hypersurfaces whose shape operators have constant norm do not exist in complex projective spaces, so we may assume that the ambient manifold is a complex hyperbolic space $\mathbb{C}H^n$. Theorem 2.4.2 also implies that M is a strongly 2-Hopf hypersurface and $|\mathcal{S}|^2 = -c/2$. Moreover, since $|\mathcal{S}|^2 = \lambda^2+2\mu^2$, we get $\mu \leq \sqrt{-c}/2$ on M.

Assume firstly that there exists a point $p \in M$ in such a way that $\mu(p) = \sqrt{-c}/2$. On the one hand, since M is a strongly 2-Hopf hypersurface, the principal curvatures α and β, or equivalently, the functions λ and μ, are constant along the integral submanifolds of $\mathcal{D} = \operatorname{span}\{U, V\} = \operatorname{span}\{J\xi, JA\}$. On the other hand, by virtue of Subsection 2.1.1 (discussion after (2.3)), we know that $\mu = \sqrt{-c}/2$ is constant along the integral submanifold of $(J\xi)^\perp$ containing p. Therefore, we deduce that $\mu = \sqrt{-c}/2$ is constant on M, and as $|\mathcal{S}|^2 = \lambda^2 + 2\mu^2$, then $\lambda = 0$ on M. Hence, M has constant principal curvatures, and thus, M is an open part of a Lohnherr hypersurface in $\mathbb{C}H^n$, which corresponds to the first case in the statement.

Assume now that $\mu < \sqrt{-c}/2$ on M. In such case, it follows from the discussion after (2.3) in Subsection 2.1.1 that the extension $\widetilde{M} = \bigcup_{p\in M} \exp_p(J\xi_p)^\perp$ of M, given by the union of every totally geodesic $\mathbb{C}H^{n-1}$ containing an integral submanifold of $(J\xi)^\perp$, is a ruled hypersurface without singular points. Let us still denote by ξ and $J\xi$ the unit normal and Hopf vector fields of $\widetilde{M}$. It follows from the expressions (2.3) that the functions λ and μ^2 are real analytic when restricted to any geodesic which is contained in a ruling of $\widetilde{M}$. Since $\lambda^2 + 2\mu^2 = -c/2$ on the open subset M of $\widetilde{M}$, this property also holds on the whole $\widetilde{M}$, and thus, $\widetilde{M}$ is a strongly 2-Hopf hypersurface by Theorem 2.4.2. Furthermore, as $\widetilde{M}$ is smooth and its rulings are complete, it follows from the discussion after (2.3) in Subsection 2.1.1 (see also [1, Lemma 1]) that there exists a point $p \in \widetilde{M}$ such that $\mu(p) = 0$, and hence, $\lambda(p) = \pm\sqrt{-c/2}$. Since $\widetilde{M}$ is strongly 2-Hopf, both λ and μ are constant along the integral curves of the Hopf vector field $J\xi$ of $\widetilde{M}$. Let us denote by γ the integral curve of $J\xi$ with $\gamma(0) = p$. Then, $\lambda(\gamma(t)) = \pm\sqrt{-c/2}$ and $\mu(\gamma(t)) = 0$ for every $t \in \mathbb{R}$ where γ is defined.

Since equation (2.25) also holds on $\widetilde{M}$, we obtain that the integral curve γ of $J\xi$ with $\gamma(0) = p$ satisfies the following differential equations

$$\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \mp\sqrt{\frac{-c}{2}}\, J\dot{\gamma}, \qquad\qquad \bar{\nabla}_{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \frac{c}{2}\,\dot{\gamma}, \tag{2.26}$$

which means that γ is a circle of curvature $\kappa = \sqrt{-c/2}$ inside a totally geodesic complex hyperbolic line $\mathbb{C}H^1$ (see [1, Section 2]). This corresponds to the second case in the statement.

To finish the proof of this result, it remains to check that the ruled real hypersurface constructed as in item (2) of the statement is smooth and its shape operator has constant norm. In order to do so, we consider a complete circle $\gamma\colon \mathbb{R} \to \mathbb{C}H^n$ of curvature $\kappa = \sqrt{-c/2}$ inside a totally geodesic $\mathbb{C}H^1$. Notice that, since $\sqrt{-c/2} < \sqrt{-c}$, γ is an equidistant curve to a geodesic in such a totally geodesic $\mathbb{C}H^1$. Let $M = \bigcup_{p\in M} \exp_{\gamma(t)}(\mathbb{C}\dot{\gamma})^\perp$ be the corresponding ruled real hypersurface. Since M and γ satisfy the relations (2.25) and (2.26), respectively, and $\dot{\gamma} = J\xi$ along γ (changing the sign of ξ if necessary), we obtain $\lambda \circ \gamma = \sqrt{-c/2}$ and $\mu \circ \gamma = 0$. Under these conditions, it follows from Subsection 2.1.1 that M is an immersed hypersurface of $\mathbb{C}H^n$ and, moreover, the functions μ and λ are given by the expressions

$$\lambda(\sigma(r)) = \sqrt{\frac{-c}{2}}\operatorname{sech}\left(\frac{\sqrt{-c}}{2}r\right), \qquad \mu(\sigma(r)) = \frac{\sqrt{-c}}{2}\tanh\left(\frac{\sqrt{-c}}{2}r\right),$$

where $r \geq 0$ and $\sigma(r) = \exp_{\gamma(t)}(rX)$ is a unit speed geodesic in the ruling of M through $\gamma(t)$, for each unit $X \in (\mathbb{C}\dot{\gamma}(t))^\perp$ and any $t \in \mathbb{R}$. Therefore, the squared norm of the shape operator of M satisfies $|\mathcal{S}|^2 = \lambda^2 + 2\mu^2 = -c/2$, as we wanted to show. Finally, notice that M is closed and embedded in $\mathbb{C}H^n$ due to the fact that the totally geodesic $\mathbb{C}H^{n-1}$ which are perpendicular to the totally geodesic $\mathbb{C}H^1$ determine a smooth foliation of $\mathbb{C}H^n$. $\square$

2.5 Biharmonic ruled hypersurfaces

The motivation for studying biharmonic submanifolds comes from the fact that they constitute a natural generalization of minimal submanifolds of a Riemannian manifold. In this section we study biharmonic ruled hypersurfaces in nonflat complex space forms. In order to do so, we firstly introduce some definitions and terminology. We refer the reader to [63] and [69] for more information on this topic.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and consider a smooth map $\varphi\colon (M_1, g_1) \to (M_2, g_2)$ between them. Let ∇^1 and R_1 be the Levi-Civita connection and the curvature tensor of M_1, and denote by ∇^2 and R_2 the corresponding objects for M_2. The smooth function $e(\varphi) = \frac{1}{2}|\varphi_*|^2$, where φ_* denotes the differential of φ, is called the *energy density* of φ. The *energy functional* with respect to the compact subset $\Omega \subset M$ is defined as

$$\mathrm{E}(\varphi) = \int_\Omega e(\varphi)dx,$$

for any smooth map φ as above. Moreover, one may define the *tension field* of φ, $\tau(\varphi)$, as the trace of its second fundamental form $\nabla\varphi_*$, which is given by

$$\nabla\varphi_*(X,Y) = \nabla^2_{\varphi_* X}\ \varphi_* Y - \varphi_*(\nabla^1_X Y),$$

for X, $Y \in TM_1$. The smooth function φ is said to be *harmonic* if it is a critical point of the energy functional for any compact subset $\Omega \subset M$. Equivalently, φ is a harmonic function if, and only if, its tension field vanishes identically. Minimal submanifolds can be characterized as those submanifolds whose defining isometric immersions $\varphi\colon M \to \bar{M}$ are harmonic maps.

The notion of harmonic function can be generalized in the following way. The *bienergy density* of φ is defined as the smooth function $e^2(\varphi) = \frac{1}{2}|\tau(\varphi)|^2$. With this notation, one may introduce the bienergy functional, which gives a measure of how far a smooth function φ is from being harmonic. More specifically, the *bienergy functional* with respect to the compact domain $\Omega \subset M$ is defined by

$$\mathrm{E}^2(\varphi) = \int_\Omega e^2(\varphi)dx,$$

for all smooth φ. Moreover, if we denote by ∇^φ the induced connection, then the *bitension field* of φ is defined by

$$\tau^2(\varphi) = \mathrm{tr}(\nabla^\varphi\nabla^\varphi - \nabla^\varphi_{\nabla^1})\tau(\varphi) - \mathrm{tr}\, R_2(\varphi_*, \tau(\varphi))\varphi_*.$$

With this notation, φ is said to be a *biharmonic function* if it is a critical point of the bienergy functional for any compact subset $\Omega \subset M$. It is possible to characterize the critical points of the bienergy functional by constructing the Euler-Lagrange equations of E^2. From these equations one deduces that φ is a biharmonic function if, and only if, its bitension field vanishes identically. A submanifold M of a Riemannian manifold $\bar{M}$ is said to be *biharmonic* if its defining isometric immersion $\varphi\colon M \to \bar{M}$ is a biharmonic function.

From now on, we focus our attention on codimension one biharmonic isometric immersions, that is, on biharmonic hypersurfaces, for which there exists an explicit formula that characterizes them.

Proposition 2.5.1. [63, Theorem 2.1] *Let $\bar{M}$ be a Riemannian manifold and $M \subset \bar{M}$ a hypersurface with unit normal vector field ξ. M is biharmonic if, and only if, it satisfies the following relations*

$$\begin{cases} \Delta\mathcal{H} - \mathcal{H}|\mathcal{S}|^2 + \mathcal{H}\overline{\mathrm{Ric}}(\xi,\xi) = 0, \\ 2\mathcal{S}(\nabla\mathcal{H}) + \mathcal{H}\nabla\mathcal{H} - 2\mathcal{H}(\overline{\mathrm{Ric}}(\xi))^\top = 0, \end{cases} \tag{2.27}$$

where $\mathcal{H} = \mathrm{tr}(\mathcal{S})$ is the mean curvature of the hypersurface, ∇ denotes the gradient, Δ is the Laplace-Beltrami operator of M, $\overline{\mathrm{Ric}}$ denotes both the (0,2) and the (1,1) Ricci tensors of $\bar{M}$, and $\top$ denotes the tangent projection on M.

It is known that minimal hypersurfaces are biharmonic. There exist some known results and conjectures claiming that the converse is, under certain assumptions, also true. For instance, Chen conjectured that any minimal hypersurface in the Euclidean space is biharmonic, and in the context of Riemannian manifolds of nonpositive curvature, it has been proved that both compact biharmonic hypersurfaces and biharmonic hypersurfaces with constant mean curvature are precisely the minimal ones [42, 63]. However, if one removes these conditions, we cannot ensure (in principle) minimality. Indeed, deciding whether biharmonicity implies minimality in ambient spaces of nonpositive curvature is the content of the generalized Chen's conjecture, proposed by Caddeo, Montaldo and Oniciuc in [19]. Ou and Tang have constructed some counterexamples which prove that this conjecture is not true [64], but due to the incompleteness of such examples, the generalized Chen's conjecture is still one of the main motivations for studying biharmonic hypersurfaces in Riemannian manifolds of nonpositive curvature. Thus, it becomes interesting to study biharmonic hypersurfaces satisfying other conditions, such as ruled hypersurfaces.

It has recently been proved that biharmonic ruled hypersurfaces in complex projective spaces are minimal [67]. The aim of this section is to extend this result to the entire context of nonflat complex space forms. In particular, we present the following result.

Theorem 2.5.2. *Let M be a biharmonic ruled real hypersurface in a nonflat complex space form. Then, M is minimal.*

This theorem, combined again with the results in [1] and [51], yields the classification of biharmonic ruled hypersurfaces in nonflat complex space forms. Such a hypersurface must then be an open part of a Kimura type hypersurface in a complex projective or hyperbolic space, a bisector in a complex hyperbolic space, or a Lohnherr hypersurface in a complex hyperbolic space.

From now on, in order to prove Theorem 2.5.2, $\bar{M}$ will denote a complex space form of constant holomorphic sectional curvature $c \neq 0$. In this case, the Ricci tensor of $\bar{M}$ satisfies $\overline{\mathrm{Ric}}(\xi,\xi) = c(n+1)/2$ and $(\overline{\mathrm{Ric}}(\xi))^\top = 0$, which follows immediately from the formula for the curvature tensor of a complex space form. Thus, in our case, equations (2.27) can be rewritten as follows (cf. [39, Proposition 2.1]):

$$\begin{cases} \Delta\mathcal{H} - \mathcal{H}|\mathcal{S}|^2 + \frac{1}{2}\mathcal{H}c(n+1) = 0, \\ 2\mathcal{S}(\nabla\mathcal{H}) + \mathcal{H}\nabla\mathcal{H} = 0. \end{cases} \tag{2.28}$$

Proof of Theorem 2.5.2. Let M be a biharmonic ruled hypersurface in a nonflat complex space form and let $\mathcal{H} = \alpha + \beta$ denote the mean curvature of the ruled hypersurface M. Recall that, according to the discussion before Proposition 2.2.2, there is an open and dense subset $\mathcal{U}$ of M where $h = 2$. Proposition 2.2.3 and relations given in Proposition 2.2.2 hold in this open subset. In what follows, we will work in terms of an orthonormal local reference of principal curvature vector fields $\{U, V, A, W_4, \dots, W_{2n-1}\}$, where $W_i \in \Gamma(T_0 \ominus \mathbb{R}A)$, $i \in \{4, \dots, 2n-1\}$.

Suppose that the mean curvature, $\mathcal{H} = \alpha + \beta$, is not zero in an open subset of $\mathcal{U}$. We will work in this open subset of M from now on. Since M is a biharmonic hypersurface, it satisfies equations (2.28).

With respect to the orthonormal basis $\{U, V, A, W_4, \ldots, W_{2n-1}\}$, we have

$$\nabla\mathcal{H} = U(\mathcal{H})U + V(\mathcal{H})V + A(\mathcal{H})A + \sum_{i=4}^{2n-1} W_i(\mathcal{H})W_i.$$

On the other hand, since U, V, A and W_i, for $i \in \{4, \ldots, 2n-1\}$, are orthogonal eigenvectors of the shape operator $\mathcal{S}$ of M associated with eigenvalues α, β and 0, respectively, we have

$$\mathcal{S}(\nabla\mathcal{H}) = \alpha U(\mathcal{H})U + \beta V(\mathcal{H})V.$$

Thus, inserting these relations into the second equation in (2.28), we obtain

$$\begin{aligned} 0 &= 2\mathcal{S}(\nabla\mathcal{H}) + \mathcal{H}\nabla\mathcal{H} \\ &= (2\alpha + \mathcal{H})U(\mathcal{H})U + (2\beta + \mathcal{H})V(\mathcal{H})V + \mathcal{H}A(\mathcal{H})A + \sum_{i=4}^{2n-1} \mathcal{H}W_i(\mathcal{H})W_i. \end{aligned}$$

Since $\mathcal{H} \neq 0$ by assumption, one can deduce that $A(\mathcal{H}) = 0$ and $W_i(\mathcal{H}) = 0$ for $i \in \{4, \ldots, 2n-1\}$. Moreover, one of the following conditions holds on an open subset:

(1) $U(\mathcal{H}) = V(\mathcal{H}) = 0$.

(2) $\alpha = \beta = -\mathcal{H}/2$.

(3) $U(\mathcal{H}) = 0$ and $2\beta + \mathcal{H} = 0$.

(4) $V(\mathcal{H}) = 0$ and $2\alpha + \mathcal{H} = 0$.

Neither case (1) nor case (2) are possible. Indeed, if $U(\mathcal{H}) = V(\mathcal{H}) = 0$ on an open subset, then such subset has constant mean curvature, and since it is ruled, $\mathcal{H} = 0$ due to Theorem 2.3.1, which gives a contradiction. Analogously, since M is ruled, $\alpha \neq \beta$ on any open subset.

Suppose that $U(\mathcal{H}) = 0$ and $2\beta + \mathcal{H} = 0$, or equivalently, $\mathcal{H} = 2\alpha/3$ (case (4) is analogous). Then, both α and β can be expressed as $\alpha = 3\mathcal{H}/2$ and $\beta = -\mathcal{H}/2$, respectively. Inserting these expressions in the formula for $A(\alpha)$ given in Proposition 2.2.3, one gets

$$\frac{3}{2}A(\mathcal{H}) = A(\alpha) = 2b(a(c + 3\mathcal{H}^2) - 3bA(\mathcal{H})).$$

Moreover, $A(\mathcal{H}) = 0$ on $\mathcal{U}$, from where $ab(3\mathcal{H}^2 + c) = 0$. Since a and b are not zero in $\mathcal{U}$, M has constant mean curvature on $\mathcal{U}$, and as M is a ruled, it must be minimal by virtue of Theorem 2.3.1, which concludes the proof. $\square$

Chapter 3

Homogeneous CR submanifolds in complex hyperbolic spaces

This chapter is devoted to presenting the classification of homogeneous CR submanifolds in complex hyperbolic spaces that arise as orbits of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$. In order to do so, we firstly review, in Section 3.1, some basic terminology concerning the notion of CR submanifold of a Hermitian manifold and study, in Section 3.2, some algebraic properties of homogeneous CR submanifolds in Hermitian symmetric spaces of noncompact type. The classification of homogeneous CR submanifolds in complex hyperbolic spaces arising as orbits of the solvable part of the Iwasawa decomposition of $SU(1,n)$ is settled in Section 3.3. Finally, in Section 3.4, we study the congruence classes of the examples that we have obtained in the classification theorem. The results of this chapter have given rise to the article [32].

3.1 CR submanifolds of a Hermitian manifold

In this section we introduce the main notation to deal with CR submanifolds of a Hermitian manifold. Consider V a complex vector space with complex structure J and inner product $\langle \cdot, \cdot \rangle$. Recall that a subspace $W \subset V$ is said to be complex if it is invariant by the complex structure, that is, if $JW \subset W$. On the other hand, W is said to be totally real if JW is orthogonal to W.

In the setting of Hermitian manifolds, one can generalize these concepts by introducing the notions of complex and totally real submanifolds. Let $\bar{M}$ be a Hermitian manifold with complex structure J. A submanifold $M \subset \bar{M}$ is said to be complex (totally real) if, at each point $p \in M$, the tangent space T_pM is a complex (totally real) vector subspace of $T_p\bar{M}$. The subspace $J(T_pM) \cap T_pM$ is a J-invariant subspace. In fact, it is the maximal complex subspace of T_pM.

The notion of CR submanifold in a Hermitian manifold includes both complex and totally real submanifolds as particular examples. A submanifold $M \subset \bar{M}$ is said to be a *CR* (*Cauchy-Riemann* or *complex-real*) submanifold if there exists a pair of orthogonal complementary distributions of the tangent bundle of M, $TM = \mathfrak{C} \oplus \mathfrak{R}$, where $\mathfrak{C}$ is complex and $\mathfrak{R}$ is totally real. In other words, M is a CR submanifold of $\bar{M}$ if the maximal complex subspaces have constant dimension and their orthogonal complements in each tangent space are totally real subspaces. We refer to [10] and [34] for more information on CR submanifolds of a Hermitian manifold.

In the setting of submanifold geometry, an interesting problem is to classify homogeneous CR submanifolds in several important families of Kähler manifolds, such as Hermitian symmetric spaces or, more specifically, complex space forms. The importance of studying homogeneous CR submanifolds in this setting comes from the fact that they include several special examples of submanifolds of interest in the context of symmetric spaces, such as real hypersurfaces, Kähler submanifolds or Lagrangian submanifolds. We review below some known classification results regarding these types of CR submanifolds in the context of complex space forms.

Homogeneous CR submanifolds in complex space forms

Homogeneous real hypersurfaces, that is, submanifolds of real codimension one, constitute an important subclass of homogeneous CR submanifolds in complex space forms that has been thoroughly studied by many authors. More specifically, the classification of homogeneous real hypersurfaces, or equivalently, of cohomogeneity one actions, in complex Euclidean spaces follows from a classical work on isoparametric hypersurfaces due to Segre [68]. The corresponding classification in complex projective spaces $\mathbb{C}P^n$ has been obtained by Takagi in [70], whereas the classification in complex hyperbolic spaces has been achieved by Berndt and Tamaru in [15].

Another important subclass of homogeneous CR submanifolds in complex space forms is that of Kähler ones. Di Scala, Ishi and Loi have proved in [26] that the only examples of homogeneous Kähler submanifolds in complex Euclidean and hyperbolic spaces $\mathbb{C}^n$ and $\mathbb{C}H^n$ are totally geodesic $\mathbb{C}^k$ and $\mathbb{C}H^k$, with $k \in \{0, \dots, n\}$, respectively. The corresponding classification in complex projective spaces, $\mathbb{C}P^n$, achieved by Takeuchi in [71], includes more examples.

Lagrangian submanifolds, that is, totally real submanifolds of maximal dimension, constitute a nice particular example of CR submanifolds in complex space forms. Although the classification of homogeneous Lagrangian submanifolds in these ambient manifolds is still an open problem, several partial results have been achieved. For instance, Bedulli and Gori have obtained the classification of homogeneous Lagrangian submanifolds in complex projective spaces induced by the action of a simple compact subgroup of $SU(n+1)$. Under additional assumptions, such as the parallelity of the second fundamental form, some results have also been derived; see [61] for a survey. In the hyperbolic case, classifying homogeneous Lagrangian submanifolds has been shown to be a very involved problem, mainly due to the noncompactness of its isometry group. However, Hashinaga and Kajigaya have obtained some partial results in [43]. In particular, they have classified homogeneous Lagrangian submanifolds in complex hyperbolic spaces induced by the action of a subgroup of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$.

In the following sections we focus on the classification of homogeneous CR submanifolds in complex hyperbolic spaces that arise as orbits of subgroups of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$.

3.2 The Lie algebra of a CR orbit

In this section we introduce a result which allows us to characterize homogeneous CR submanifolds in a Hermitian symmetric space of noncompact type in terms of the Lie subalgebras associated with the Lie subgroups that determine such CR submanifolds. With the notation established in Section 1.4 and Subsection 1.5.1, let $\bar{M} = G/K$ be a Hermitian symmetric space of noncompact type, where G is (up to finite covering) the connected component of the identity element of the isometry group of $\bar{M}$, and $K = G_o$ is the isotropy at some fixed point $o \in \bar{M}$. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and consider the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with respect to $o \in \bar{M}$, as well as the Iwasawa decomposition at the level of Lie groups, $G = KAN$.

As we have pointed out in Subsection 1.5.1, every symmetric space of noncompact type can be regarded as a solvable Lie group endowed with a left-invariant metric. We recall some basic facts and notation in relation to such identification. Consider the smooth function $\phi\colon G \to \bar{M}$ given by $g \mapsto g(o)$. The restriction $\phi|_{AN}\colon AN \to \bar{M}$ is a diffeomorphism, so $\mathfrak{a} \oplus \mathfrak{n}$ can be identified with the tangent space $T_o\bar{M}$ using ϕ_*. By pulling back the metric $\bar{g}$ of $\bar{M}$, one can endow AN with a left-invariant metric $(\phi|_{AN})^*\bar{g}$. Furthermore, AN can also be equipped with the complex structure induced by the one in $\bar{M}$ by means of $\phi|_{AN}$.

Under these conditions, we prove the following result.

Lemma 3.2.1. *Let H be a connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then, the H-orbit through o, $H \cdot o$, is a CR submanifold if, and only if, the projection of $\mathfrak{h}$ onto $\mathfrak{a} \oplus \mathfrak{n}$ with respect to the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ splits as an orthogonal direct sum of a complex subspace and a totally real one.*

Proof. By definition, the orbit $H \cdot o$ is a CR submanifold if at each point $h(o) \in \bar{M}$, with $h \in H$, the tangent space to $H \cdot o$ at $h(o)$ splits as an orthogonal direct sum of a totally real subspace of and a complex one.

Consider the smooth map $\phi\colon G \to \bar{M}$, $g \mapsto g(o)$, defined above, whose differential is given as follows:

$$\phi_{*_e}\colon \mathfrak{g} \to T_o\bar{M}, \qquad\qquad X \mapsto \phi_{*_e}X = \frac{d}{dt}\Big|_{t=0} \operatorname{Exp}(tX)(o),$$

where Exp denotes the Lie exponential map and e is the identity element of G. The restriction of ϕ_{*_e} to $\mathfrak{a} \oplus \mathfrak{n}$ is an isomorphism, so we can identify $\mathfrak{a} \oplus \mathfrak{n} \cong T_o\bar{M}$. In particular, if we denote by $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ the projection of $\mathfrak{h}$ onto $\mathfrak{a} \oplus \mathfrak{n}$ with respect to the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, the tangent space to the orbit $H \cdot o$ at o is $T_o(H \cdot o) = \phi_*\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$. Moreover, if L denotes the left translation, the tangent space of $H \cdot o$ at any other point $h(o) \in \bar{M}$, with $h \in H$, can be calculated as follows:

$$T_{h(o)}(H \cdot o) = h_*T_o(H \cdot o) = h_*\phi_*\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = (h \circ \phi)_*\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \phi_* \operatorname{L}_{h*} \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}} = \phi_*\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}},$$

since $h \circ \phi = \phi \circ \operatorname{L}_h$. In particular, $T_{h(o)}(H \cdot o) = h_*T_o(H \cdot o) = \phi_*\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ for each $h \in H$. Moreover, since the restriction of ϕ_* to $\mathfrak{a} \oplus \mathfrak{n}$ is a linear holomorphic isometry, it preserves both maximal holomorphic subspaces and totally real ones. Taking these facts into account, we deduce that the orbit $H \cdot o$ is a CR submanifold if, and only if, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$ splits into an orthogonal direct sum of a complex subspace and a totally real one. □

By virtue of this result, it follows that the problem of classifying homogeneous CR submanifolds in a Hermitian symmetric space of noncompact type $\bar{M} = G/K$ reduces to finding all the Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ such that the projection onto $\mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{n}}$, can be decomposed into an orthogonal direct sum of a totally real subspace and a complex one.

From now on, we will focus on the study of homogeneous CR submanifolds in complex hyperbolic spaces. We briefly discuss below some important examples. In order to do so, we will use the notation established in Subsection 1.4.2. Recall that, if $G = SU(1,n)$ is the connected component of the identity of the isometry group of $\mathbb{C}H^n$ and $\mathfrak{g}$ denotes its corresponding Lie algebra, then the Iwasawa decomposition (at the level of Lie algebras) reads $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{a}$ is a one-dimensional abelian subspace and $\mathfrak{n}$ is a nilpotent Lie subalgebra which can be decomposed as $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, with $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ and $\mathfrak{g}_{2\alpha} \cong \mathbb{R}$.

The Berndt-Brück submanifolds of $\mathbb{C}H^n$

Let $\mathfrak{w}^\perp$ be a totally real $k-$dimensional vector subspace of $\mathfrak{g}_\alpha$ and denote by $\mathfrak{w} = \mathfrak{g}_\alpha \ominus \mathfrak{w}^\perp$ its orthogonal complement in $\mathfrak{g}_\alpha$. Consider the Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ given by $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, and let S be the connected Lie subgroup of AN whose Lie algebra is $\mathfrak{s}$. Since the exponential map $\operatorname{Exp}\colon \mathfrak{a} \oplus \mathfrak{n} \to AN$ is a diffeomorphism, S is simply connected and closed in AN. The *Berndt-Brück submanifold* with totally real normal bundle of rank k in $\mathbb{C}H^n$, commonly denoted by W^{2n-k}, is defined as the orbit of S through o. In particular, when $k = 1$, W^{2n-1} is the Lohnherr hypersurface of $\mathbb{C}H^n$, which we have defined in Section 2.1. Berndt-Brück submanifolds with $k > 1$ arise as singular orbits of cohomogeneity one actions on complex hyperbolic spaces, whereas the Lohnherr hypersurface is the only minimal orbit of a cohomogeneity one action without singular orbits [13].

We now show that W^{2n-k} is a CR submanifold. Indeed, $\mathfrak{s}$ can be decomposed into an orthogonal direct sum of a complex subspace and a totally real one as follows:

$$\mathfrak{s} = (\mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{g}_{2\alpha}) \oplus J\mathfrak{w}^\perp,$$

where $\mathfrak{c} = \mathfrak{w} \ominus J\mathfrak{w}^\perp$ is a complex subspace of $\mathfrak{g}_\alpha$. Thus, Lemma 3.2.1 ensures that the Berndt-Brück submanifold W^{2n-k} is a CR-submanifold.

CR submanifolds given by polar actions

With a similar description to the one presented above, one can easily construct more examples of homogeneous CR submanifolds in complex hyperbolic spaces. Indeed, consider the Lie subalgebra of $\mathfrak{su}(1,n)$ defined by $\mathfrak{s} = \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ denotes a subspace of $\mathfrak{g}_\alpha$. If $\mathfrak{w}$ is a CR subspace of $\mathfrak{g}_\alpha$, that is, if $\mathfrak{w}$ splits as an orthogonal direct sum of a totally real subspace of $\mathfrak{g}_\alpha$ and a complex one, then the orbit through o of the associated connected Lie subgroup, $S \cdot o$, is a CR submanifold contained in the horosphere $N \cdot o$, where N denotes the nilpotent part of the Iwasawa decomposition of $SU(1,n)$. These submanifolds are not orbits of cohomogeneity one actions (except in the particular case $\mathfrak{w} = \mathfrak{g}_\alpha$), but orbits of polar actions on $\mathbb{C}H^n$ [30, Theorem A (ii)]. Similarly, the Lie subalgebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is a CR subspace of $\mathfrak{g}_\alpha$, gives rise to a minimal (singular, except if $\mathfrak{w}$ has codimension one in $\mathfrak{g}_\alpha$) orbit of a polar action on $\mathbb{C}H^n$ [30, Theorem A (ii)].

Homogeneous real hypersurfaces in $\mathbb{C}H^n$

Homogeneous real hypersurfaces, that is, homogeneous submanifolds of real codimension one, also constitute a particular example of homogeneous CR submanifolds in $\mathbb{C}H^n$. The classification of homogeneous real hypersurfaces, or equivalently, of cohomogeneity one actions, in complex hyperbolic spaces has been obtained by Berndt and Tamaru in [15]. Some of the examples given in this classification result are induced by the action of a connected subgroup of the solvable part of the Iwasawa decomposition of the isometry group of $\mathbb{C}H^n$, AN. More specifically, such homogeneous real hypersurfaces of $\mathbb{C}H^n$ are horospheres, Lohnherr hypersurfaces and their equidistant hypersurfaces.

3.3 The classification

The aim of this section is to present the classification of homogeneous CR submanifolds in complex hyperbolic spaces which arise as orbits of connected subgroups of the solvable part of the Iwasawa decomposition of G, AN. To tackle this problem, we proceed as follows: first of all, in Section 3.3.1, we study which are the subgroups of AN that produce a CR orbit. Secondly, in Section 3.3.2, in order to get a classification result, we check whether any of the remaining orbits of these subgroups is also a CR submanifold or not. In order to do so, we will use the notation established in Subsection 1.5.4.

3.3.1 Actions with a CR orbit

This subsection is devoted to determining the connected Lie subgroups H of AN which act on $\mathbb{C}H^n$ producing a CR orbit. Since AN acts transitively on $\mathbb{C}H^n$, we can assume, without loss of generality, that such CR orbit is precisely $H \cdot o$.

Proposition 3.3.1. *Let $H \subset AN$ be a connected Lie subgroup of the solvable part of the Iwasawa decomposition of G acting on $\mathbb{C}H^n$ in such a way that the orbit through o, $H \cdot o$, is a CR-submanifold. Then, its Lie algebra $\mathfrak{h}$ is conjugate by an element of AN to one of the following subalgebras:*

1. $\mathfrak{h} = \mathfrak{r}$, *or*
2. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$, *or*

3. $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, *or*

4. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$,

where $\mathfrak{r}$ *is totally real subspace of* $\mathfrak{g}_\alpha$ *and* $\mathfrak{c}$ *is a complex one.*

To prove this result, let $\mathfrak{h}$ denote the Lie algebra of H and consider the projection onto $\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$, $\pi\colon \mathfrak{g} \to \mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$. We denote by $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} := \pi(\mathfrak{h})$ the image of $\mathfrak{h}$ under π. Then, $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}}$ must be one of the following subspaces:

$$\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} \in \{0, \mathfrak{a}, \mathfrak{g}_{2\alpha}, \mathbb{R}(aB + bZ), \mathfrak{a} \oplus \mathfrak{g}_{2\alpha}\}, \tag{3.1}$$

for some nonzero $a, b \in \mathbb{R}$. Notice that, since AN consists of holomorphic isometries, a homogeneous submanifold of $\mathbb{C}H^n$ is CR if, and only if, its tangent space is a CR subspace of the tangent space of $\mathbb{C}H^n$ at some point. Thus, and taking into account Lemma 3.2.1, it will be enough to study these cases separately, trying to find those subalgebras of $\mathfrak{a} \oplus \mathfrak{n}$ that can be decomposed into an orthogonal direct sum of a totally real subspace and a complex one and whose projection onto $\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$ is one of the subspaces given in (3.1).

Case (i): $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} = 0$**.**

Let U, $V \in \mathfrak{h}$. Then, due to the definition of the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$, $[U, V] = \langle JU, V\rangle Z$, which lies in $\mathfrak{h}$ since $\mathfrak{h}$ is a Lie algebra, but also in $\mathfrak{g}_{2\alpha} = \mathbb{R}Z$. As $\mathfrak{g}_{2\alpha} \cap \mathfrak{h} = 0$, one gets that $\langle JU, V\rangle = 0$, from where we deduce that $\mathfrak{h}$ is a totally real subspace. This corresponds to Case 1 in the statement.

Case (ii): $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} = \mathfrak{a}$**.**

In this case, $\mathfrak{h} = \mathbb{R}(B + X) \oplus \mathfrak{w}$, where $\mathfrak{w}$ is a subspace of $\mathfrak{g}_\alpha$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Given U, $V \in \mathfrak{w}$, using the expression for the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$, we have

$$\begin{aligned} &[U, V] = \langle JU, V\rangle Z \in \mathfrak{g}_{2\alpha} \cap \mathfrak{h} = 0, \\ &[B + X, U] = [B, U] + [X, U] = \frac{1}{2}U + \langle JX, U\rangle Z \in \mathfrak{h}. \end{aligned}$$

As above, from the first equation we obtain that $\mathfrak{w}$ is a totally real subspace. From the second one, since $\mathfrak{h}$ is a Lie subalgebra, we get that $\langle JX, U\rangle = 0$, and consequently, $JX \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Thus, one gets $\langle J(B+X), U\rangle = \langle Z+JX, U\rangle = 0$. Hence, the Lie algebra $\mathfrak{h}$ is a totally real subspace of $\mathfrak{a} \oplus \mathfrak{n}$. Moreover, $\operatorname{Ad}(\operatorname{Exp}(2X))\mathfrak{h} = e^{2\operatorname{ad} X}\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{w}$, since

$$e^{2\operatorname{ad} X}(B + X) = B \qquad \text{and} \qquad e^{2\operatorname{ad} X}(U) = U.$$

This corresponds to Case 2 in the statement.

Case (iii): $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} = \mathfrak{g}_{2\alpha}$.

In this case, there exists $X \in \mathfrak{g}_\alpha$ such that $\mathfrak{h} = \mathfrak{w} \oplus \mathbb{R}(X+Z)$, where $\mathfrak{w} \subset \mathfrak{g}_\alpha$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Let U, $V \in \mathfrak{w}$. Due to the definition of the Lie bracket of $\mathfrak{a} \oplus \mathfrak{n}$,

$$\begin{aligned}&[U,V] = \langle JU, V\rangle Z \in \mathfrak{g}_{2\alpha} \cap \mathfrak{h},\\ &[U, X+Z] = [U,X] + [U,Z] = \langle JU, X\rangle Z \in \mathfrak{g}_{2\alpha} \cap \mathfrak{h}.\end{aligned}$$

Since $\mathfrak{h}$ is a Lie algebra, from the first equation we deduce that $\langle JU, V\rangle = 0$ or $X = 0$. Analogously, from the second one, we get that $X = 0$ or $\langle JU, X\rangle = 0$. In particular, if $X = 0$ it is also true that $\langle JU, X\rangle = 0$. Thus, from the second relation we deduce, in any case, that $JX \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Moreover, $\langle J(X+Z), U\rangle = \langle JX - B, U\rangle = 0$, so $J(X+Z)$ is orthogonal to $\mathfrak{h}$.

We distinguish two possibilities, depending on whether $\mathfrak{w}$ contains a complex subspace or $\mathfrak{w}$ is a totally real subspace. If $\mathfrak{w}$ contains a complex subspace, then it is clear that $X = 0$. Consequently, the Lie algebra $\mathfrak{h}$ is of the form $\mathfrak{h} = \mathfrak{c}\oplus\mathfrak{r}\oplus\mathfrak{g}_{2\alpha}$, where $\mathfrak{c}$ denotes a complex subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{r}$ a totally real one. This corresponds to Case 3 in the statement. Otherwise, if $\mathfrak{w}$ is a totally real subspace, then $\mathfrak{h} = \mathfrak{w} \oplus \mathbb{R}(X+Z)$, where $X \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{w}$. Moreover, for $\tau = 1/|X|^2$, $\operatorname{Ad}(\operatorname{Exp}(\tau JX))\mathfrak{h} = e^{\tau \operatorname{ad} JX}\mathfrak{h} = \mathfrak{w} \oplus \mathbb{R}X$. Indeed,

$$e^{\tau \operatorname{ad} JX} U = U \qquad \text{and} \qquad e^{\tau \operatorname{ad} JX}(X+Z) = X.$$

Thus, the Lie algebra $\mathfrak{h}$ is conjugate to a totally real subspace of $\mathfrak{g}_\alpha$. This corresponds again to Case 1 in the statement.

Case (iv): $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} = \mathbb{R}(aB + bZ)$ for some nonzero $a, b \in \mathbb{R}$.

In this case, $\mathfrak{h} = \mathbb{R}(aB + X + bZ) \oplus \mathfrak{w}$, where $\mathfrak{w} \subset \mathfrak{g}_\alpha$ and $X \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Let U, $V \in \mathfrak{w}$. Then, taking brackets,

$$\begin{aligned}&[U,V] = \langle JU, V\rangle Z \in \mathfrak{g}_{2\alpha} \cap \mathfrak{h},\\ &[aB + X + bZ, U] = a[B,U] + [X,U] + b[Z,U] = \frac{a}{2}U + \langle JX, U\rangle Z \in \mathfrak{h}.\end{aligned}$$

From the first equation, we get that $\langle JU, V\rangle = 0$, which implies that $\mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$. From the second one, taking into account that $aU/2 \in \mathfrak{w} \subset \mathfrak{h}$, we deduce that $\langle JX, U\rangle = 0$, from where $JX \in \mathfrak{g}_\alpha \ominus \mathfrak{w}$. Moreover, for $\tau = b/|X|^2$, we have

$$\begin{aligned}&e^{\tau \operatorname{ad} JX}(U) = U,\\ &e^{\tau \operatorname{ad} JX}(aB + X + bZ) = aB + X + bZ - \frac{a\tau}{2}JX - \tau|X|^2 Z = aB + X - \frac{ab}{2|X|^2}JX.\end{aligned}$$

Thus, $\operatorname{Ad}(\operatorname{Exp}(\tau JX))\mathfrak{h} = \mathfrak{w} \oplus \mathbb{R}(aB + Y)$, where $Y = X - \frac{ab}{2|X|^2}JX \in \mathfrak{w}$, which has been shown to be conjugate to $\mathfrak{a} \oplus \mathfrak{w}$. This corresponds again to Case 2 in the statement.

Case (v): $\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{g}_{2\alpha}} = \mathfrak{a}\oplus\mathfrak{g}_{2\alpha}$.

In such case, there exist a vector subspace $\mathfrak{w}\subset\mathfrak{g}_\alpha$ and X, $Y\in\mathfrak{g}_\alpha\ominus\mathfrak{w}$ such that $\mathfrak{h}=\mathbb{R}(B+X)\oplus\mathfrak{w}\oplus\mathbb{R}(Y+Z)$. Due to the definition of the Lie bracket of $\mathfrak{a}\oplus\mathfrak{n}$, we have:

$$
\begin{aligned}
&[U,V]=\langle JU,V\rangle Z\in\mathfrak{h};\\
&[B+X,U]=[B,U]+[X,U]=\frac{1}{2}U+\langle JX,U\rangle Z\in\mathfrak{h};\\
&[Y+Z,U]=[Y,U]+[Z,U]=-\langle JU,Y\rangle Z\in\mathfrak{h};\\
&[B+X,Y+Z]=[B,Y]+[B,Z]+[X,Y]+[X,Z]=\frac{1}{2}Y+(1+\langle JX,Y\rangle)\,Z\in\mathfrak{h}.
\end{aligned}
$$

From these relations, we obtain the following conclusions:

- From the first equation, $\langle JU,V\rangle=0$ for all U, $V\in\mathfrak{w}$ (and so $\mathfrak{w}$ is a totally real subspace), or $Y=0$.
- From the second one, we deduce that $\langle JX,U\rangle=0$ for all $U\in\mathfrak{w}$, or $Y=0$.
- Third equation yields $\langle JU,Y\rangle=0$ (the other possibility is $Y=0$, which implies $\langle JU,Y\rangle=0$).
- The last equation implies that $Y=0$ or $\frac{1}{2}Y+(1+\langle JX,Y\rangle)\,Z$ is proportional to $Y+Z$, from where we deduce that $\langle JX,Y\rangle=-1/2$.

We distinguish two cases depending on whether $Y=0$ or $Y\neq0$.

Subcase (v)-(a). Assume that $Y=0$. Then, $\mathfrak{h}=\mathbb{R}(B+X)\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$. We firstly show that $\mathfrak{w}$ is a CR subspace of $\mathfrak{g}_\alpha$. In order to do so, let $\mathfrak{c}=\mathfrak{w}\cap J\mathfrak{w}$ be the maximal holomorphic subspace of $\mathfrak{w}$. Notice that $\mathfrak{h}\ominus\mathfrak{c}$ is not a totally real subspace since $\langle J(B+X),Z\rangle\neq0$, where $B+X$, $Z\in\mathfrak{h}\ominus\mathfrak{c}$. Then, it follows that there exists $\xi'\in(\mathfrak{h}\cap J\mathfrak{h})\ominus\mathfrak{c}$, $\xi'\neq0$. Let us put $\xi'=a(B+X)+W'+bZ$, for some $W'\in\mathfrak{w}$, and where $a,b\in\mathbb{R}$ cannot vanish simultaneously. With this notation,

$$J\xi'=aZ+aJX+JW'-bB=-b(B+X)+bX+aJX+JW'+aZ\in\mathfrak{h}.$$

Hence, $bX+aJX+JW'\in\mathfrak{w}$. We take $\xi=(a\xi'-bJ\xi')/(a^2+b^2)\in(\mathfrak{h}\cap J\mathfrak{h})\ominus\mathfrak{c}$, which has the form $\xi=B+X+W$, with $W=(aW'-bJW'-b^2X-abJX)/(a^2+b^2)\in\mathfrak{w}$. Then, $J\xi=JX+JW+Z\in\mathfrak{h}$, where $JX+JW\in\mathfrak{w}$. In this situation, $\eta:=JX+JW-(|X|^2+|W|^2)Z\in\mathfrak{h}\ominus\mathbb{C}\xi$. Let us decompose $\eta=\eta_c+\eta_r$, where $\eta_c\in\mathfrak{h}\cap J\mathfrak{h}$ and $\eta_r\in\mathfrak{h}\ominus(\mathfrak{h}\cap J\mathfrak{h})$. Since $\mathfrak{h}$ is a CR subspace of $\mathfrak{a}\oplus\mathfrak{n}$, $J\eta=J\eta_c+J\eta_r$, where $J\eta_c\in\mathfrak{h}$ and $J\eta_r\in(\mathfrak{a}\oplus\mathfrak{n})\ominus\mathfrak{h}$. However,

$$
\begin{aligned}
J\eta&=-X-W+(|X|^2+|W|^2)B\\
&=\left(\frac{|W|^2}{1+|X|^2}(B+X)-W\right)+\frac{1+|X|^2+|W|^2}{1+|X|^2}(|X|^2B-X),
\end{aligned}
$$

where the first addend belongs to $\mathfrak{h}$ and the second one is orthogonal to $\mathfrak{h}$. In particular, one deduces that

$$J\eta_c = \frac{|W|^2}{1+|X|^2}(B+X) - W \quad \text{and} \quad -\frac{|W|^2}{1+|X|^2}(Z+JX) + JW = \eta_c \in \mathfrak{h}.$$

Since Z, $JX + JW \in \mathfrak{h}$, we get that JX, $JW \in \mathfrak{h}$. In particular, $JX \in \mathfrak{w}$. Then,

$$\mathfrak{h} = \mathbb{C}(B+X) \oplus \mathbb{R}(|X|^2 Z - JX) \oplus (\mathfrak{w} \ominus \mathbb{R}JX)$$

is a $\mathbb{C}$-orthogonal direct sum, from where we deduce that $\mathfrak{w}$ is a CR subspace of $\mathfrak{g}_\alpha$. Now, taking into account that

$$e^{2\,\mathrm{ad}\,X}(B+X) = B, \qquad e^{2\,\mathrm{ad}\,X}U = U, \qquad e^{2\,\mathrm{ad}\,X}Z = Z,$$

one can deduce that $\mathrm{Ad}(\mathrm{Exp}(2X))\mathfrak{h} = e^{2\,\mathrm{ad}(X)}\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$. Thus, since $\mathfrak{w}$ is a CR subspace of $\mathfrak{g}_\alpha$, say $\mathfrak{w} = \mathfrak{c} \oplus \mathfrak{r}$, we obtain the last case in the statement.

Subcase (v)-(b). To finish, assume that $Y \neq 0$. In this situation,

- $\langle JU, V\rangle = 0$ for all U, $V \in \mathfrak{w}$, that is, $\mathfrak{w}$ is a totally real subspace of $\mathfrak{g}_\alpha$;
- $\langle JX, U\rangle = 0$ for all $U \in \mathfrak{w}$, that is, $X \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{w}$;
- $\langle JY, U\rangle = 0$ for all $U \in \mathfrak{w}$, that is, $Y \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{w}$;
- $\langle JX, Y\rangle = -1/2$.

By assumption, $\mathfrak{h}$ is a CR subspace of $\mathfrak{a}\oplus\mathfrak{n} \cong \mathbb{C}^n$. Since $\mathfrak{w}$ is totally real, it follows that $\mathbb{R}(B+X) \oplus \mathbb{R}(Y+Z)$ is either complex or totally real. Observe that if it is totally real, then

$$0 = \langle J(B+X), Y+Z\rangle = \langle Z+JX, Y+Z\rangle = 1 + \langle JX, Y\rangle.$$

This equation yields $\langle JX, Y\rangle = -1$, which contradicts $\langle JX, Y\rangle = -1/2$. Consequently, $\mathbb{R}(B+X) \oplus \mathbb{R}(Y+Z)$ is a complex subspace. Since $J(B+X) = Z+JX$, then $Y = JX$ necessarily. Hence, $-1/2 = \langle JX, Y\rangle = |X|^2$, which gives a contradiction. Thus, this case is not possible.

3.3.2 Homogeneous CR submanifolds in $\mathbb{C}H^n$

Now that we know the subgroups H of AN that have a CR orbit (which has been assumed to be the one through $o \in \mathbb{C}H^n$), we must study which of the remaining H-orbits are CR submanifolds in order to get a classification result.

Observe that Proposition 3.3.1 can be rephrased by saying that any homogeneous CR submanifold induced by a Lie subgroup of AN is congruent to an orbit of the action of one of the four possible types in Proposition 3.3.1. Thus, since AN acts transitively on the complex hyperbolic space, it will be enough to decide which elements $g \in AN$ satisfy that the orbit $H \cdot g(o)$ is a CR submanifold for each of the four types of subgroups. This is what we address in the following result.

Theorem 3.3.2. *An orbit of the action of a connected Lie subgroup H of AN with Lie algebra $\mathfrak{h}$ is a CR submanifold of $\mathbb{C}H^n$ if, and only if, it is congruent to the orbit $H \cdot g(o)$ for one of the following cases:*

1. $\mathfrak{h} = \mathfrak{r}$ *and* $g \in AN$*; in this case, all the H-orbits are totally real submanifolds that constitute a homogeneous subfoliation of a horosphere foliation.*

2. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$ *and* $g \in \operatorname{Exp}((\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathfrak{g}_{2\alpha})$*; in this case, the CR orbits are totally real equidistant submanifolds to a totally geodesic $\mathbb{R}H^k$, for $k \in \{1, \dots, n\}$.*

3. $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ *and* $g \in AN$*; in this case, all the orbits of H are CR submanifolds that are congruent to each other, and constitute a homogeneous subfoliation of a horosphere foliation.*

4. $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ *and* $g \in \operatorname{Exp}(J\mathfrak{r})$*; in this case, the CR orbits are the leaves of a homogeneous polar foliation with exactly one minimal leaf (which turns out to be a Berndt-Brück submanifold) on a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$, for $k \in \{2, \dots, n\}$.*

Here, $\mathfrak{r}$ stands for a totally real subspace of $\mathfrak{g}_\alpha$, and $\mathfrak{c}$ for a complex subspace of $\mathfrak{g}_\alpha$.

The rest of this section is devoted to proving Theorem 3.3.2. In order to do so, recall that, by definition, the orbit $H \cdot g(o)$ is a CR submanifold of $\mathbb{C}H^n$ if, and only if, its tangent space at each point splits as an orthogonal direct sum of a totally real subspace of $T_{g(o)}(H \cdot g(o))$ and a complex one. Due to the homogeneity hypothesis and the fact that H, as a subgroup of G, consists of holomorphic transformations, it is actually enough to check the CR condition at the point $g(o)$. Notice that, since $H \cdot g(o) = g(g^{-1}Hg \cdot o)$, the tangent space to the orbit $H \cdot g(o)$ at $g(o)$ can be written in terms of the Lie algebra $\mathfrak{h}$ as follows:

$$T_{g(o)}\left(H \cdot g(o)\right) = g_* \left(T_o\left(g^{-1}H \cdot g(o)\right)\right) = g_* \operatorname{Ad}(g^{-1})\mathfrak{h}.$$

Thus, since g is a holomorphic isometry, it is enough to study which $g \in AN$ satisfy that $\operatorname{Ad}(g^{-1})\mathfrak{h}$ decomposes into an orthogonal direct sum of a totally real subspace and a complex one. By virtue of Proposition 3.3.1, the Lie algebras $\mathfrak{h}$ we have to work with are the following:

$$\mathfrak{h} \in \{\mathfrak{r},\ \mathfrak{a} \oplus \mathfrak{r},\ \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha},\ \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}\},$$

where $\mathfrak{r}$ denotes a totally real subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{c}$ is a complex one.

Case 1

We deal firstly with the case $\mathfrak{h} = \mathfrak{r}$, where $\mathfrak{r}$ denotes a totally real subspace of $\mathfrak{g}_\alpha$. In order to do so, we compute $\operatorname{Ad}(g^{-1})\mathfrak{h}$, where the element $g^{-1} \in AN$ can be written as $g^{-1} = \operatorname{Exp}(T)$, with $T = aB + W + U + V + bZ$ for some $W \in \mathfrak{r}$, $U \in J\mathfrak{r}$, $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and a, $b \in \mathbb{R}$.

If $a \neq 0$, given $\widetilde{W} \in \mathfrak{r}$,

$$\begin{aligned}\mathrm{Ad}(g^{-1})(\widetilde{W}) = e^{\mathrm{ad}(T)}(\widetilde{W}) &= \sum_{k=0}^{\infty} \frac{a^k}{2^k k!}\widetilde{W} + \sum_{k=1}^{\infty} \frac{a^{k-1}(2^k-1)}{2^{k-1}k!}\langle JU, \widetilde{W}\rangle Z \\ &= e^{a/2}\widetilde{W} + \frac{2}{a}e^{a/2}(e^{a/2}-1)\langle JU, \widetilde{W}\rangle Z.\end{aligned}$$

Thus, if $a \neq 0$, we identify the tangent space to the orbit $g^{-1}Hg \cdot o$ at o with $\mathrm{Ad}(g^{-1})\mathfrak{h} = \{e^{a/2}\widetilde{W} + \frac{2}{a}e^{a/2}(e^{a/2}-1)\langle JU, \widetilde{W}\rangle Z : \widetilde{W} \in \mathfrak{r}\}$, which can be written as the following orthogonal direct sum:

$$\mathrm{Ad}(g^{-1})\mathfrak{h} = (\mathfrak{r} \ominus \mathbb{R}JU) \oplus \mathbb{R}\left(JU + \frac{2(e^{a/2}-1)|U|^2}{a}Z\right). \tag{3.2}$$

Otherwise, if $a = 0$, $\mathrm{Ad}(g^{-1})(\widetilde{W}) = \widetilde{W} + \langle JU, \widetilde{W}\rangle Z$, so the tangent space to the orbit $g^{-1}Hg \cdot o$ at o can be identified with

$$\mathrm{Ad}(g^{-1})\mathfrak{h} = (\mathfrak{r} \ominus \mathbb{R}JU) \oplus (JU + |U|^2 Z). \tag{3.3}$$

In any case, $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is totally real since, for any $R \in \mathfrak{r} \ominus \mathbb{R}JU$,

$$\langle JR, JU + \frac{2(e^{a/2}-1)|U|^2}{a}Z\rangle = 0 \qquad \text{and} \qquad \langle JR, JU + |U|^2 Z\rangle = 0.$$

Therefore, the corresponding orbit $g^{-1}Hg \cdot o$ is a CR submanifold. As $g^{-1} \in AN$ is arbitrary, one concludes that every H-orbit is totally real, and thus, a CR submanifold. Moreover, since $\mathfrak{h} \subset \mathfrak{n}$, each H-orbit is contained in one of the leaves of the horosphere foliation induced by the nilpotent Lie group N, from where Case 1 in the statement follows.

Case 2

We study now the case $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$, where $\mathfrak{r}$ denotes a totally real subspace of $\mathfrak{g}_\alpha$. Notice that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to the orbit $H \cdot o$ at o, $T_o(H \cdot o)$, and then, the normal space to $H \cdot o$ at o can be identified with the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{a} \oplus \mathfrak{n}$, $\nu_o(H \cdot o) = (\mathfrak{g}_\alpha \ominus \mathfrak{r}) \oplus \mathfrak{g}_{2\alpha} =: \mathfrak{l}$. This normal subspace $\nu_o(H \cdot o)$ is in fact a Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and since $\mathrm{Exp}\colon \mathfrak{a} \oplus \mathfrak{n} \to AN$ is a diffeomorphism, $L := \mathrm{Exp}\left((\mathfrak{g}_\alpha \ominus \mathfrak{r}) \oplus \mathfrak{g}_{2\alpha}\right)$ is a Lie subgroup of AN. The orbit of this Lie subgroup through o, $L \cdot o$, turns out to be a submanifold of $\mathbb{C}H^n$ which intersects every orbit of the H-action. To show this, it is enough to check that the smooth map

$$\begin{aligned}\varphi\colon H \times L &\to AN \\ (h, \ell) &\mapsto h\ell\end{aligned}$$

is surjective. On the one hand, L acts polarly on AN with section $H \cdot e$ (see [30]), which implies that each L-orbit intersects $H \cdot e$ orthogonally. Then, for each $g \in AN$

there exist $\ell \in L$ and $h \in H$ such that $\ell g = h$, or equivalently, $g = \ell^{-1}h$. This actually shows that the map $\phi\colon L \times H \to AN$, $(\ell, h) \mapsto \ell h$, is surjective. On the other hand, H normalizes L, that is, for each $h \in H$ and each $\ell \in L$, $h\ell h^{-1} \in L$, or equivalently, $\mathrm{I}_h\, L \subset L$, where I_h denotes the conjugation by $h \in H$. To prove this fact, we show that $\mathrm{Ad}(h)\mathfrak{l} = (\mathrm{I}_h)_*\mathfrak{l} \subset \mathfrak{l}$. Let $h = \mathrm{Exp}(aB + W) \in H$, with $a \in \mathbb{R}$ and $W \in \mathfrak{r}$. Then, given $U + xZ \in \mathfrak{l}$, with $U \in \mathfrak{g}_\alpha \ominus \mathfrak{r}$ and $x \in \mathbb{R}$,

$$\mathrm{Ad}(h)(U + xZ) = e^{\mathrm{ad}(aB+W)}(U + xZ) = \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k(aB + W)(U + xZ)}{k!},$$

which lies in $\mathfrak{l}$ since each addend does, as $[aB+W, U+xZ] = \frac{a}{2}U + (ax + \langle JW, U\rangle)Z \in \mathfrak{l}$. This proves that $(\mathrm{I}_h)_*\mathfrak{l} \subset \mathfrak{l}$ for each $h \in H$. Hence, the connected Lie subgroup with Lie algebra $(\mathrm{I}_h)_*\mathfrak{l}$, $\mathrm{I}_h\, L = hLh^{-1}$ is contained in L.

Considering these two facts, it easily follows that φ is a surjective map. Indeed, since ϕ is surjective, there exist $h \in H$ and $\ell \in L$ in such a way that $g = \ell h \in Lh$, that also lies in hL since H normalizes L. Then, there exists $\tilde{\ell} \in L$ such that $g = h\tilde{\ell}$, which finally proves that φ is surjective.

Then, taking into account that $L \cdot o$ intersects any H-orbit, in order to know if the remaining orbits of the H-action are CR submanifolds it is enough to decide which $g \in \mathrm{Exp}(\nu_o(H \cdot o))$ satisfy that $\mathrm{Ad}(g^{-1})\mathfrak{h}$ splits into an orthogonal direct sum of a totally real subspace and a complex one. Consider then $g^{-1} = \mathrm{Exp}(U + V + xZ)$, where $U \in J\mathfrak{r}$, $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x \in \mathbb{R}$. It is easy to check that

$$\mathrm{Ad}(g^{-1})(B) = B - \frac{1}{2}(U + V) - xZ \qquad \text{and} \qquad \mathrm{Ad}(g^{-1})(\widetilde{W}) = \widetilde{W} + \langle JU, \widetilde{W}\rangle Z,$$

for each $\widetilde{W} \in \mathfrak{r}$. Then, the tangent space to the orbit $g^{-1}Hg \cdot o$ at o can be identified with the Lie subalgebra

$$\mathrm{Ad}(g^{-1})\mathfrak{h} = \mathbb{R}\left(B - \frac{1}{2}(U + V) - xZ\right) \oplus \mathbb{R}\left(JU + |U|^2 Z\right) \oplus (\mathfrak{r} \ominus \mathbb{R}JU). \tag{3.4}$$

Now we determine when $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is a CR subspace of $\mathfrak{a} \oplus \mathfrak{n}$. Note that $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is a totally real subspace if, and only if, the inner product

$$\begin{aligned}
&\langle J\big(B - \frac{1}{2}(U + V) - xZ\big), JU + |U|^2 Z\rangle \\
&\quad = \langle Z - \frac{1}{2}J(U + V) + xB, JU + |U|^2 Z\rangle = \frac{|U|^2}{2}
\end{aligned}$$

vanishes, which happens if, and only if, $U = 0$. Then, the orbit $g^{-1}Hg \cdot o$ is a totally real submanifold if and only if $g \in \mathrm{Exp}\big((\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathfrak{g}_{2\alpha}\big)$.

On the other hand, in order to check if $\mathrm{Ad}(g^{-1})\mathfrak{h}$ produces a not totally real CR submanifold for some $g \in \mathrm{Exp}(\nu_o(H \cdot o))$, we compute the maximal holomorphic subspace $\mathfrak{m}$ of $\mathrm{Ad}(g^{-1})\mathfrak{h}$, and study whether its orthogonal complement in $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is a totally real subspace. In this case, since we are looking for CR submanifolds that are not totally real, we get $U \neq 0$. Notice that $\mathbb{R}\left(JU + |U|^2 Z\right) \oplus (\mathfrak{r} \ominus \mathbb{R}JU)$ is a totally real subspace and that $B - \frac{1}{2}(U+V) - xZ$ is complex orthogonal to $\mathfrak{r} \ominus \mathbb{R}JU$. Moreover, $J(B - \frac{1}{2}(U+V) - xZ) = xB + Z - \frac{1}{2}(JU + JV)$ cannot be proportional to $JU + |U|^2 Z$, from where we deduce that $\mathrm{Ad}(g^{-1})\mathfrak{h}$ does not contain a nontrivial complex vector subspace. Then, the maximal holomorphic subspace is, in this case, $\mathfrak{m} = J(\mathrm{Ad}(g^{-1})\mathfrak{h}) \cap \mathrm{Ad}(g^{-1})\mathfrak{h} = 0$, and its orthogonal complement in $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is precisely $\mathrm{Ad}(g^{-1})\mathfrak{h}$, which is not a totally real subspace.

Thus, we conclude that the only CR H-orbits, which are in fact totally real submanifolds, are of the form $g^{-1}Hg \cdot o$, where $g \in \mathrm{Exp}((\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathfrak{g}_{2\alpha})$. Furthermore, the orbit $H \cdot o$ is a totally geodesic $\mathbb{R}H^k$, with $k = \dim(\mathfrak{r}) + 1$, and thus the remaining H-orbits are equidistant to it. This corresponds to Case 2 in the statement.

Case 3

We study now the case $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{c} \oplus \mathfrak{r}$ denotes a CR subspace of $\mathfrak{g}_\alpha$. In order to determine which of the remaining orbits of the action of the corresponding connected Lie subgroup H of AN are CR submanifolds, we compute $\mathrm{Ad}(g^{-1})\mathfrak{h}$, where $g \in AN$ is arbitrary. Consider $g^{-1} = \mathrm{Exp}(T)$, where $T = aB + R + JR' + U + W + bZ$ for some R, $R' \in \mathfrak{r}$, $U \in \mathfrak{c}$, $W \in \mathfrak{g}_\alpha \ominus (\mathbb{C}\mathfrak{r} \oplus \mathfrak{c})$ and a, $b \in \mathbb{R}$.

If $a \neq 0$, given $\widetilde{U} \in \mathfrak{c}$ and $\widetilde{R} \in \mathfrak{r}$,

$$\begin{aligned}
\mathrm{Ad}(g^{-1})(Z) &= e^{\mathrm{ad}(T)}(Z) = \sum_{k=0}^{\infty} \frac{a^k}{k!} Z = e^a Z, \\
\mathrm{Ad}(g^{-1})(\widetilde{U}) &= e^{\mathrm{ad}(T)}(\widetilde{U}) = \sum_{k=0}^{\infty} \frac{a^k}{2^k k!}\widetilde{U} + \sum_{k=1}^{\infty} \frac{a^{k-1}(2^k-1)}{2^{k-1}k!}\langle JU, \widetilde{U}\rangle Z \\
&= e^{a/2}\widetilde{U} + \frac{2e^{a/2}(e^{a/2}-1)}{a}\langle JU, \widetilde{U}\rangle Z, \\
\mathrm{Ad}(g^{-1})(\widetilde{R}) &= e^{\mathrm{ad}(T)}(\widetilde{R}) = \sum_{k=0}^{\infty} \frac{a^k}{2^k k!}\widetilde{R} - \sum_{k=1}^{\infty} \frac{a^{k-1}(2^k-1)}{2^{k-1}k!}\langle R', \widetilde{R}\rangle Z \\
&= e^{a/2}\widetilde{R} - \frac{2e^{a/2}(e^{a/2}-1)}{a}\langle R', \widetilde{R}\rangle Z.
\end{aligned}$$

Otherwise, if $a = 0$, given $\widetilde{U} \in \mathfrak{c}$ and $\widetilde{R} \in \mathfrak{r}$,

$$\mathrm{Ad}(g^{-1})(Z) = Z, \quad \mathrm{Ad}(g^{-1})(\widetilde{U}) = \widetilde{U} + \langle JU, \widetilde{U}\rangle Z, \quad \mathrm{Ad}(g^{-1})(\widetilde{R}) = \widetilde{R} - \langle R', \widetilde{R}\rangle Z.$$

Thus, in any case, $\mathrm{Ad}(g^{-1})\mathfrak{h} = \mathfrak{h}$ for any $g \in AN$. It follows immediately that every H-orbit is then a CR submanifold of $\mathbb{C}H^n$ and, moreover, that all H-orbits are mutually congruent. Since $\mathfrak{h} \subset \mathfrak{n}$, the H-orbits are contained in the leaves of the horosphere foliation induced by N. This corresponds to Case 3 in the statement.

Case 4

Finally, we study the last case in the statement, that is, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{c} \oplus \mathfrak{r}$ denotes a CR subspace of $\mathfrak{g}_\alpha$. The Lie algebra $\mathfrak{h}$ can be identified with the tangent space to the orbit $H\cdot o$ at o, and then, the corresponding normal space can be identified with the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{a}\oplus\mathfrak{n}$, $\nu_o(H\cdot o)=\mathfrak{g}_\alpha\ominus(\mathfrak{c}\oplus\mathfrak{r})=J\mathfrak{r}\oplus\mathfrak{c}'$, where $\mathfrak{c}'=\mathfrak{g}_\alpha\ominus(\mathfrak{c}\oplus\mathfrak{r}\oplus J\mathfrak{r})$ is a complex subspace of $\mathfrak{g}_\alpha$. We denote $\Sigma=\operatorname{Exp}(J\mathfrak{r}\oplus\mathfrak{c}')$, which is a submanifold of AN (since $\operatorname{Exp}\colon\mathfrak{a}\oplus\mathfrak{n}\to AN$ is a diffeomorphism) but not a subgroup in general. Proceeding as in Case 2, we will prove that Σ intersects every orbit of the H-action, and thus, it will be enough to decide which $g\in\operatorname{Exp}(\nu_o(H\cdot o))$ satisfy that the orbit $g^{-1}Hg\cdot o$ is a CR submanifold of $\mathbb{C}H^n$.

Again, to see that Σ intersects each H-orbit, we show that the smooth map

$$\begin{aligned}\varphi\colon H\times\Sigma&\to AN\\ (h,p)&\mapsto(h,p)=hp.\end{aligned}$$

is surjective. To do so, let $g=hp\in AN$, where $h=\operatorname{Exp}(aB+U+V+xZ)\in H$ and $p=\operatorname{Exp}(JV'+W)\in\Sigma$ for some $U\in\mathfrak{c}$, $V,V'\in\mathfrak{r}$, $W\in\mathfrak{c}'$ and $x\in\mathbb{R}$.

If $a\neq 0$ then, by [17, Subsections 4.1.3 and 4.1.4],

$$\begin{aligned}g=&\operatorname{Exp}(aB+U+V+xZ)\cdot\operatorname{Exp}(JV'+W)\\ =&\left(\operatorname{Exp}_{\mathfrak{n}}\left(\frac{2(e^{a/2}-1)}{a}(U+V)+\frac{x(e^a-1)}{a}Z\right),a\right)\cdot\left(\operatorname{Exp}_{\mathfrak{n}}(JV'+W),0\right)\\ =&\left(\operatorname{Exp}_{\mathfrak{n}}\left(\frac{2(e^{a/2}-1)}{a}(U+V)+e^{a/2}(JV'+W)+\frac{x(e^a-1)}{a}Z\right.\right.\\ &\left.\left.+\frac{e^{a/2}}{2}\left[\frac{2(e^{a/2}-1)}{a}(U+V),JV'+W\right],a\right)\right)\\ =&(\operatorname{Exp}_{\mathfrak{n}}(\widetilde{W}+yZ),a),\end{aligned}$$

where

$$\widetilde{W}=\frac{2(e^{a/2}-1)}{a}(U+V)+e^{a/2}(JV'+W),\quad y=\frac{e^{a/2}(e^{a/2}-1)}{a}\langle V,V'\rangle+\frac{e^a-1}{a}x.$$

Analogously, if $a=0$, $g=hp=\operatorname{Exp}(U+V+xZ)\cdot\operatorname{Exp}(JV'+W)$ then, by [17, Subsections 4.1.3 and 4.1.4],

$$\begin{aligned}&\operatorname{Exp}(U+V+xZ)\cdot\operatorname{Exp}_{\mathfrak{a}\oplus\mathfrak{n}}(JV'+W)\\ &\quad=(\operatorname{Exp}_{\mathfrak{n}}(U+V+xZ),0)\cdot(\operatorname{Exp}_{\mathfrak{n}}(JV'+W),0)\\ &\quad=\left(\operatorname{Exp}_{\mathfrak{n}}\left(U+V+JV'+W+xZ+\frac{1}{2}[U+V,JV'+W]\right),0\right)\\ &\quad=\left(\operatorname{Exp}_{\mathfrak{n}}(\widetilde{W}+yZ),0\right),\end{aligned}$$

where $\widetilde{W}=U+V+JV'+W$ and $y=x+\frac{1}{2}\langle V,V'\rangle$. From these expressions it is straightforward to check that φ is a surjective map.

Therefore, it is enough to study which $g^{-1}=\operatorname{Exp}(JR+C)$, with $R\in\mathfrak{r}$ and $C\in\mathfrak{c}'$, satisfy that $g^{-1}Hg \cdot o$ is a CR submanifold. Let $V \in \mathfrak{r}$, $U \in \mathfrak{c}$. Then,

$$\mathrm{Ad}(g^{-1})(B) = B - \frac{1}{2}(JR + C), \qquad \mathrm{Ad}(g^{-1})(U) = U,$$
$$\mathrm{Ad}(g^{-1})(V) = V - \langle R, V\rangle Z, \qquad \mathrm{Ad}(g^{-1})(Z) = Z,$$

from where

$$\mathrm{Ad}(g^{-1})\mathfrak{h} = \mathbb{R}\left(B - \frac{1}{2}(JR + C)\right) \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}. \tag{3.5}$$

Now we check if this Lie algebra corresponds to the tangent space of a CR submanifold. Proceeding as in Case 2, we compute the maximal holomorphic subspace $\mathfrak{m} = J(\mathrm{Ad}(g^{-1})\mathfrak{h}) \cap \mathrm{Ad}(g^{-1})\mathfrak{h}$ and check if its orthogonal complement in $\mathrm{Ad}(g^{-1})\mathfrak{h}$ is a totally real subspace. After some easy calculations, one obtains that the maximal holomorphic subspace is, in this case,

$$\mathfrak{m} = \begin{cases} \mathfrak{c}, & \text{if } C \neq 0; \\ \mathbb{R}\left(Z + \frac{1}{2}R\right) \oplus \mathfrak{c} \oplus \mathbb{R}\left(B - \frac{1}{2}JR\right), & \text{if } C = 0. \end{cases}$$

We study these two cases separately, depending on whether $C = 0$ or $C \neq 0$.

If $C \neq 0$, then the orthogonal complement to the maximal complex subspace is $\mathrm{Ad}(g^{-1})\mathfrak{h} \ominus \mathfrak{m} = \mathbb{R}\left(B - \frac{1}{2}(JR + C)\right) \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, which is not a totally real subspace since

$$\langle J\left(B - \frac{1}{2}(JR + C)\right), Z\rangle = \langle Z + \frac{1}{2}(R - JC), Z\rangle = 1 \neq 0.$$

Then, $g^{-1}Hg \cdot o$ is not a CR submanifold when $C \neq 0$. Otherwise, if $C = 0$, the orthogonal complement to the maximal complex subspace is

$$\mathrm{Ad}(g^{-1})\mathfrak{h} \ominus \mathfrak{m} = (\mathfrak{r} \oplus \mathbb{R}Z) \ominus \mathbb{R}\left(Z + \frac{1}{2}R\right) = (\mathfrak{r} \ominus \mathbb{R}R) \oplus \mathbb{R}\left(\frac{|R|^2}{2}Z - R\right).$$

Given $\hat{R} \in \mathfrak{r} \ominus \mathbb{R}R$, it follows that

$$\langle \hat{R}, J\left(\frac{|R|^2}{2}Z - R\right)\rangle = -\langle \hat{R}, \frac{|R|^2}{2}B + JR\rangle = 0,$$

and so $\mathrm{Ad}(g^{-1})\mathfrak{h} \ominus \mathfrak{m}$ is a totally real subspace. Thus we conclude that the orbit $g^{-1}Hg \cdot o$ is a CR submanifold if, and only if, $g \in \mathrm{Exp}(J\mathfrak{r})$. Moreover, it cannot be a totally real submanifold since the maximal holomorphic subspace $\mathfrak{m}$ is not trivial in this case. Notice that $\mathrm{Exp}(\mathfrak{h} \oplus J\mathfrak{r}) \cdot o$ is a totally geodesic $\mathbb{C}H^k$, with $k = \dim_{\mathbb{C}}(\mathfrak{h} \oplus J\mathfrak{r})$. The H-orbits that foliate this $\mathbb{C}H^k$ constitute a homogeneous polar regular foliation whose leaf $H \cdot o$ is minimal and indeed it is a Berndt-Brück submanifold of such $\mathbb{C}H^k$, which follows from [30, Theorem A and Corollary 6.2]. This corresponds to Case 4 in the statement.

3.4 Congruence classes

This section is devoted to classifying the examples that we have obtained in Theorem 3.3.2, up to congruence. This problem is settled in the following result.

Theorem 3.4.1. *Let H_1 and H_2 be two Lie subgroups of AN and denote by $\mathfrak{h}_1$ and $\mathfrak{h}_2$ their Lie algebras, respectively. Assume that H_1 and H_2 act on $\mathbb{C}H^n$ in such a way that $H_1 \cdot g_1(o)$ and $H_2 \cdot g_2(o)$ are CR submanifolds, with $g_1, g_2 \in AN$ as given by Theorem 3.3.2. Then, $H_1 \cdot g_1(o)$ and $H_2 \cdot g_2(o)$ are congruent orbits if, and only if, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ correspond to the same type in Theorem 3.3.2 and, according to the type:*

1. *$g_i = \operatorname{Exp}(a_iB+W_i+U_i+V_i+b_iZ)$, with $a_i, b_i \in \mathbb{R}$, $W_i \in \mathfrak{r}$, $U_i \in J\mathfrak{r}$, $V_i \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$, for $i \in \{1,2\}$, and $\rho(a_2)|U_1| = \rho(a_1)|U_2|$, where $\rho\colon \mathbb{R} \to (0,\infty)$ is the analytic function defined by $\rho(0) = 1$ and $\rho(t) = 2(e^{t/2}-1)/t$ for any $t \neq 0$;*
2. *$g_i = \operatorname{Exp}(V_i + x_iZ)$, with $x_i \in \mathbb{R}$, $V_i \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$, for $i \in \{1,2\}$, and $|V_1| = |V_2|$, $|x_1| = |x_2|$;*
3. *In this case, all the orbits are congruent;*
4. *$g_i = \operatorname{Exp}(JR_i)$, with $R_i \in \mathfrak{r}$, for $i \in \{1,2\}$, and $|R_1| = |R_2|$.*

The rest of this section is devoted to proving Theorem 3.4.1. We start by studying each case separately.

Case 1

We firstly study the congruence among the orbits of the form $H \cdot g(o)$, where $\mathfrak{h} = \mathfrak{r}$ is a totally real subspace of $\mathfrak{g}_\alpha$ and $g^{-1} = \operatorname{Exp}(aB + W + U + V + bZ)$ for some $W \in \mathfrak{r}$, $U \in J\mathfrak{r}$, $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and a, $b \in \mathbb{R}$. Recall that two totally real subspaces of $\mathfrak{g}_\alpha$ are congruent by an element of $K_0 \cong S(U(1)U(n))$, and hence the actions of the corresponding connected Lie subgroups of AN are conjugate if, and only if, both have the same dimension. Then, we can fix $\mathfrak{r}$ in the rest of the proof.

First of all, consider the element $\tilde{g}^{-1} = \operatorname{Exp}(\widetilde{U})$, where $\widetilde{U} \in J\mathfrak{r}$. Then, given $\widetilde{W} \in \mathfrak{r}$, we have $\operatorname{Ad}(\tilde{g}^{-1})(\widetilde{W}) = e^{\operatorname{ad}(\widetilde{U})}(\widetilde{W}) = \widetilde{W} + \langle J\widetilde{U}, \widetilde{W}\rangle Z$, from where

$$\operatorname{Ad}(\tilde{g}^{-1})\mathfrak{h} = (\mathfrak{r} \ominus \mathbb{R}J\widetilde{U}) \oplus \mathbb{R}(J\widetilde{U} + |\widetilde{U}|^2 Z).$$

Notice that, in particular, if $a \neq 0$, one may consider the element

$$\widetilde{U} = \frac{2}{a}(e^{a/2}-1)U \in J\mathfrak{r}. \tag{3.6}$$

Then, we obtain

$$\begin{aligned}\operatorname{Ad}(\tilde{g}^{-1})\mathfrak{h} &= (\mathfrak{r} \ominus \mathbb{R}JU) \oplus \mathbb{R}\left(\frac{2}{a}(e^{a/2}-1)JU + \left|\frac{2}{a}(e^{a/2}-1)U\right|^2 Z\right) \\ &= (\mathfrak{r} \ominus \mathbb{R}JU) \oplus \mathbb{R}\left(JU + \frac{2}{a}(e^{a/2}-1)|U|^2 Z\right) = \operatorname{Ad}(g^{-1})\mathfrak{h},\end{aligned} \tag{3.7}$$

where the last equality follows from the relation (3.2). Otherwise, if $a = 0$, taking $\tilde{g}^{-1} = \operatorname{Exp}(U)$, it follows from (3.3) that

$$\operatorname{Ad}(\tilde{g}^{-1})\mathfrak{h} = \operatorname{Ad}(g^{-1})\mathfrak{h} = (\mathfrak{w} \ominus \mathbb{R}JU) \oplus \mathbb{R}(JU + |U|^2 Z). \tag{3.8}$$

Thus, we deduce that $g^{-1}Hg \cdot o$ is congruent to $\tilde{g}^{-1}H\tilde{g} \cdot o$, or equivalently, that $H \cdot g(o)$ is congruent to $H \cdot \tilde{g}(o)$, where $\tilde{g}^{-1} = \operatorname{Exp}(\rho(a)U)$ and $\rho \colon \mathbb{R} \to (0, \infty)$ is the analytic function defined by

$$\rho(t) = \begin{cases} 1, & \text{if } t = 0, \\ \frac{2(e^{t/2}-1)}{t}, & \text{if } t \neq 0. \end{cases} \tag{3.9}$$

Therefore, in order to settle the congruence problem for Case 1, we just have to consider elements of the form $g \in \operatorname{Exp}(J\mathfrak{r})$. In the following result, we study the mean curvature of these orbits.

Lemma 3.4.2. *Let $\mathfrak{h} = \mathfrak{r}$ be a totally real subspace of $\mathfrak{g}_\alpha$ and let $g^{-1} = \operatorname{Exp}(U)$, with $U \in J\mathfrak{r}$. If $s = \dim(\mathfrak{r} \ominus \mathbb{R}JU)$, the squared norm of the mean curvature vector $\mathcal{H}$ of the orbit $H \cdot g(o)$ is given by*

$$|\mathcal{H}|^2 = \frac{4|U|^2 + (1 + s + (2+s)|U|^2)^2}{4(1 + |U|^2)^2}.$$

Proof. Let $g^{-1} = \operatorname{Exp}(U)$, with $U \in J\mathfrak{r}$. Since $H \cdot g(o)$ is congruent to $g^{-1}Hg \cdot o$, we compute the mean curvature for the latter. By homogeneity, it is enough to do so at $o \in \mathbb{C}H^n$. Recall that the tangent space of the orbit $g^{-1}Hg \cdot o$ at o, which we identify with $\operatorname{Ad}(g^{-1})\mathfrak{h}$, is

$$T_o(g^{-1}Hg \cdot o) = (\mathfrak{r} \ominus \mathbb{R}JU) \oplus \mathbb{R}(JU + |U|^2 Z).$$

Analogously, the normal space at o, which can be identified with the orthogonal complement of $\operatorname{Ad}(g^{-1})\mathfrak{h}$ in $\mathfrak{a} \oplus \mathfrak{n}$, is

$$\nu_o(g^{-1}Hg \cdot o) = \mathfrak{a} \oplus J\mathfrak{r} \oplus (\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathbb{R}(Z - JU). \tag{3.10}$$

Let $W \in \mathfrak{r} \ominus \mathbb{R}JU$ with $|W| = 1$ and $T = \frac{JU + |U|^2 Z}{|U|\sqrt{1+|U|^2}}$ be left-invariant unit tangent vector fields to $g^{-1}Hg \cdot o$ at o. Using the expression for the Levi-Civita connection of the complex hyperbolic space in terms of left-invariant vector fields of AN (Subsection 1.5.4), it follows that

$$\bar{\nabla}_W W = \frac{1}{2}B \qquad \text{and} \qquad \bar{\nabla}_T T = \frac{1 + 2|U|^2}{2(1 + |U|^2)}B + \frac{1}{1 + |U|^2}U.$$

Recall that, given an orthonormal basis $\{E_i\}_i$ of $T_o(g^{-1}Hg \cdot o)$, the mean curvature of $g^{-1}Hg \cdot o$ at o can be computed as follows:

$$\mathcal{H} = \sum_i I\!I(E_i, E_i) = \sum_i (\bar{\nabla}_{E_i} E_i)^\perp,$$

where II is the second fundamental form of $g^{-1}Hg\cdot o$. If we denote $s=\dim(\mathfrak{r}\ominus\mathbb{R}JU)$, using the expressions of the Levi-Civita connection above, and projecting onto the normal space according to (3.10), it follows that the mean curvature of the orbit $g^{-1}Hg\cdot o$ is

$$\mathcal{H}=\left(\frac{s}{2}+\frac{1+2|U|^2}{2(1+|U|^2)}\right)B+\frac{1}{1+|U|^2}U. \tag{3.11}$$

The result follows after computing the squared norm of this vector. $\square$

In order to finish the proof of this case, we consider $g_i^{-1}=\operatorname{Exp}(U_i)$, with $U_i\in J\mathfrak{r}$, for $i\in\{1,2\}$. We distinguish two possibilities, depending on whether $|U_1|=|U_2|$ or $|U_1|\neq|U_2|$.

Case 1.a

Assume firstly $|U_1|=|U_2|$. The connected component of the identity of the normalizer of $\mathfrak{r}$ in K_0, which is given by $N^0_{K_0}(\mathfrak{r})\simeq SO(\dim(\mathfrak{r}))\times U(n-\dim(\mathfrak{r})-1)$, acts transitively on the spheres of $\mathfrak{r}$ centered at the origin. Hence, there exists an element $k\in N^0_{K_0}(\mathfrak{r})$ satisfying $\operatorname{Ad}(k)(U_1)=U_2$. As $k\in N^0_{K_0}(\mathfrak{r})$ and $K_0\simeq U(n-1)$, then $k\in N_{K_0}(\mathfrak{g}_\alpha\ominus\mathbb{C}\mathfrak{r})$. Considering these facts, it follows that

$$\begin{aligned}\operatorname{Ad}(k)\operatorname{Ad}(g_1^{-1})\mathfrak{h}&=\operatorname{Ad}(k)\left((\mathfrak{r}\ominus\mathbb{R}JU_1)\oplus\mathbb{R}(JU_1+|U_1|^2Z)\right)\\&=\left((\mathfrak{r}\ominus\mathbb{R}JU_2)\ominus\mathbb{R}(JU_2+|U_2|^2Z)\right)=\operatorname{Ad}(g_2^{-1})\mathfrak{h}.\end{aligned}$$

Since k fixes $o\in\mathbb{C}H^n$, it follows that $k(g_1^{-1}Hg_1\cdot o)=g_2^{-1}Hg_2\cdot o$, which shows that $H\cdot g_1(o)$ is congruent to $H\cdot g_2(o)$.

Case 1.b

Now we study the congruence between the orbits $H\cdot g_1(o)$ and $H\cdot g_2(o)$ when $|U_1|\neq|U_2|$. The expression for $|\mathcal{H}|^2$ given in Lemma 3.4.2 allows us to conclude that, if $|U_1|\neq|U_2|$, the orbits $H\cdot g_1(o)$ and $H\cdot g_2(o)$ cannot be congruent since the squared norms of their associated mean curvatures are different. Indeed, if we denote by $|\mathcal{H}_i|^2$ the squared norm of the mean curvature of the orbit $H\cdot g_i(o)$, for $i\in\{1,2\}$, then $|\mathcal{H}_1|^2=|\mathcal{H}_2|^2$ provided that

$$(|U_1|^2-|U_2|^2)\left(2s(1+|U_1|^2)(1+|U_2|^2)+3(2+|U_1|^2+|U_2|^2)\right)=0,$$

which happens if, and only if, $|U_1|=|U_2|$.

To finish with this case, let now $g_i^{-1}=\operatorname{Exp}(a_iB+W_i+U_i+V_i+b_iZ)$, where a_i, $b_i\in\mathbb{R}$, $W_i\in\mathfrak{r}$, $U_i\in J\mathfrak{r}$, $V_i\in\mathfrak{g}_\alpha\ominus\mathbb{C}\mathfrak{r}$, for $i\in\{1,2\}$. Then, taking into account relations (3.7) and (3.8), as well as the expressions (3.6) and (3.9) of the definition of $\widetilde{U}$ and the function ρ, respectively, one concludes that the orbits $H\cdot g_1(o)$ and $H\cdot g_2(o)$ are congruent if, and only if, $\rho(a_2)|U_1|=\rho(a_1)|U_1|$.

Case 2

We now analyze the congruence among the orbits of the form $H \cdot g(o)$, where $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$, $\mathfrak{r}$ is a totally real subspace of $\mathfrak{g}_\alpha$, and $g = \operatorname{Exp}(2V + xZ)$, with $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x \in \mathbb{R}$. Let $g_1 = \operatorname{Exp}(2V_1 + x_1 Z)$, $g_2 = \operatorname{Exp}(2V_2 + x_2 Z)$, with $V_i \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x_i \in \mathbb{R}$ for $i \in \{1, 2\}$. Again, recall that two totally real subspaces of $\mathfrak{g}_\alpha$ are congruent if, and only if, both have the same dimension. Then, we can fix $\mathfrak{r}$ in the rest of the proof.

To start with, we compute the squared norm of both the mean curvature and the second fundamental form of a Type 2 orbit $H \cdot g(o)$.

Lemma 3.4.3. *Let $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$, where $\mathfrak{r}$ denotes a totally real subspace of $\mathfrak{g}_\alpha$, and let $g = \operatorname{Exp}(2V + xZ)$, with $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x \in \mathbb{R}$. Then, if $s = \dim(\mathfrak{r})$, the squared norms of the mean curvature and the second fundamental form of the orbit $H \cdot g(o)$ are given by the following expressions:*

$$|\mathcal{H}|^2 = \frac{(1+s)^2|V|^4 + (2+s)^2x^2(1+x^2) + |V|^2\left(1 + 8x^2 + s^2(1+2x^2) + 2s(1+3x^2)\right)}{4(1+x^2+|V|^2)^2},$$
$$|II|^2 = \frac{(1+s)|V|^4 + (4+3s)x^2(1+x^2) + |V|^2\left(1 + s + 4x^2(2+s)\right)}{4(1+x^2+|V|^2)^2}.$$

Proof. Let $g = \operatorname{Exp}(2V + xZ)$, with $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x \in \mathbb{R}$. Since $H \cdot g(o)$ and $g^{-1}Hg \cdot o$ are congruent orbits, we compute the mean curvature and the second fundamental form for the latter. From relation (3.4), the Lie subalgebra $\operatorname{Ad}(g^{-1})\mathfrak{h} = \mathbb{R}(B + V + xZ) \oplus \mathfrak{r}$ can be identified with the tangent space $T_o(g^{-1}Hg \cdot o)$. Consider an arbitrary $W \in \mathfrak{r}$ with $|W| = 1$ and

$$X = \frac{B + V + xZ}{\sqrt{1 + x^2 + |V|^2}},$$

which are left-invariant unit tangent vector fields to $g^{-1}Hg \cdot o$ at o. Consider also $J\widetilde{W}$, with $\widetilde{W} \in \mathfrak{r}$ and $|\widetilde{W}| = 1$, $JV/|V|$ (in case $V \neq 0$), $V' \in (\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \ominus \mathbb{C}V$ and

$$\xi_1 = \frac{xB - Z}{\sqrt{1+x^2}}, \qquad \xi_2 = \frac{(1+x^2)V - |V|^2 B - x|V|^2 Z}{|V|\sqrt{(1+x^2)\left(1+x^2+|V|^2\right)}},$$

which are normal vectors. Now, we compute the shape operators with respect to these unit normal vectors. Using again the formula for the Levi-Civita connection of the complex hyperbolic space, one obtains

$$\begin{aligned}
\mathcal{S}_{\xi_1} W &= \frac{x}{2\sqrt{1+x^2}}\, W, & \mathcal{S}_{\xi_1} X &= \frac{x(2+2x^2+|V|^2)}{2(1+x^2+|V|^2)\sqrt{1+x^2}}\, X,\\
\mathcal{S}_{\xi_2} W &= -\frac{|V|}{2\sqrt{(1+x^2)\big(1+x^2+|V|^2\big)}}\, W, & \mathcal{S}_{\xi_2} X &= -\frac{|V|}{2\sqrt{(1+x^2)\big(1+x^2+|V|^2\big)}}\, X,\\
\mathcal{S}_{J\widetilde{W}} W &= -\frac{x\langle W, \widetilde{W}\rangle}{2\sqrt{1+x^2+|V|^2}}\, X, & \mathcal{S}_{J\widetilde{W}} X &= -\frac{x}{2\sqrt{1+x^2+|V|^2}}\, W,\\
\mathcal{S}_{\frac{JV}{|V|}} W &= 0, & \mathcal{S}_{\frac{JV}{|V|}} X &= -\frac{x|V|}{1+x^2+|V|^2}\, X,\\
\mathcal{S}_{V'} W &= 0, & \mathcal{S}_{V'} X &= 0.
\end{aligned}$$

Given orthonormal bases $\{E_i\}_i$ and $\{\xi_k\}_k$ of $T_o(g^{-1}Hg\cdot o)$ and $\nu_o(g^{-1}Hg\cdot o)$, respectively, the squared norms of the mean curvature and the second fundamental can be calculated as follows:

$$|\mathcal{H}|^2 = \sum_k \operatorname{tr}(\mathcal{S}_{\xi_k})^2, \qquad |II|^2 = \sum_{i,j,k} \langle \mathcal{S}_{\xi_k} E_i, E_j\rangle^2.$$

In particular, if we denote $s = \dim(\mathfrak{r})$, inserting the relations of the shapes operators above, one gets the explicit expressions for these two geometric invariants given in the statement. $\square$

Subtracting the two equalities given in Lemma 3.4.3, we obtain a third geometric invariant:

$$|\mathcal{H}|^2 - |II|^2 = \frac{s(1+s)(x^2+|V|^2)}{4(1+x^2+|V|^2)}.$$

We will use these expressions in order to prove that there exists a 2-parameter family of orbits of Type 2, up to congruence. More specifically, if $g_1 = \operatorname{Exp}(2V_1 + x_1 Z)$, $g_2 = \operatorname{Exp}(2V_2 + x_2 Z)$, with $V_i \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x_i \in \mathbb{R}$, for $i \in \{1,2\}$, we will show that the orbits $H\cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent if, and only if, $|V_1| = |V_2|$ and $|x_1| = |x_2|$.

Case 2.a

We firstly show that if $|V_1| \neq |V_2|$ or $|x_1| \neq |x_2|$, the orbits $H\cdot g_1(o)$ and $H\cdot g_2(o)$ cannot be congruent. We distinguish two possibilities, depending on whether $s = 0$ or $s \neq 0$.

Case $s \neq 0$. To tackle this problem, we prove that the system in $|V|$ and x^2 given by the expressions for $S := |\mathcal{H}|^2 - |II|^2$ and $T := |II|^2$,

$$
\begin{aligned}
S &= \frac{s(1+s)(x^2+|V|^2)}{4(1+x^2+|V|^2)}, \\
T &= \frac{(1+s)|V|^4 + (4+3s)x^2(1+x^2) + |V|^2\left(1+s+4x^2(2+s)\right)}{4(1+x^2+|V|^2)^2},
\end{aligned}
$$

has a unique nonnegative solution. From the first equation we get

$$
|V|^2 = \frac{-s(1+s)x^2 + 4S(1+x^2)}{s^2+s-4S}. \tag{3.12}
$$

Inserting this expression for $|V|^2$ into the second equality of the system, we obtain a quadratic equation in x^2 with necessarily nonnegative discriminant, which implies that there exist two solutions for x^2 (which may coincide). However, inserting these two values of x^2 into (3.12), one gets that one of the resulting values of $|V|^2$ is strictly negative. Indeed, the denominator is a positive number (since $s^2+s-4S = (s^2+s)/(1+x^2+|V|^2)$), but the coefficient of x^2 is negative, so $|V|^2$ would be a negative number provided that $-s(1+s)x^2+4S(1+x^2) < 0$ for some value of x^2. In particular, this happens if such value of x^2 is higher than $\frac{4S}{s^2+s-4S}$. The highest value of x^2 obtained from (3.12) satisfies this inequality, and hence, the corresponding value of $|V|^2$ is negative, which is a contradiction. Thus, the previous system has a unique nonnegative solution $(|V|, x^2)$.

Case $s = 0$. In this case, $\mathfrak{h} = \mathfrak{a}$ and, with the notation above, $\mathrm{Ad}(g)\mathfrak{h} = \mathbb{R}X$. In particular, every orbit of the corresponding Lie subgroup H is one-dimensional. Notice that, since $s = 0$, $|\mathcal{H}|^2 = |II|^2$. So, in order to get a system of two equations in $|V|$ and x, we compute yet another geometric invariant: the complex curvature $\langle \bar{\nabla}_X X, JX\rangle$ [55, Section 5]. Again, using the formula for the Levi-Civita connection of the complex hyperbolic space, we obtain

$$
\begin{aligned}
\bar{\nabla}_X X &= \frac{1}{1+x^2+|V|^2}\bar{\nabla}_{B+V+xZ}(B+V+xZ) \\
&= \frac{1}{2(1+x^2+|V|^2)}\left((|V|^2+2x^2)B - V - 2xJV - 2xZ\right).
\end{aligned}
$$

Then, since $JX = (Z+JV-xB)/\sqrt{1+x^2+|V|^2}$,

$$
\begin{aligned}
\langle \bar{\nabla}_X X, JX\rangle &= \frac{1}{2(1+x^2+|V|^2)^{3/2}}\langle (|V|^2+2x^2)B - V - 2xJV - 2xZ, Z+JV-xB\rangle \\
&= \frac{-x}{2(1+x^2+|V|^2)^{3/2}}\left(3|V|^2 + 2(1+x^2)\right).
\end{aligned}
$$

Let $\lambda = x^2$, $\mu = |V|^2$ and, to shorten notation, let us denote $S = \langle \bar{\nabla}_X X, JX\rangle^2$, $T = |\mathcal{H}|^2$. Then we have the following system in λ and μ:

$$
S = \frac{\lambda(2+2\lambda+3\mu)^2}{4(1+\lambda+\mu)^3}, \qquad T = \frac{4\lambda(1+\lambda)+\mu(1+8\lambda+\mu)}{4(1+\lambda+\mu)^2}.
$$

We now show that this system has only one nonnegative solution for nonnegative S and T, or equivalently, that the map

$$\begin{aligned} F\colon [0,+\infty)\times[0,+\infty) &\to [0,+\infty)\times[0,+\infty) \\ (\lambda,\mu) &\mapsto \left(\frac{\lambda(2+2\lambda+3\mu)^2}{4(1+\lambda+\mu)^3}, \frac{4(\lambda+\lambda^2)+\mu(1+8\lambda+\mu)}{4(1+\lambda+\mu)^2}\right) \end{aligned}$$

is injective. Suppose that, on the contrary, F is not an injective map. In such case, there exists a pair of distinct points, (λ_1,μ_1) and (λ_2,μ_2) that have the same image under F. Consider the straight line joining both points. Then, each one of both components of F should have a critical point in the open segment between (λ_1,μ_1) and (λ_2,μ_2).

Suppose firstly that these two points satisfy $\lambda_1=\lambda_2=:\lambda$. Hence, there exists a straight vertical line (λ,t) joining them. Restricted to the points of this vertical segment, the function F has the form

$$F(\lambda,t)=\left(\frac{\lambda(2+2\lambda+3t)^2}{4(1+\lambda+t)^3}, \frac{4(\lambda+\lambda^2)+t(1+8\lambda+t)}{4(1+\lambda+t)^2}\right),$$

and its derivative with respect to t is

$$\frac{dF}{dt}(\lambda,t)=\left(-\frac{3\lambda t(2+2\lambda+3t)}{4(1+\lambda+t)^4}, \frac{1+\lambda+t-6\lambda t}{4(1+\lambda+t)^3}\right).$$

If $\lambda\neq 0$, since $t>0$, the first component of this derivative is strictly negative, so there is no critical point, which yields a contradiction. Otherwise, if $\lambda=0$, the second component of this derivative, $t/4(1+t)^2$, is strictly positive and hence there is no critical point, which yields again a contradiction.

Suppose now that there exists a pair of distinct points, (λ_1,μ_1) and (λ_2,μ_2), having the same image under the function F, and assume that there exists a nonvertical straight line $(t,mt+n)$ joining such points. Restricted to this segment, F has the form

$$F(t,mt+n)=\left(\frac{t(2+3n+(2+3m)t)^2}{4(1+n+t+mt)^3}, \frac{4t(1+t)+(mt+n)(1+n+(8+m)t)}{4(1+n+t+mt)^2}\right).$$

Moreover, the corresponding derivative with respect to t is

$$\begin{aligned}\frac{dF}{dt}(t,mt+n)=\Bigg(&\frac{((3m+2)t+3n+2)(n(3mt+3n+5)+(5m+2)t+2)}{4(mt+n+t+1)^4}, \\ &\frac{t(m(m+6n+11)+4)+(n+1)(m+6n+4)}{4(mt+n+t+1)^3}\Bigg).\end{aligned}$$

If $m\geq 0$ and $n\geq 0$, since $t\geq 0$, the first component of this derivative is strictly positive. Thus, there is no critical point and we get a contradiction.

On the other hand, assume $m\leq 0$ and consider the change of variable $t=\lambda$, $n=\mu-m\lambda$. Taking into account that λ, $\mu\geq 0$, it follows that $n\geq 0$, and thus, the first component of the derivative of F is strictly positive. Again, this means that there is no critical point, from where we get a contradiction.

We finally deal with the case $m > 0$ and $n < 0$, checking the injectivity of F directly. Suppose that there exist two values of t in such a way that the corresponding images under F take the same value (S, T), for some fixed S, $T \geq 0$. We will get a contradiction in order to prove that these two values for t are, necessarily, equal.

We firstly determine such values of t. In order to do so, we use the second component of F to derive the following equation:

$$\begin{aligned} t^2(4+8m+m^2-4(1+m)^2T)+t((4+m)(1+2n)-8(1+m)(1+n)T) \\ +(1+n)(n-4(1+n)T)=0. \end{aligned} \tag{3.13}$$

We assume that $4+8m+m^2-4(1+m)^2T \neq 0$ because, otherwise, this equation would have only one solution for t and we would be done. Moreover, notice that the two solutions coincide whenever this equation has vanishing discriminant, that is, if

$$T = \frac{m(m+8)+16(1+3n(1+n))}{48(n-m)(1+n)}. \tag{3.14}$$

Now, inserting the two solutions of (3.13) into the first component of F, we get two expressions that must coincide, which happens if and only if

$$\begin{gathered} \sqrt{48(n+1)T(m-n)+m(m+8)+16(3n(n+1)+1)} = 0 \quad \text{or} \\ m^2+m(12(n+1)T-9n-4)+4(1-3n(n+1)(T-1)) = 0. \end{gathered}$$

Notice that the vanishing of the first relation is equivalent to the vanishing of the discriminant (3.14). In this case, the two values of t must be the same and we are finished. So we assume that the second expression vanishes, or equivalently,

$$T = \frac{4-4m+m^2+12n-9mn+12n^2}{12(n-m)(1+n)}. \tag{3.15}$$

Inserting this value of T into (3.14), the expression of the discriminant above reduces to $-3m(-8+m-12n)$, which must be strictly positive in order to obtain two different values for t. Since we are assuming that $m > 0$ and $n < 0$, it follows immediately that $n+1 > 0$ and $m-n > 0$. However, considering these facts, an elementary calculation using (3.15) shows that $T < 0$, which is a contradiction. Then, there exists a unique possible value for t. Thus, we finally conclude that F is an injective map.

Remark 3.4.4. We give now an alternative geometric argument to study the congruence classes of orbits of the one-dimensional Lie group A. Recall, from Subsection 1.4.2, that $A \cdot o$ is a geodesic. Let $\gamma\colon \mathbb{R} \to \mathbb{C}H^n$ be a unit speed parametrization of $A \cdot o$, and assume that $\lim_{t\to\infty} \gamma(t) = x$, the point at infinity determined by $\mathfrak{a}$ and the fact that α is a positive root. If $A \cdot g(o)$, with $g \in \operatorname{Exp}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha})$, is another orbit of the action of A, it can be parametrized as $\beta(t) = \exp_{\gamma(t)}(r\xi_{\gamma(t)})$, where ξ is an equivariant unit normal vector field along $A \cdot o$ (see [12, Subsection 2.1.8]), and

r is a positive constant (it is, in fact, the distance to $A \cdot o$). Now, we apply the law of cosines [38, Corollary 1.4.4(3)] to the points o, $\gamma(t)$ and $\beta(t)$. Notice that $\lim_{t\to\infty} d(o,\gamma(t)) = \infty$, but $d(\gamma(t),\beta(t))$ is bounded, which comes from the fact that $A \cdot o$ and $A \cdot g(o)$ are equidistant submanifolds. Then, the angle $\sphericalangle_o(\gamma(t),\beta(t))$ subtended from o between $\gamma(t)$ and $\beta(t)$ approaches 0 as $t \to \infty$. According to the definition of the cone topology of $\mathbb{C}H^n \cup \mathbb{C}H^n(\infty)$ (see, for example, [38, Proposition 1.7.6]), we conclude that $\lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = x$. Analogously, one can show that $\lim_{t\to-\infty} \beta(t) = \lim_{t\to-\infty} \gamma(t) = -x$, the other point at infinity of the geodesic $A \cdot o$.

Let now $g_i = \operatorname{Exp}(2V_i + x_i Z) \in \operatorname{Exp}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha})$, where $V_i \in \mathfrak{g}_\alpha$ and $x_i \in \mathbb{R}$, for $i \in \{1,2\}$, and write, according to the relation (3.4), $\operatorname{Ad}(g_i^{-1}) = \mathbb{R}(B + V_i + x_i Z)$. Assume that there exists an isometry ϕ of the full isometry group of $\mathbb{C}H^n$ mapping $A \cdot g_1(o)$ to $A \cdot g_2(o)$. Due to the discussion above, ϕ must map the limit points of one orbit to the limit points of the other orbit. Since these limit points are x and $-x$, we deduce that ϕ leaves the subset $\{-x, x\} \subset \mathbb{C}H^n(\infty)$ invariant. In particular, the unique geodesic of $\mathbb{C}H^n$ whose limit set is $\{-x,x\}$ is $A \cdot o$. Thus, ϕ maps $A \cdot o$ to itself.

Let c denote the complex conjugation of projective coordinates of $\mathbb{C}H^n$ as a quotient of the pseudo-Hermitian flat space $\mathbb{C}^{(1,n)} - \{0\}$, as explained in Subsection 1.5.3. Thus, c is an anti-holomorphic isometry of $\mathbb{C}H^n$ fixing o. Considering the matrix expressions for B and Z given in Subsection 1.4.2, it follows that

$$\operatorname{Ad}(c)(B) = B \qquad \text{and} \qquad \operatorname{Ad}(c)(Z) = -Z.$$

In particular, c maps x to itself.

There exists an element $a \in A$ in such a way that $\phi a(o) = o$. Therefore, $k = \phi a$ maps $A \cdot o$ to itself, $A \cdot g_1(o)$ to $A \cdot g_2(o)$, and fixes $o \in \mathbb{C}H^n$. Define $h = fk$, where f is the identity transformation if $k(x) = x$, or it is the geodesic symmetry at o, s_o, if $k(x) = -x$. Then, $h(x) = x$, which implies that $h \in \widetilde{K}_0 := K_0 \sqcup cK_0$. Since f normalizes A, we have $h(A \cdot g_1(o)) = fk(A \cdot g_1(o)) = f(A \cdot g_2(o)) = A \cdot f(g_2(o))$. It is not difficult to check that there exists a unique $g \in N$ such that $g(o) \in A \cdot f(g_2(o))$, and if $g = \operatorname{Exp}(2V + xZ)$, with $V \in \mathfrak{g}_\alpha$, $x \in \mathbb{R}$, then $|V| = |V_2|$ and $|x| = |x_2|$. Indeed, this is clear when f is the identity transformation. Otherwise, if $f = s_o$ is the geodesic symmetry at o, we have

$$A \cdot s_o(g_2(o)) = A \cdot s_o g_2 s_o^{-1} s_o(o) = A \cdot \sigma(g_2)(o) = A \cdot \operatorname{Exp}(\theta(2V_2 + x_2 Z)),$$

where $\sigma = \mathrm{I}_{s_o} \in \operatorname{Aut}(G)$ and $\theta = \sigma_*$ is the Cartan involution, given by $\theta = -(\cdot)^*$. Then, using the matrix expression of elements of $\mathfrak{a} \oplus \mathfrak{n}$ (see Subsection 1.5.1), and working in projective coordinates of $\mathbb{C}H^n$, one can check that the system

$$e^{2V+xZ} e_1 = e^{i\varphi} e^{tB} e^{-(2V_2 + x_2 Z)^*} e_1,$$

where $e_1 = (1,0,\dots,0)^t$, or equivalently, the system

$$\begin{pmatrix} 1 + ix + 2|V|^2 \\ ix + 2|V|^2 \\ 2V \end{pmatrix} = e^{i\varphi} \begin{pmatrix} \frac{1}{2} e^{-t}(1 + e^{2t} + 2ix_2 + 4|V_2|^2) \\ \frac{1}{2} e^{-t}(-1 + e^{2t} - 2ix_2 - 4|V_2|^2) \\ -2V_2, \end{pmatrix}$$

admits a unique solution for $V \in \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, x, t, $\varphi \in \mathbb{R}$ in terms of V_2 and x_2. Such solution is given by:

$$x = -x_2, \qquad t = \frac{1}{2}\log\left(4x_2^2 + (1+4|V_2|^2)^2\right),$$
$$\varphi = \arctan\left(\frac{-2x_2}{1+4|V_2|^2}\right), \qquad V = -\frac{1+4|V_2|^2 - 2x_2 i}{\sqrt{4x_2^2 + (1+4|V_2|^2)^2}}V_2.$$

Now, as $\widetilde{K}_0$ normalizes AN, we have $h_*|_{T_o\mathbb{C}H^n} \equiv \mathrm{Ad}(h)|_{\mathfrak{a}\oplus\mathfrak{n}}$. Since $h(A \cdot g_1(o)) = A \cdot g(o)$, and $\widetilde{K}_0$ acts trivially on $\mathfrak{a}$ and leaves $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ invariant, we have

$$\begin{aligned}\mathbb{R}(B + V + xZ) &= \mathrm{Ad}(g^{-1})\mathfrak{a} = \mathrm{Ad}(h)\,\mathrm{Ad}(g_1^{-1})\mathfrak{a}\\ &= \mathrm{Ad}(h)(\mathbb{R}(B + V_1 + x_1 Z)) = \mathbb{R}(B + \mathrm{Ad}(h)V_1 \pm x_1 Z).\end{aligned}$$

As $\widetilde{K}_0$ acts transitively on the spheres of $\mathfrak{g}_\alpha$, we get $|V_1| = |V| = |V_2|$ and $|x_1| = |x| = |x_2|$, which finishes the argument for $s = 0$.

Case 2.b

Assume now that $|V_1| = |V_2|$ and $x_1 = x_2$. The connected component of identity of the normalizer of $\mathfrak{r}$ in K_0, which is given by

$$N^0_{K_0}(\mathfrak{r}) \cong SO(\dim(\mathfrak{r})) \times U(n - 1 - \dim(\mathfrak{r})),$$

acts transitively on the spheres of $\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ centered at the origin. Thus, if $|V_1| = |V_2|$ and $x_1 = x_2$, the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent.

The congruence classes do not depend on the sign of x

To finish with this case, we show that the congruence class does not depend on the sign of x, that is, if $g_1 = \mathrm{Exp}(2V + xZ)$ and $g_2 = \mathrm{Exp}(2V - xZ)$, with $V \in \mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}$ and $x \in \mathbb{R}$, then the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent. In order to do so, let c denote the complex conjugation considered above, which is an anti-holomorphic isometry, but fixes $o \in \mathbb{C}H^n$. We can find an element of K_0 mapping the totally real subspace $\mathfrak{r}$ to a subspace of $\mathfrak{g}_\alpha$ whose elements are real vectors. Thus, we can assume that $\mathrm{Ad}(c)|_{\mathfrak{r}} = \mathrm{Id}_{\mathfrak{r}}$.

Again, from the matrix expressions for B and Z one gets $\mathrm{Ad}(c)(B) = B$ and $\mathrm{Ad}(c)(Z) = -Z$. Moreover, assuming without loss of generality that V has only real entries, we have $\mathrm{Ad}(c)(V) = V$. Thus, $\mathrm{Ad}(c)(B + V + xZ) = B + V - xZ$ and $\mathrm{Ad}(c)\mathfrak{r} = \mathfrak{r}$, as we wanted to show.

Case 3

Let H be the connected Lie subgroup of AN with Lie algebra $\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$. It follows from Theorem 3.3.2 that the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent for every g_1, $g_2 \in AN$ since $\mathrm{Ad}(g^{-1})\mathfrak{h} = \mathfrak{h}$ for any $g \in AN$.

Let now H_1 and H_2 be connected subgroups of G of Type 3, and denote by $\mathfrak{h}_i = \mathfrak{c}_i \oplus \mathfrak{r}_i \oplus \mathfrak{g}_{2\alpha}$, with $i \in \{1, 2\}$, their corresponding Lie algebras. Since isometries of $SU(1, n)$ are holomorphic, it follows that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate if, and only if, $\dim(\mathfrak{c}_1) = \dim(\mathfrak{c}_2)$ and $\dim(\mathfrak{r}_1) = \dim(\mathfrak{r}_2)$. Therefore, $H_1 \cdot o$ and $H_2 \cdot o$ are congruent orbits if, and only if, $\dim(\mathfrak{c}_1) = \dim(\mathfrak{c}_2)$ and $\dim(\mathfrak{r}_1) = \dim(\mathfrak{r}_2)$.

In the following result, we compute the mean curvature of these orbits.

Lemma 3.4.5. *Under these conditions, the squared norm of the mean curvature vector $\mathcal{H}$ of any H-orbit is given by the expression*

$$|\mathcal{H}|^2 = \frac{(2 + \dim(\mathfrak{c} \oplus \mathfrak{r}))^2}{4}.$$

Proof. Using [30, Corollary 6.2], it is easy to check that the mean curvature of an orbit of Type 3, $H \cdot g(o)$, with $g \in AN$, is given by

$$\mathcal{H} = \frac{2 + \dim(\mathfrak{c} \oplus \mathfrak{r})}{2} B. \tag{3.16}$$

The result follows taking squared norm. □

Case 4

We now compare Type 4 orbits. Recall from Theorem 3.3.2 that any Type 4 orbit has the form $H \cdot g(o)$, where $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, $\mathfrak{c}$ is a complex subspace of $\mathfrak{g}_\alpha$ and $\mathfrak{r}$ is a totally real one, and $g \in \mathrm{Exp}(J\mathfrak{r})$.

Let H_1 and H_2 be connected subgroups of G with Lie algebras $\mathfrak{h}_i = \mathfrak{a} \oplus \mathfrak{c}_i \oplus \mathfrak{r}_i \oplus \mathfrak{g}_{2\alpha}$, for $i \in \{1, 2\}$. Since isometries of $SU(1, n)$ are holomorphic, the Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate if, and only if, $\dim(\mathfrak{c}_1) = \dim(\mathfrak{c}_2)$ and $\dim(\mathfrak{r}_1) = \dim(\mathfrak{r}_2)$. Then, the orbits $H_1 \cdot o$ and $H_2 \cdot o$ are congruent if, and only if, $\dim(\mathfrak{c}_1) = \dim(\mathfrak{c}_2)$ and $\dim(\mathfrak{r}_1) = \dim(\mathfrak{r}_2)$. Thus, from now on we fix $\mathfrak{c}$ and $\mathfrak{r}$.

The next result is devoted to computing the squared norm of the mean curvature of an orbit $H \cdot g(o)$.

Lemma 3.4.6. *Let $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{c} \oplus \mathfrak{r}$ denotes a CR subspace of $\mathfrak{g}_\alpha$, and let $g^{-1} = \mathrm{Exp}(JR)$, with $R \in \mathfrak{r}$. Then, the squared norm of the mean curvature vector $\mathcal{H}$ of the orbit $H \cdot g(o)$ is given by the expression*

$$|\mathcal{H}|^2 = \frac{|R|^2 \left(3 + \dim(\mathfrak{c} \oplus \mathfrak{r})\right)^2}{4(4 + |R|^2)}.$$

Proof. Let $g^{-1} = \mathrm{Exp}(JR)$, where $R \in \mathfrak{r}$. Since the orbits $H \cdot g(o)$ and $g^{-1}Hg \cdot o$ are congruent, we calculate the mean curvature of the latter. In this case, it follows from relation (3.5) that the tangent space at o can be identified with $\mathrm{Ad}(g^{-1})\mathfrak{h} = \mathbb{R}(B - JR/2) \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$. Then, by virtue of [30, Lemma 6.1], the mean curvature of $g^{-1}Hg \cdot o$ reads:

$$\mathcal{H} = \frac{3 + \dim(\mathfrak{c} \oplus \mathfrak{r})}{2(4 + |R|^2)} \left(|R|^2 B + 2JR\right). \tag{3.17}$$

The formula in the statement follows after calculating the squared norm of $\mathcal{H}$. □

We now analyze if the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$, where $g_1^{-1} = \mathrm{Exp}(JR_1)$, $g_2^{-1} = \mathrm{Exp}(JR_2)$, and R_1, $R_2 \in \mathfrak{r}$, are congruent.

Case 4.a

We firstly show that if $|R_1| = |R_2|$, then $H \cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent orbits. Recall from (3.5) that the tangent space to the orbit $H \cdot g_i(o)$ at $g_i(o)$ can be identified with $\mathrm{Ad}(g_i^{-1})\mathfrak{h} = \mathbb{R}(B + JR_i/2) \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$, for $i \in \{1, 2\}$. The connected component of the identity of the normalizer of $\mathfrak{r}$ in K_0, which is given by

$$N^0_{K_0}(\mathfrak{r}) \simeq SO(\dim(\mathfrak{r})) \times U(n - 1 - \dim(\mathfrak{r})),$$

acts transitively on the spheres of $\mathfrak{r}$. Thus, there exists an element $k \in N^0_{K_0}(\mathfrak{r})$ in such a way that $\mathrm{Ad}(k)(JR_1) = JR_2$ and $\mathrm{Ad}(k)\mathfrak{c} = \mathfrak{c}$. Then,

$$\mathrm{Ad}(k)\,\mathrm{Ad}(g_1^{-1})\mathfrak{h} = \mathrm{Ad}(g_2^{-1})\mathfrak{h},$$

so the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$ are congruent.

Case 4.b

We now show that if $|R_1| \neq |R_2|$, then the corresponding orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$ cannot be congruent. Indeed, denoting by $\mathcal{H}_i$ the mean curvature of the submanifold $H \cdot g_i(o)$, for $i \in \{1, 2\}$, we have that $|\mathcal{H}_1|^2 = |\mathcal{H}_2|^2$ if, and only if,

$$|R_2|^2(4 + |R_1|^2) = |R_1|^2(4 + |R_2|^2),$$

which only occurs when $|R_1| = |R_2|$.

Thus, the orbits $H \cdot g_1(o)$ and $H \cdot g_2(o)$, with $g_1^{-1} = \mathrm{Exp}(JR_1)$, $g_2^{-1} = \mathrm{Exp}(JR_2)$, R_1, $R_2 \in \mathfrak{r}$, are congruent if, and only if, $|R_1| = |R_2|$.

Noncongruence of the different types

We finally study the noncongruence among the four different types of orbits listed in Theorem 3.3.2. In order to do so, we firstly notice that orbits of Type 1 and Type 3 are contained in horospheres while none of the orbits of Type 2 and Type 4 satisfies this property. Indeed, the limit set of the orbits of an action of Type 2 or 4 has, at least, two points at infinity (corresponding to the geodesic $A \cdot o$), whereas the limit set of an orbit contained in a horosphere has, at most, one point at infinity. Considering this fact, it follows that none of the orbits of Type 1 or Type 3 is congruent to any orbit of Type 2 or Type 4.

On the other hand, every Type 2 orbit is a totally real submanifold, while any orbit of Type 4 has nontrivial holomorphic part. Thus, none of the orbits of Type 2 is congruent to any Type 4 orbit.

It only remains to analyze the congruence between orbits of Type 1 and Type 3. In order to do so, we denote by H_i the connected Lie subgroup of G with Lie algebra

$\mathfrak{h}_i$, $i \in \{1,3\}$, where $\mathfrak{h}_1 = (\mathfrak{r}_1 \ominus \mathbb{R}JU) \oplus \mathbb{R}(JU + |U|^2 Z)$, with $U \in J\mathfrak{r}_1$, and $\mathfrak{h}_3 = \mathfrak{c}_3 \oplus \mathfrak{r}_3 \oplus \mathfrak{g}_{2\alpha}$. Here, $\mathfrak{r}_i$ denotes a totally real subspace of $\mathfrak{g}_\alpha$ for each $i \in \{1,3\}$, and $\mathfrak{c}_3 \subset \mathfrak{g}_\alpha$ denotes a complex one.

Suppose that an H_1-orbit is congruent to an H_3-orbit. In such case, since every Type 1 orbit is a totally real submanifold and the maximal holomorphic subspace of a Type 3 orbit is precisely the complex subspace $\mathfrak{c}_3$, then $\dim(\mathfrak{c}_3) = 0$. Moreover, the dimensions of their totally real subspaces must coincide. In particular,

$$\dim(\mathfrak{r}_1 \ominus \mathbb{R}JU) = \dim(\mathfrak{r}_3) =: s.$$

With this notation, the squared norms of the associated mean curvatures read

$$|\mathcal{H}_1|^2 = \frac{4|U|^2 + (1+s+(2+s)|U|^2)^2}{4(1+|U|^2)^2} \qquad \text{and} \qquad |\mathcal{H}_3|^2 = \frac{(2+s)^2}{4}.$$

As we are assuming that the orbits are congruent, $|\mathcal{H}_1|^2 = |\mathcal{H}_3|^2$, or equivalently,

$$3 + 2s(1+|U|^2) = 0,$$

which never happens. Thus, none of the orbits of Type 1 is congruent to any Type 3 orbit. This concludes the proof of Theorem 3.4.1.

Chapter 4
Cohomogeneity one actions on Minkowski spacetimes

The aim of this chapter is to present some structural results for cohomogeneity one actions on Minkowski spacetimes. We also give a classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime $\mathbb{L}^4$. In order to do so, we firstly settle, in Section 4.1, some known results and notation that we will need throughout this chapter. After that, in Section 4.2, we give an alternative proof to the classical classification result of cohomogeneity one actions on the n-dimensional Euclidean space, up to orbit equivalence. Finally, Section 4.3 is devoted to presenting a classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime, up to orbit equivalence. The results of this chapter can be found in [33].

4.1 Motivation and main tools

One of the main purposes of this thesis is to get a better understanding of isometric actions on Lorentzian manifolds. In this context, several results have been achieved. For instance, transitive isometric actions on Lorentzian manifolds have been studied by Adams and Stuck in [2] and [3]. It is common to assume, in the Riemannian setting, that isometric actions are proper, mainly due to the nice properties that these actions satisfy. However, this is not a natural assumption when studying isometric actions on Lorentzian manifolds. Indeed, the natural action of the Lie group $SO^0(1,n)$ on the $(n+1)$-dimensional Minkowski spacetime $\mathbb{L}^{n+1}$ is not proper; we refer to Section 1.6 for more information on the Minkowski spacetime, its isometry group and the notation we use in this chapter. Ahmadi and Kashani have investigated proper cohomogeneity one actions on Minkowski spacetimes in [4]. A study of not necessarily proper cohomogeneity one actions on Minkowski spacetimes of dimensions 2 and 3 has been developed by Berndt, Díaz-Ramos and Vanaei in [14]. More specifically, the next theorem deals with the classification result of such actions in dimension 2.

Theorem 4.1.1. [14, Theorem 5.1] *Let G be a connected Lie subgroup of $I^0(\mathbb{L}^2) = SO^0(1,1) \times_\Phi \mathbb{L}^2$ acting on $\mathbb{L}^2$ with cohomogeneity one. Then, the action of G is orbit equivalent to one of the following:*

(i) *The action of a line ℓ by translations. There exist three equivalence classes, depending on whether ℓ is a spacelike, timelike or lightlike line;*

(ii) *The action of $SO^0(1,1)$.*

The classification of cohomogeneity one actions on the three-dimensional Minkowski spacetime is settled in the next result.

Theorem 4.1.2. [14, Theorem 6.1] *Let G be a connected Lie subgroup of $I^0(\mathbb{L}^3) = SO^0(1,2) \times_\Phi \mathbb{L}^3$ acting on $\mathbb{L}^3$ with cohomogeneity one. Consider the Iwasawa decomposition $SO^0(1,2) = KAN$, and let $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie subalgebras of K, A and N, respectively. Then, the action of G is orbit equivalent to one of the following:*

(i) *The action of a plane $\mathfrak{v}$ by translations. There exist three equivalence classes, depending on whether the metric on $\mathfrak{v}$ is Riemannian, Lorentzian or degenerate;*

(ii) *The action of $SO(2) \times \ell$, where ℓ denotes a timelike line in $\mathbb{L}^3$;*

(iii) *The action of $SO^0(1,1) \times \ell$, where ℓ denotes a spacelike line in $\mathbb{L}^3$;*

(iv) *The action of $SO^0(1,1) \times \ell$, where ℓ is a lightlike line in $\mathbb{L}^3$;*

(v) *The action of $N \times \ell$, where ℓ is a lightlike line in $\mathbb{L}^3$;*

(vi) *The action of the Lie group whose Lie algebra is $\mathbb{R}(B + (0,0,y)) \oplus \mathbb{R}(1,1,0)$, where $y > 0$ and B is a generator of $\mathfrak{a}$;*

(vii) *The action of the Lie group whose Lie algebra is $\mathbb{R}(U + (y,0,0)) \oplus \mathbb{R}(1,1,0)$, where U is a generator of $\mathfrak{n}$;*

(viii) *The action of $SO^0(1,2)$;*

(ix) *The action of the solvable part of the Iwasawa decomposition of $SO^0(1,2)$, AN.*

In this chapter we continue the study of cohomogeneity one actions on Minkowski spacetimes. We develop several structural results, and then provide explicit calculations to get a classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime $\mathbb{L}^4$. The four-dimensional Minkowski spacetime is the mathematical model for Special Relativity; thus, it is interesting not only from the mathematical viewpoint, but also from the point of view of physics. In the rest of this section we present some known results that we are going to need throughout this chapter.

First of all, we present a brief proposition, due to M. Alexandrino, which states that, in order to determine whether two isometric actions are orbit equivalent, it is enough to check that the tangent spaces to the orbits are the same. This is what we settle in the following result.

Proposition 4.1.3. *Let G_1 and G_2 act isometrically on a semi-Riemannian manifold M. Then, the actions of G_1 and G_2 are orbit equivalent if, and only if, there exists an isometry $\phi\colon M \to M$ in such a way that $\phi_{*p}(T_p(G_1 \cdot p)) = T_{\phi(p)}(G_2 \cdot \phi(p))$, for all $p \in M$.*

Remark 4.1.4. Let G act isometrically on the semi-Riemannian manifold M, and let $\mathfrak{g}$ be its Lie algebra. As a matter of notation, if $X \in \mathfrak{g}$, we will denote by X^* the vector field on M defined as follows:

$$X_p^* = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Exp}(tX)(p),$$

where $p \in M$, and Exp is the Lie exponential map. Since the flow of X^* is by isometries, X^* turns out to be a Killing vector field.

Proof. Let G_1 and G_2 act isometrically on a semi-Riemannian manifold M and denote by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ their corresponding Lie algebras.

If G_1 and G_2 are orbit equivalent, the relation between the corresponding tangent spaces follows after differentiation.

Conversely, after conjugation, one can assume $T_p(G_1 \cdot p) = T_p(G_2 \cdot p)$ for all $p \in M$. We have to show that $G_1 \cdot p = G_2 \cdot p$ for all $p \in M$. In order to do so, we will use some concepts of singular Riemannian foliations; we refer to [6] for further information.

Let $p \in M$ and consider $X_1, \ldots, X_k \in \mathfrak{g}_1$ such that $\{X_1^*(p), \ldots, X_k^*(p)\}$ is a basis of $T_p(G_1 \cdot p)$. We consider the foliation $\mathcal{F}$ induced by $X_1^*, \ldots, X_k^*$ in a neighborhood of p. By making this neighborhood small enough, we may assume that this foliation is regular. For any $q \in G_2 \cdot p$ in this neighborhood, we have $T_q\mathcal{F}_q \subset T_q(G_1 \cdot q) = T_q(G_2 \cdot q)$, where $\mathcal{F}_q$ is the leaf of $\mathcal{F}$ through q. Moreover, by construction and hypothesis, we have the following chain of equalities:

$$\dim(T_q\mathcal{F}_q) = \dim(T_p\mathcal{F}_p) = \dim(G_1 \cdot p) = \dim(G_2 \cdot p) = \dim(G_2 \cdot q).$$

Hence, $T_q\mathcal{F}_q = T_q(G_2 \cdot q)$. This implies that $G_2 \cdot p$ is tangent to a leaf of the regular foliation $\mathcal{F}$. Then, it is known that $G_2 \cdot p$ and $\mathcal{F}_p$ coincide locally. Since $\mathcal{F}_p$ and $G_1 \cdot p$ coincide locally by construction, the result follows. $\square$

We now introduce three well-known results which turn out to be useful when dealing with compact Lie groups and Lie algebras. A Lie algebra $\mathfrak{g}$ is said to be compact if there exists a compact Lie group G whose Lie algebra is $\mathfrak{g}$. For example, $\mathfrak{g} = \mathbb{R}$ is a compact Lie algebra since the Lie algebra of the compact Lie group $G = S^1$ is, precisely, $\mathbb{R}$.

Theorem 4.1.5 (Cartan's fixed point theorem)**.** [38, Theorem 1.4.6] *Let M be a complete, simply connected manifold of nonpositive curvature and let G be a subgroup of the isometry group of M. If there exists a point $p \in M$ in such a way that the orbit $G \cdot p$ is bounded, then G fixes a point $q \in M$.*

In particular, notice that this result applies when G is a compact subgroup of the isometry group $I(M)$.

Proposition 4.1.6. [18, Chapter IX, Section 1, Proposition 2] *A Lie subalgebra of a compact Lie algebra is compact.*

Theorem 4.1.7. [18, Chapter IX, Section 1, Proposition 5] *Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is compact. If G is connected, then it has a maximal compact Lie subgroup K, which is connected. There exists a closed abelian subgroup T of G in such a way that $G = T \times K$.*

This result can be reformulated at the level of Lie algebras. Let $\mathfrak{t}$ and $\mathfrak{k}$ denote the Lie algebras of T and K, respectively. Then, $\mathfrak{t}$ turns out to be an abelian Lie algebra, $\mathfrak{k}$ is compact and semisimple and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$.

4.2 A proof in the Euclidean case

In this section we present an alternative proof of the classification of cohomogeneity one actions on Euclidean spaces $\mathbb{R}^n$ using Lie group theory. This is done in order to illustrate an easier case than the one we are dealing with. See, for example, [22, pp. 96–97], [24] or [68] for other proofs.

It is well-known that the connected component of the identity of the isometry group of the Euclidean space $\mathbb{R}^n$ is the semi-direct product $I^0(\mathbb{R}^n) = SO^0(n) \times_\Phi \mathbb{R}^n$, where

$$\Phi\colon SO(n) \to \mathrm{Aut}(\mathbb{R}^n) \text{ is given by } \Phi(a)(v) = av.$$

The natural operation of this Lie group is given by $(a, v)(b, w) = (ab, v+aw)$, for (a, v), $(b, w) \in I^0(\mathbb{R}^n)$, and any Lie subgroup of $I^0(\mathbb{R}^n)$ acts on $\mathbb{R}^n$ by $(a, v)x = ax+v$, where now $x \in \mathbb{R}^n$. The Lie algebra of $I^0(\mathbb{R}^n)$ is $\mathfrak{i}(\mathbb{R}^n) = \mathfrak{so}(n) \oplus_\phi \mathbb{R}^n$, where

$$\phi\colon \mathfrak{so}(n) \to \mathrm{Der}(\mathbb{R}^n) \text{ is given by } \phi(X)(v) = Xv.$$

The corresponding Lie bracket is given by $[X+v, Y+w] = (XY-YX)+(Xw-Yv)$, for $X + v$, $Y + w \in \mathfrak{i}(\mathbb{R}^n)$. With this notation, we prove the following result.

Theorem 4.2.1. *Let G be a connected Lie subgroup of $I^0(\mathbb{R}^n)$ acting isometrically with cohomogeneity one on the Euclidean space $\mathbb{R}^n$. Then, the action of G is orbit equivalent to one of:*

1. *the action of $SO(n)$, whose orbits are concentric spheres around a point;*
2. *the action of $SO(n-k) \times \mathbb{R}^k$, for some $k \in \{1, \dots, n-2\}$, whose orbits are coaxial cylinders around an affine $\mathbb{R}^k$;*
3. *the action of $\mathbb{R}^{n-1}$, whose orbits are parallel hyperplanes.*

Proof. In order to prove Theorem 4.2.1, let $\mathfrak{g}$ denote the Lie algebra of G and define $\mathfrak{v} = \mathfrak{g} \cap \mathbb{R}^n$ to be its pure translational part, which is an ideal of $\mathfrak{g}$.

The same proof as in Lemma 4.3.2 in the next section can be used to show that the action of G is orbit equivalent to the action of a subgroup $H \times \mathfrak{v}$, where $H \subset SO(\mathfrak{v}^\perp) \times_\Phi \mathfrak{v}^\perp$, and $\mathfrak{v}^\perp$ denotes the orthogonal complement of $\mathfrak{v}$ in $\mathbb{R}^n$. Thus, it is enough to study the action of H on $\mathfrak{v}^\perp$. In view of this assertion, we can assume that $\mathfrak{g} \cap \mathbb{R}^n = 0$.

Consider the projection $\pi\colon \mathfrak{g} \to \mathfrak{so}(n)$, which is a homomorphism of Lie algebras. Since $\ker(\pi) = \mathfrak{g} \cap \mathbb{R}^n = 0$, $\mathfrak{g}$ and $\pi(\mathfrak{g})$ are isomorphic Lie algebras, and by virtue of Proposition 4.1.6, $\mathfrak{g}$ is compact.

Using Theorem 4.1.7, one can write $\mathfrak{g}$ as the direct sum $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}$, where $\mathfrak{t}$ is an abelian Lie algebra and $\mathfrak{k}$ is compact and semisimple. At the level of Lie groups, we have the corresponding decomposition $G = T \times K$, where T is a closed abelian Lie subgroup, K is a compact semisimple Lie subgroup of G, and both T and K are connected.

Taking into account that K is compact, Theorem 4.1.5 (Cartan's fixed point theorem) ensures that there exists an element $p \in \mathbb{R}^n$ such that $K \cdot p = p$, and thus, $\mathfrak{k} \cdot p = 0$.

If $\mathfrak{t} = 0$, then $G = K \subset SO(n)$ since K fixes a point of $\mathbb{R}^n$, so the orbits of G would be contained in spheres. Since G acts on $\mathbb{R}^n$ with cohomogeneity one, we conclude that G is orbit equivalent to the action of $SO(n)$. Therefore, we henceforth assume $\mathfrak{t} \neq 0$.

Let $Y + u \in \mathfrak{k}$. Conjugating by $\operatorname{Ad}(I, -p)$, we get

$$\operatorname{Ad}(I,-p)(Y+u) = Y + Yp + u = Y + (Y+u)\cdot p = Y,$$

from where we can assume $\mathfrak{k} \subset \mathfrak{so}(n)$.

Since the elements of $\mathfrak{so}(n)$ are skew-symmetric matrices, they have pure imaginary eigenvalues, so we can define the following spaces:

$$\mathfrak{t}_\lambda = \{v \in \mathbb{R}^n : X^2 v = -\lambda(X)^2 v, \text{ for all } X \in \pi(\mathfrak{t})\}, \qquad (\lambda \in \pi(\mathfrak{t})^*).$$

We denote $\Delta = \{\lambda \in \pi(\mathfrak{t})^* - \{0\} : \mathfrak{t}_\lambda \neq 0\}$. With this notation,

$$\mathfrak{t}_0 = \bigcap_{X \in \pi(\mathfrak{t})} \ker(X), \qquad \text{and} \qquad \mathfrak{t}_0^\perp = \bigoplus_{\lambda \in \Delta} \mathfrak{t}_\lambda.$$

Let $Y \in \mathfrak{k}$, $v \in \mathfrak{t}_\lambda$ and $X \in \pi(\mathfrak{t})$. Since $\mathfrak{t}$ and $\mathfrak{k}$ commute, $X^2 Y v = Y X^2 v = -\lambda(X)^2 Y v$, and thus $Yv \in \mathfrak{t}_\lambda$. Then, we deduce that, for each $\lambda \in \Delta \cup \{0\}$, $\mathfrak{k} \cdot \mathfrak{t}_\lambda \subset \mathfrak{t}_\lambda$.

Now, since $\mathfrak{t}$ is an abelian Lie algebra, a result by Di Scala [25, Proof of Theorem 2.1] yields

$$\mathfrak{t} = \operatorname{span}\{X_1 + d_1 - X_1 c, \dots, X_k + d_k - X_k c\},$$

for some $X_i \in \mathfrak{so}(\mathfrak{t}_0^\perp)$, $d_i \in \mathfrak{t}_0$ and $c \in \mathfrak{t}_0^\perp$.

Let $X + d - Xc \in \mathfrak{t}$ and $Y \in \mathfrak{k}$. Since $\mathfrak{t}$ and $\mathfrak{k}$ commute, $0 = [X + d - Xc, Y] = [X, Y] - Yd + YXc$, which implies $[X, Y] = 0$ and $YXc - Yd = 0$. As $Yd \in \mathfrak{t}_0$ and $YXc \in \mathfrak{t}_0^\perp$, we get $YXc = Yd = 0$. In particular, since $X + d - Xc \in \mathfrak{t}$ is arbitrary, $XYc = YXc = 0$ implies $Yc \in \mathfrak{t}_0$. As $c \in \mathfrak{t}_0^\perp$ and $\mathfrak{k} \cdot \mathfrak{t}_0 \subset \mathfrak{t}_0$, we get $Yc = 0$.

Now, conjugating by $\operatorname{Ad}(I, -c)$, we obtain

$$\begin{aligned}
&\operatorname{Ad}(I,-c)(X + d - Xc) = X + Xc + d - Xc = X + d,\\
&\operatorname{Ad}(I,-c)(Y) = Y + Yc = Y,
\end{aligned}$$

whenever $X + d - Xc \in \mathfrak{t}$ and $Y \in \mathfrak{k}$. Thus, one deduces that $\mathfrak{g}$ is conjugate to $\operatorname{span}\{X_1 + d_1, \dots, X_k + d_k\} \oplus \mathfrak{k}$, with $\mathfrak{k} \subset \mathfrak{so}(n)$.

Given an element $p \in \mathbb{R}^n$, we set $p = p_0 + \sum_{\lambda \in \Delta} p_\lambda$, where $p_0 \in \mathfrak{t}_0$ and $p_\lambda \in \mathfrak{t}_\lambda$ for each $\lambda \in \Delta$. Taking $Y \in \mathfrak{k}$ and $X + d \in \mathfrak{t}$, we can write $Y = \sum_{\lambda \in \Delta \cup \{0\}} Y_\lambda$ and $X = \sum_{\lambda \in \Delta} X_\lambda$, with Y_λ, $X_\lambda \in \mathfrak{so}(\mathfrak{t}_\lambda)$. Then,

$$Yp = \sum_{\lambda \in \Delta \cup \{0\}} Y_\lambda p_\lambda \qquad \text{and} \qquad (X+d) \cdot p = \sum_{\lambda \in \Delta} X_\lambda p_\lambda + d.$$

Since X_λ and Y_λ are skew-symmetric and $d \in \mathfrak{t}_0$, it follows that $\mathfrak{g} \cdot p$ is orthogonal to p_λ for each $\lambda \in \Delta$. Thus, the cohomogeneity of the action of G is at least the cardinal of Δ. Since by hypothesis the cohomogeneity of G is one, it follows that Δ has just one element, from where $\mathbb{R}^n = \mathfrak{t}_0 \oplus \mathfrak{t}_\lambda$. Since the elements of $\pi(\mathfrak{t})$ have only one nonzero eigenvalue and they have a simultaneous diagonalization, it follows that $\dim(\pi(\mathfrak{t})) = 1$, and thus, $\dim(\mathfrak{t}) = 1$. We can write $\mathfrak{t} = \mathbb{R}(X + d)$ for some $X \in \mathfrak{so}(\mathfrak{t}_\lambda)$ and $d \in \mathfrak{t}_0 = \ker(X)$. We can assume $d \neq 0$; otherwise G would be compact and its orbits would be contained in a sphere, and thus, for dimension reasons, the action of G would be orbit equivalent to the action of $SO(n)$.

For any $Y \in \mathfrak{k}$, we have $0 = [X+d, Y] = -Yd$. Thus, since $\mathfrak{k} \cdot \mathfrak{t}_\lambda \subset \mathfrak{t}_\lambda$ and $\mathfrak{k} \cdot \mathfrak{t}_0 \subset \mathfrak{t}_0$, we have $\mathfrak{k} \subset \mathfrak{so}(\mathfrak{t}_\lambda) \oplus \mathfrak{so}(\mathfrak{t}_0 \ominus \mathbb{R}d)$. If we take $p \in \mathbb{R}^n$, we can write $p = xd + p_0 + p_\lambda$, where $x \in \mathbb{R}$, $p_0 \in \mathfrak{t}_0 \ominus \mathbb{R}d$ and $p_\lambda \in \mathfrak{t}_\lambda$. If $Y \in \mathfrak{k}$, we can also write $Y = Y_0 + Y_\lambda$, with $Y_0 \in \mathfrak{so}(\mathfrak{t}_0 \ominus \mathbb{R}d)$ and $Y_\lambda \in \mathfrak{so}(\mathfrak{t}_\lambda)$. Thus, $(X+d) \cdot p = Xp_\lambda + d$ and $Y \cdot p = Y_0 p_0 + Y_\lambda p_\lambda$. By skew-symmetry, p_λ and p_0 are orthogonal to $\mathfrak{g} \cdot p$. If p_λ or p_0 vanishes, then the codimension of $\mathfrak{g} \cdot p$ is greater than two. In any case, the codimension of $\mathfrak{g} \cdot p$ is greater or equal than two. Since G acts with cohomogeneity one, we must have $\mathfrak{t}_0 \ominus \mathbb{R}d = 0$, which implies $\mathfrak{t}_0 = \mathbb{R}d$, $\mathfrak{so}(\mathfrak{t}_\lambda) = \mathfrak{so}(\mathbb{R}^n \ominus \mathbb{R}d) \cong \mathfrak{so}(n-1)$.

We have therefore proved that, up to conjugation, $\mathfrak{g} \subset \mathfrak{so}(\mathbb{R}^n \ominus \mathbb{R}d) \oplus \mathbb{R}d$, which implies that the orbits of G are contained in coaxial cylinders. Since the action of G is of cohomogeneity one, they must be cylinders. This concludes the proof. □

Notice that we can summarize Theorem 4.2.1 by saying that any cohomogeneity one action on $\mathbb{R}^n$ is orbit equivalent to the action of $SO(n-k) \times \mathbb{R}^k$, for some $k \in \{0, \dots, n-1\}$. Moreover, the only cohomogeneity one action on $\mathbb{R}^n$ that is neither ‘reducible’ (not a product) nor ‘full’ (orbits are not contained in parallel lower dimensional Euclidean spaces) is the action of $SO(n)$.

We conclude this section by exhibiting an example that shows that the singular homogeneous foliation given by an axis and its coaxial cylinders can be induced by an action of a group without pure translational part.

Example 4.2.2. We introduce an example of a Lie subgroup of $I^0(\mathbb{R}^5)$ without pure translational part (that is, not of the form $K \times \mathfrak{v}$) acting on $\mathbb{R}^5$ with cohomogeneity one, and whose orbits are cylinders. We consider

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and define the following matrices by blocks:

$$A = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} J & 0 & 0 \\ 0 & -J & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & J & 0 \\ J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the Lie algebra given by

$$\mathfrak{g} = \operatorname{span}_{\mathbb{R}}\{A + (0,0,0,0,1), B, C, D\}.$$

The connected Lie subgroup of $I^0(\mathbb{R}^5)$ whose Lie algebra is $\mathfrak{g}$ acts with cohomogeneity one on $\mathbb{R}^5$. Its orbits are coaxial cylinders around the axis determined by $(0,0,0,0,1)$. Thus, the action of G is orbit equivalent to the action of $SO(4) \times r$, where $r = \mathbb{R}(0,0,0,0,1)$.

4.3 Cohomogeneity one actions on $\mathbb{L}^{n+1}$

The aim of this section is to present some partial results related to the study of cohomogeneity one actions on Minkowski spacetimes $\mathbb{L}^{n+1}$. Moreover, we give a classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime, up to orbit equivalence. This result has recently been obtained independently by Ahmadi, Safari and Hassani in [5] by a different method.

Theorem 4.3.1. *Let G be a connected Lie subgroup of $I^0(\mathbb{L}^4) = SO^0(1,3) \times_\Phi \mathbb{L}^4$ with Lie algebra $\mathfrak{g}$ acting with cohomogeneity one on $\mathbb{L}^4$. Consider the Iwasawa decomposition of the rotational part of $I^0(\mathbb{L}^4)$, $SO^0(1,3) = KAN$, and also the corresponding decomposition at the level of Lie algebras, $\mathfrak{so}(1,3) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then, the action of G is orbit equivalent to one of the following:*

1. *Actions with nondegenerate translational part:*
   - (a) *The action of $SO(k) \times \mathbb{L}^{4-k}$, with $k \in \{1,2,3\}$;*
   - (b) *The action of $SO^0(1,k) \times \mathbb{R}^{3-k}$, with $k \in \{0,1,2,3\}$;*
   - (c) *The action of $A\widetilde{N} \times \mathbb{R}$, where $\mathbb{R}$ is a spacelike line in $\mathbb{L}^4$, and $A\widetilde{N}$ is the solvable part of the Iwasawa decomposition of $SO^0(1,2)$;*
   - (d) *The action of QAN, where $Q \in \{\{I\}, K_0\}$;*
   - (e) *The action of the Lie group whose Lie algebra is $\mathbb{R}(E + \mathbf{e}) \oplus \mathfrak{n}$, where*
   $$\mathbb{R}(E + \mathbf{e}) \oplus \mathfrak{n} = \mathbb{R}\left(\begin{pmatrix} 0 & 0 & 0^t \\ 0 & 0 & 0^t \\ 0 & 0 & E \end{pmatrix} + \mathbf{e}\right) \oplus \mathfrak{n};$$
   - (f) *The action of the Lie group whose Lie algebra is $\mathbb{R}(E + 1) \oplus \mathfrak{n}$, where*
   $$\mathbb{R}(E + 1) \oplus \mathfrak{n} = \mathbb{R}\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & E \end{pmatrix} \oplus \mathfrak{n};$$

2. *Actions with degenerate translational part:*
   (a) *The action of* $\mathbb{W}^3$*;*
   (b) *The action of* $\operatorname{Exp}(\mathfrak{v}) \times \mathbb{W}^2$*, where* $\mathfrak{v}$ *is the subspace of* $\mathfrak{n}$ *generated by the element* $(0,1) \in \mathfrak{n} \cong \mathbb{R}^2$*;*
   (c) *The action of the Lie group whose Lie algebra is* $\mathfrak{g} = \mathbb{R}(v+(1,0,0,0)) \oplus \mathbb{W}^2$*, where* $v = (0,1) \in \mathfrak{n} \cong \mathbb{R}^2$*, and*
$$\mathbb{R}\left(v + (1,0,0,0)\right) \oplus \mathbb{W}^2 = \mathbb{R}\left(\begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \oplus \mathbb{W}^2;$$
   (d) *The action of the Lie group whose Lie algebra is* $\mathfrak{g} = \mathbb{R}(1+(0,0,0,\lambda)) \oplus \mathbb{W}^2$*, where* $\lambda > 0$*, and*
$$\mathbb{R}\left(1 + (0,0,0,\lambda)\right) \oplus \mathbb{W}^2 = \mathbb{R}\left(\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda \end{pmatrix}\right) \oplus \mathbb{W}^2;$$
   (e) *The action of* $QN \times \mathbb{W}^1$*, where* $Q \in \{\{I\}, K_0\}$*;*
   (f) *The action of* $K_0 A \times \mathbb{W}^1$*;*
   (g) *The action of* $A \operatorname{Exp}(\mathfrak{v}) \times \mathbb{W}^1$*, where* $\mathfrak{v}$ *is a one-dimensional subspace of* $\mathfrak{n}$*;*
   (h) *The action of the Lie group whose Lie algebra is* $\mathbb{R}(1+(0,0,b)) \oplus \mathfrak{v} \oplus_\phi \mathbb{W}^1$*, where* $\mathfrak{v}$ *is a one-dimensional subspace of* $\mathfrak{n}$*,* $b \in \mathbb{R}^2$*, and*
$$\mathbb{R}\left(1 + (0,0,b)\right) \oplus \mathfrak{v} \oplus_\phi \mathbb{W}^1 = \mathbb{R}\left(\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}\right) \oplus \mathfrak{v} \oplus_\phi \mathbb{W}^1;$$
   (i) *The action of the Lie group whose Lie algebra is given by* $\big(\mathbb{R}(u+(0,0,x)) \oplus \mathbb{R}(v+(0,0,y))\big) \oplus_\phi \mathbb{W}^1$*, where* $\{u,v\}$ *is an orthonormal basis of* $\mathfrak{n}$*,* x*,* $y \in \mathbb{R}^2$*, and*
$$\begin{aligned}&\big(\mathbb{R}(u + (0,0,x)) \oplus \mathbb{R}(v + (0,0,y))\big) \oplus_\phi \mathbb{W}^1 \\ &= \mathbb{R}\left(\begin{pmatrix} 0 & 0 & u^t \\ 0 & 0 & u^t \\ u & -u & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}\right) \oplus_\phi \mathbb{W}^1.\end{aligned}$$

In the statement of the previous theorem, as well as in its proof, we use the notation settled in Subsection 1.4.1 and Section 1.6. Consider the Iwasawa decomposition of the semisimple Lie group $SO^0(1,n)$, KAN, and also the corresponding decomposition at the level of Lie algebras, $\mathfrak{so}(1,n) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{k} \cong \mathfrak{so}(n)$, $\mathfrak{a} \cong \mathbb{R}$ and

$\mathfrak{n} = \mathfrak{g}_\alpha \cong \mathbb{R}^{n-1}$. In particular, if $n = 3$, the Iwasawa decomposition of $\mathfrak{so}(1,3)$ reads $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{k} \cong \mathfrak{so}(3)$, $\mathfrak{a} \cong \mathbb{R}$ and $\mathfrak{n} = \mathfrak{g}_\alpha \cong \mathbb{R}^2$. In this case, $\mathfrak{k}_0$ is isomorphic to the one-dimensional Lie algebra $\mathfrak{so}(2)$. A typical generator of $\mathfrak{so}(2)$ is

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that any maximal proper subalgebra of $\mathfrak{so}(1,n)$ is either a reductive subalgebra, $\mathfrak{so}(1,k) \oplus \mathfrak{so}(n-k)$, with $k \in \{0, 1, \ldots, n-1\}$, or a parabolic subalgebra, $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Let now $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbb{L}^{n+1}$, where $\langle \mathbf{e}_0, \mathbf{e}_0 \rangle = -1$ and $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$ for $i \in \{1, \ldots, n\}$. In general, for $1 \leq k \leq n$, $\mathbb{L}^k$ will denote the subspace of $\mathbb{L}^{n+1}$ defined by $\mathbb{L}^k = \operatorname{span}\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{k-1}\}$. With this notation,

$$\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1$$

is a lightlike vector of $\mathbb{L}^{n+1}$, and we will denote by $\mathbb{W}^1$ the lightlike line that it determines. We also define the degenerate subspace $\mathbb{W}^k = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_k$, for $k \in \{2, \ldots, n\}$. In the particular case $n = 3$, we will consider

$$\mathbb{W}^2 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2, \qquad \mathbb{W}^3 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3.$$

Now, we present some structural results related to the study of cohomogeneity one actions on $\mathbb{L}^{n+1}$, which will allow us to prove Theorem 4.3.1. Let then G be a connected Lie subgroup of $I^0(\mathbb{L}^{n+1}) = SO^0(1,n) \times_\Phi \mathbb{L}^{n+1}$ acting on $\mathbb{L}^{n+1}$ with cohomogeneity one. We emphasize the fact that the action of G is not necessarily proper. Let $\mathfrak{g}$ denote the Lie algebra of G, which is a Lie subalgebra of $\mathfrak{i}(\mathbb{L}^{n+1}) = \mathfrak{so}(1,n) \oplus_\phi \mathbb{L}^{n+1}$, and define $\mathfrak{v} = \mathfrak{g} \cap \mathbb{L}^{n+1}$ the pure translational part of $\mathfrak{g}$. Note that $\mathfrak{v}$ is an ideal of $\mathfrak{g}$, which can also be identified with a vector subspace of $\mathbb{L}^{n+1}$.

Consider the projection onto the first factor $\pi\colon \mathfrak{so}(1,n) \oplus_\phi \mathbb{L}^{n+1} \to \mathfrak{so}(1,n)$, which is a homomorphism of Lie algebras whose kernel is, precisely, $\ker(\pi|_\mathfrak{g}) = \mathfrak{v}$. We proceed separately, depending on whether the translational part of $\mathfrak{g}$ is degenerate or not.

4.3.1 Nondegenerate translational part

We assume firstly that $\mathfrak{v}$ is a nondegenerate subspace of $\mathbb{L}^{n+1}$, that is, $\mathfrak{v}$ has either a Riemannian or Lorentzian metric, and let $\mathfrak{v}^\perp$ denote its orthogonal complement in $\mathbb{L}^{n+1}$. Since $\mathfrak{v}$ is nondegenerate, it follows that $\mathbb{L}^{n+1} = \mathfrak{v} \oplus \mathfrak{v}^\perp$.

Let $X + \mathbf{u} \in \mathfrak{g} \subset \mathfrak{i}(\mathbb{L}^{n+1})$ and $\mathbf{v} \in \mathfrak{v}$. Since $\mathfrak{v}$ is an ideal of $\mathfrak{g}$, $X\mathbf{v} = [X + \mathbf{u}, \mathbf{v}] \in \mathfrak{v}$. Thus, $X\mathfrak{v} \subset \mathfrak{v}$ and, as $\mathfrak{v}$ is a nondegenerate subspace, $X\mathfrak{v}^\perp \subset \mathfrak{v}^\perp$. This implies $\pi(\mathfrak{g}) \subset \mathfrak{so}(\mathfrak{v}) \oplus \mathfrak{so}(\mathfrak{v}^\perp)$, and thus, $\mathfrak{g} \subset (\mathfrak{so}(\mathfrak{v}^\perp) \oplus_\phi \mathfrak{v}^\perp) \oplus (\mathfrak{so}(\mathfrak{v}) \oplus_\phi \mathfrak{v})$. Notice that $\mathfrak{v}^\perp$ is Lorentzian when $\mathfrak{v}$ is Riemannian and viceversa, so exactly one of $\mathfrak{so}(\mathfrak{v})$ or $\mathfrak{so}(\mathfrak{v}^\perp)$ is isomorphic to an $\mathfrak{so}(k)$ and the other to an $\mathfrak{so}(1,l)$, for some k and l.

Consider now the projection $\sigma\colon (\mathfrak{so}(\mathfrak{v}^\perp) \oplus_\phi \mathfrak{v}^\perp) \oplus (\mathfrak{so}(\mathfrak{v}) \oplus_\phi \mathfrak{v}) \to \mathfrak{so}(\mathfrak{v}^\perp) \oplus_\phi \mathfrak{v}^\perp$, which is a homomorphism of Lie algebras. Thus, $\mathfrak{h} = \sigma(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}^\perp) \oplus_\phi \mathfrak{v}^\perp$. Let us denote by H the connected Lie subgroup of $I^0(\mathbb{L}^{n+1})$ whose Lie algebra is $\mathfrak{h}$.

Lemma 4.3.2. *Under these conditions, the actions of $H \times \mathfrak{v}$ and G are orbit equivalent.*

Proof. By virtue of Proposition 4.1.3, it is actually enough to show $T_{(p,q)}(G\cdot(p,q)) = T_{(p,q)}((H\times\mathfrak{v})\cdot(p,q))$, for each $(p,q)\in\mathfrak{v}^\perp\times\mathfrak{v}=\mathbb{L}^{n+1}$.

Let $X+\mathbf{w}\in\mathfrak{h}\subset\mathfrak{so}(\mathfrak{v}^\perp)\oplus_\phi\mathfrak{v}^\perp$ and $\mathbf{v}\in\mathfrak{v}$. By definition, there exists $Y+\mathbf{u}\in\mathfrak{g}$ in such a way that $\sigma(Y+\mathbf{u})=X+\mathbf{w}$. We write $Y=X+Y^\top$, with $Y^\top\in\mathfrak{so}(\mathfrak{v})$. Since $\mathfrak{v}\subset\mathfrak{g}$, we can assume, without loss of generality, that $\mathbf{u}\in\mathfrak{v}^\perp$, that is, $\mathbf{u}=\mathbf{w}$. Then,

$$(X+\mathbf{w}+\mathbf{v})\cdot(p,q) = Xp+\mathbf{w}+\mathbf{v} = Xp+Y^\top q+\mathbf{u}+\mathbf{v}-Y^\top q = (Y+\mathbf{u}+(\mathbf{v}-Y^\top q))\cdot(p,q),$$

which belongs to $\mathfrak{g}\cdot(p,q)$ because $\mathbf{v}-Y^\top q\in\mathfrak{v}\subset\mathfrak{g}$.

Conversely, let $Y+\mathbf{u}\in\mathfrak{g}\subset(\mathfrak{so}(\mathfrak{v}^\perp)\oplus_\phi\mathfrak{v}^\perp)\oplus(\mathfrak{so}(\mathfrak{v})\oplus_\phi\mathfrak{v})$. We can write $Y = Y^\perp+Y^\top$, with $Y^\perp\in\mathfrak{so}(\mathfrak{v}^\perp)$, $Y^\top\in\mathfrak{so}(\mathfrak{v})$, and $\mathbf{u}=\mathbf{u}^\perp+\mathbf{u}^\top$, with $\mathbf{u}^\perp\in\mathfrak{v}^\perp$ and $\mathbf{u}^\top\in\mathfrak{v}$. Then, by definition, $Y^\perp+\mathbf{u}^\perp\in\mathfrak{h}$, and

$$(Y+\mathbf{u})\cdot(p,q) = Y^\perp p+Y^\top q+\mathbf{u}^\perp+\mathbf{u}^\top = (Y^\perp+\mathbf{u}^\perp)\cdot p+(Y^\top q+\mathbf{u}^\top),$$

which belongs to $(\mathfrak{h}\oplus\mathfrak{v})\cdot(p,q)$, as we wanted to prove. □

In order to prove Theorem 4.3.1, we will mainly focus on the case $n=3$. In view of Lemma 4.3.2, one deduces that the action of G reduces to a cohomogeneity one action on $\mathfrak{v}^\perp$. We proceed separately, depending on whether $\mathfrak{v}$ is Riemannian or Lorentzian.

Lorentzian translational part

Assume firstly that $\mathfrak{v}$ is Lorentzian. Then, $\mathfrak{v}^\perp$ is Riemannian. Note that, in this case, $\dim(\mathfrak{v})\geq 1$. The classification of cohomogeneity one actions on Euclidean spaces follows from Theorem 4.2.1. More specifically, a cohomogeneity one action on a Euclidean space in $\mathbb{R}^n$ is orbit equivalent to the action of $SO(k)\times\mathbb{R}^{n-k}$, for $k\in\{1,\dots,n\}$. Combining this result with Lemma 4.3.2, we get that the action of G on $\mathbb{L}^4$ is orbit equivalent to $SO(k)\times\mathbb{L}^{4-k}$, for some $k\in\{1,2,3\}$. The orbits of this action are coaxial elliptic cylinders for $k\in\{2,3\}$, and parallel Lorentzian hyperplanes if $k=1$. This corresponds to Case 1.(a) of Theorem 4.3.1.

Riemannian translational part

Assume now that $\mathfrak{v}$ is a Riemannian subspace such that $\dim(\mathfrak{v})\geq 1$. Then, $\mathfrak{v}^\perp$ is Lorentzian. In this case, the classification, up to orbit equivalence, reduces to the classifications in $\mathbb{L}^2$ or $\mathbb{L}^3$, that is, to Theorem 4.1.1 and Theorem 4.1.2. In these classifications we only consider the cases with no pure translational part in order to avoid repetitions. Thus, in this case, the action of G is orbit equivalent to one of the following:

(i) the actions of $SO^0(1,2)\times\ell$ and $\widetilde{AN}\times\ell$, where ℓ is a spacelike line and $\widetilde{AN}$ is the solvable part of the Iwasawa decomposition of $SO^0(1,2)$;

(ii) the action of $SO^0(1,1) \times \mathbb{R}^2$;

(iii) the action of $\mathbb{R}^3$ by translations.

This corresponds to Cases 1.(b) and 1.(c) of Theorem 4.3.1.

Finally, it remains to deal with the case $\dim(\mathfrak{v}) = 0$, that is, when there is no pure translational part. In this case, the projection $\pi\colon \mathfrak{so}(1,n) \oplus_\phi \mathbb{L}^{n+1} \to \mathfrak{so}(1,n)$ is an injective map, so $\mathfrak{g}$ is isomorphic to $\pi(\mathfrak{g})$, which is a Lie subalgebra of $\mathfrak{so}(1,n)$. We will use the following notation for elements of $\mathfrak{so}(1,n)$:

$$(X,v) + (x_0,x) \equiv \begin{pmatrix} 0 & v^t \\ v & X \end{pmatrix} + \begin{pmatrix} x_0 \\ x \end{pmatrix},$$

where $X \in \mathfrak{so}(n)$, x, $v \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}$. We firstly focus on the case $\pi(\mathfrak{g}) = \mathfrak{so}(1,n)$, which we analyze in the following result.

Lemma 4.3.3. *If $\pi(\mathfrak{g}) = \mathfrak{so}(1,n)$, then the action of G is orbit equivalent to the action of $SO(1,n)$ on $\mathbb{L}^{n+1}$.*

Proof. Let $\mathfrak{h} = \pi^{-1}(\mathfrak{so}(n))$ and denote by H the connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. Then, $\mathfrak{h}$ is a compact Lie algebra, and it follows from Theorem 4.1.7 that $\mathfrak{h}$ can be written as a direct sum of Lie algebras $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{k}$, where $\mathfrak{t}$ is abelian and $\mathfrak{k}$ is compact and semisimple. We also have, at the level of Lie groups, $H = T \times K$, where T is a closed abelian Lie subgroup and K is compact and semisimple. In this case, $\pi\colon \mathfrak{g} \to \mathfrak{so}(1,n)$ is an isomorphism of Lie algebras. In particular, $\mathfrak{h}$ is semisimple, and hence, $\mathfrak{t} = 0$ and $\mathfrak{h} = \mathfrak{k} \cong \mathfrak{so}(n)$.

Notice that the action of K on $\mathbb{L}^{n+1}$ induces an isometric action on the space $\mathbb{R}^n = \{(x_0, \dots, x_n) \in \mathbb{L}^{n+1} : x_0 = 0\}$ as follows:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} x_0 \\ x \end{pmatrix} \right) \cdot p = ap + x.$$

Since K is compact, by virtue of Theorem 4.1.5, there exists an element $p \in \mathbb{R}^n$ in such a way that $K \cdot p = p$. Thus, $\mathfrak{k} \cdot p = 0$, and it follows that

$$\mathrm{Ad}(I, -(0,p))\left((X,0) + (x_0,x)\right) = (X,0) + (x_0,0) + (0,Xp) + (0,x) = (X,0) + (x_0,0),$$

for any $(X,0) + (x_0,x) \in \mathfrak{k}$. Thus, after doing such a conjugation, we can assume that the elements of $\mathfrak{k}$ are of the form $(X,0) + (x_0,0)$, with $X \in \mathfrak{so}(n)$ and $x_0 \in \mathbb{R}$. Moreover, if $(X,0) + (x_0,0)$, $(Y,0) + (y_0,0) \in \mathfrak{k}$, we have

$$[(X,0) + (x_0,0), (Y,0) + (y_0,0)] = ([X,Y],0) + (0,0) \in \mathfrak{k} \cap \mathfrak{so}(n).$$

Since $\mathfrak{k}$ is a semisimple Lie algebra, we have $[\mathfrak{k},\mathfrak{k}] = \mathfrak{k}$, so the above equality implies that $\mathfrak{k} \subset \mathfrak{so}(n)$, and indeed $\mathfrak{k} = \mathfrak{so}(n)$ by the semisimplicity of $\mathfrak{k}$.

Now, let $(0,u) + (x_0,x)$, $(0,v) + (y_0,y) \in \mathfrak{g}$, with u and v orthogonal, that is, $\langle u,v \rangle = u^t v = 0$. Since $\mathfrak{g}$ is a Lie algebra and $\pi(\mathfrak{k}) = \mathfrak{k} = \mathfrak{so}(n)$,

$$[(0,u) + (x_0,x), (0,v) + (y_0,y)] = (uv^t - vu^t, 0) + (u^t y - v^t x, y_0 u - x_0 v) \in \mathfrak{k}.$$

Hence, $\langle u, y\rangle = \langle v, x\rangle$, and since u and v are linearly independent, we get $x_0 = y_0 = 0$. Thus, if $(0, u) + (x_0, x) \in \mathfrak{g}$, we have $x_0 = 0$. Moreover, if we take $(0, u) + (0, x)$, $(0, u) + (0, x') \in \mathfrak{g}$, then $(0, x - x') \in \mathfrak{g} \cap \mathbb{L}^{n+1} = 0$, so x is uniquely determined.

Taking now another element $(0, w) + (0, z) \in \mathfrak{g}$, we obtain

$$\begin{aligned}\big[\big[(0,u)+(0,x),(0,v)+(0,y)\big],(0,w)+(0,z)\big] &=\\ \big(0,(uv^t-vu^t)w\big)+\big(0,(uv^t-vu^t)z-w(u^ty-v^tx)\big) &=\\ \big(0,\langle v,w\rangle u-\langle u,w\rangle v\big)+\big(0,\langle v,z\rangle u-\langle u,z\rangle v\big) &=\\ \langle v,w\rangle\big((0,u)+(0,x)\big)-\langle u,w\rangle\big((0,v)+(0,y)\big)&,\end{aligned}$$

where the last equality follows from the uniqueness of the translational part. Thus, we have

$$\langle v,w\rangle x-\langle u,w\rangle y=\langle v,z\rangle u-\langle u,z\rangle v.$$

In particular, setting $(0, w)+(0, z) = (0, u)+(0, x)$ and $(0, w)+(0, z) = (0, v)+(0, y)$, respectively, in the previous equation, and taking into account that $\langle u, v\rangle = 0$, one obtains

$$\langle u,u\rangle y=-\langle v,x\rangle u+\langle u,x\rangle v, \qquad \langle v,v\rangle x=\langle v,y\rangle u-\langle u,y\rangle v.$$

Finally, taking inner product with u and v in the previous two equations, we get

$$\langle v,x\rangle=0, \qquad \langle u,y\rangle=0, \qquad \langle v,y\rangle\langle u,u\rangle=\langle u,x\rangle\langle v,v\rangle,$$

and therefore,

$$x=\lambda u, \quad y=\lambda v, \quad \text{with} \quad \lambda=\frac{\langle v,y\rangle}{\langle v,v\rangle}=\frac{\langle u,x\rangle}{\langle u,u\rangle}.$$

In other words, if $(0, u) + (0, x) \in \mathfrak{g}$, then $x = \lambda u$, where $\lambda \in \mathbb{R}$ is a constant that is independent of $(0, u) + (0, x) \in \mathfrak{g}$.

All in all this means that $\mathrm{Ad}(I, (\lambda, 0))\mathfrak{g} = \mathfrak{so}(1, n)$, and thus, the action of G on $\mathbb{L}^{n+1}$ is orbit equivalent to the action of $SO^0(1, n)$. □

For $n = 3$, Lemma 4.3.3 yields an example in Case 1.(b) (with $k = 3$) of Theorem 4.3.1.

Now we assume that $\pi(\mathfrak{g}) \subsetneq \mathfrak{so}(1, 3)$, and consider a maximal subalgebra of $\mathfrak{so}(1, 3)$ containing $\pi(\mathfrak{g})$, say $\mathfrak{l}$. Recall that, up to conjugation, $\mathfrak{l}$ is either a reductive Lie algebra $\mathfrak{l} = \mathfrak{so}(1, k) \oplus \mathfrak{so}(3 - k)$, with $k \in \{0, 1, 2\}$, or $\mathfrak{l}$ is parabolic, $\mathfrak{l} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Reductive case

To start with, we focus on the reductive case, that is, $\mathfrak{l} = \mathfrak{so}(1, k) \oplus \mathfrak{so}(3 - k)$, with $k \in \{0, 1, 2\}$.

Assume firstly that $\pi(\mathfrak{g}) \subset \mathfrak{so}(3)$, up to conjugation. Since G acts with cohomogeneity one and $\dim(\mathfrak{so}(3)) = 3$, we must have $\pi(\mathfrak{g}) = \mathfrak{so}(3)$, and as π is injective, then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(3)$. Since $\mathfrak{g}$ is compact and semisimple, it follows that

G is also a compact and semisimple Lie group. Moreover, since $\mathfrak{g}$ is a compact Lie algebra, Cartan's fixed point theorem ensures that there exists a point $\mathbf{p} \in \mathbb{L}^4$ such that $\mathfrak{g} \cdot \mathbf{p} = 0$. Then, given $X + \mathbf{v} \in \mathfrak{g}$,

$$\mathrm{Ad}(I, -\mathbf{p})(X + \mathbf{v}) = X + \mathbf{v} - X\mathbf{p} = X + (X + \mathbf{v}) \cdot \mathbf{p} = X.$$

All in all, G is conjugate to $SO(3)$. However, $SO(3)$ acts on $\mathbb{L}^4$ with cohomogeneity two, which follows from the fact that its action on $\mathbb{R}^3 \subset \mathbb{L}^4$ is of cohomogeneity one.

Assume now that $\pi(\mathfrak{g}) \subset \mathfrak{so}(1,1) \oplus \mathfrak{so}(2)$. In this case, $\dim(\pi(\mathfrak{g})) \leq 2$ and, thus, G cannot act with cohomogeneity one on $\mathbb{L}^4$.

Finally, assume that $\pi(\mathfrak{g}) \subset \mathfrak{so}(1,2)$. Since G acts on $\mathbb{L}^4$ with cohomogeneity one and $\dim(\mathfrak{so}(1,2)) = 3$, we must have that $\pi(\mathfrak{g}) = \mathfrak{so}(1,2)$. Lemma 4.3.4 shows that, in this case, the action of G cannot be of cohomogeneity one.

Lemma 4.3.4. *If $\pi(\mathfrak{g}) = \mathfrak{so}(1,2)$, the action of G is orbit equivalent to the action of $SO(1,2)$ on $\mathbb{L}^4$, which is of cohomogeneity two.*

Proof. First of all, if $\pi(\mathfrak{g}) = \mathfrak{so}(1,2)$, we can assume, after conjugation by an element of $SO(1,3)$ if necessary, that $\mathbb{L}^3 = \{(x_0, x_1, x_2, 0) \in \mathbb{L}^4 : x_i \in \mathbb{R}\}$ is invariant by the action of $SO(1,2)$.

Let $u = (1,0,0)^t$ and $v = (0,1,0)^t$, and define $X = vu^t - uv^t \in \mathfrak{so}(2) \subset \mathfrak{so}(3)$. In particular, $Xu = v$. Then, the set $\{(X,0), (0,u), (0,v)\}$ constitutes a basis of $\mathfrak{so}(1,2)$. Since $\mathfrak{g} \cap \mathbb{L}^4 = 0$, one deduces that there exist unique (x_0, x), (y_0, y), $(z_0, z) \in \mathbb{L}^4$ such that

$$\{(X,0) + (x_0, x), (0,u) + (y_0, y), (0,v) + (z_0, z)\}$$

is a basis of $\mathfrak{g}$. Then, we have the brackets

$$\begin{aligned}
[(X,0) + (x_0,x), (0,u) + (y_0,y)] &= (0,v) + (-\langle u, x\rangle, Xy - x_0 u),\\
[(X,0) + (x_0,x), (0,v) + (z_0,z)] &= (0,-u) + (-\langle v, x\rangle, Xz - x_0 v),\\
[(0,u) + (y_0,y), (0,v) + (z_0,z)] &= (-X,0) + (\langle u, z\rangle - \langle v, y\rangle, z_0 u - y_0 v).
\end{aligned}$$

Since the right-hand sides of these equations are elements of $\mathfrak{g}$, we must have

$$\begin{aligned}
z_0 &= -\langle u, x\rangle, & y_0 &= \langle v, x\rangle, & x_0 &= -\langle u, z\rangle + \langle v, y\rangle,\\
z &= Xy - x_0 u, & y &= -Xz + x_0 v, & x &= -z_0 u + y_0 v.
\end{aligned}$$

As $X(\mathbb{R}^3) = \mathbb{R}^2$, and $u, v \in \mathbb{R}^2$, the last row readily implies $x, y, z \in \mathbb{R}^2$. Moreover, $x_0 = 0$, $y_0 = \langle x, v\rangle$, $z_0 = -\langle x, u\rangle$, $\langle z, u\rangle = \langle y, v\rangle = 0$, and $\langle y, u\rangle = \langle z, v\rangle$. Using these facts, we obtain

$$\begin{aligned}
&\mathrm{Ad}\big(I, (\langle y,u\rangle, \langle x,v\rangle, -\langle x,u\rangle, 0)\big)\big((X,0) + (0,x)\big) = (X,0),\\
&\mathrm{Ad}\big(I, (\langle y,u\rangle, \langle x,v\rangle, -\langle x,u\rangle, 0)\big)\big((0,u) + (\langle x,v\rangle, \langle y,u\rangle u)\big) = (0,u),\\
&\mathrm{Ad}\big(I, (\langle y,u\rangle, \langle x,v\rangle, -\langle x,u\rangle, 0)\big)\big((0,v) + (-\langle x,u\rangle, \langle y,u\rangle v)\big) = (0,v).
\end{aligned}$$

Therefore, $\mathrm{Ad}\big(I, (\langle y,u\rangle, \langle x,v\rangle, -\langle x,u\rangle, 0)\big)\mathfrak{g} = \mathfrak{so}(1,2)$, and thus the action of G on $\mathbb{L}^4$ is orbit equivalent to the action of $SO(1,2)$ on $\mathbb{L}^4$. Since $SO(1,2)$ acts with cohomogeneity one on $\mathbb{L}^3$, the action of $SO(1,2)$ is of cohomogeneity two on $\mathbb{L}^4$. □

All in all we have proved that, if $\pi(\mathfrak{g})$ is contained in a maximal reductive subalgebra of $\mathfrak{so}(1,3)$, then the action of G on $\mathbb{L}^4$ cannot be of cohomogeneity one.

Parabolic case

We turn our attention to the parabolic case. We assume then that $\pi(\mathfrak{g}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. In the rest of this section, we will use the following notation:

$$(X + a + u) + (x_0, x_1, x) \equiv \begin{pmatrix} 0 & a & u^t \\ a & 0 & u^t \\ u & -u & X \end{pmatrix} + \begin{pmatrix} x_0 \\ x_1 \\ x \end{pmatrix},$$

where $X \in \mathfrak{so}(2)$, u, $x \in \mathbb{R}^2$, and a, x_0, $x_1 \in \mathbb{R}$. Recall from the beginning of this section that E is the generator of $\mathfrak{so}(2)$ defined by

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $\mathbf{e}$ is the lightlike vector of $\mathbb{L}^4$ given by $\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 = (1, 1, 0, 0)$.

The following result is devoted to determining the connected Lie subgroups G of $I^0(\mathbb{L}^4)$ acting with cohomogeneity one on $\mathbb{L}^4$ whose Lie algebra $\mathfrak{g}$ satisfies $\pi(\mathfrak{g}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Lemma 4.3.5. *Under the previous hypotheses, the action of G is orbit equivalent to the action of the connected Lie subgroup of $I^0(\mathbb{L}^4)$ whose Lie algebra is given by one of the following possibilities:*

(i) $\mathbb{R}(E + \mathbf{e}) \oplus \mathfrak{n}$,

(ii) $\mathfrak{a} \oplus \mathfrak{n}$,

(iii) $\mathbb{R}(E + 1) \oplus \mathfrak{n}$,

(iv) $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Proof. Let us denote by $\sigma\colon (\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}) \oplus_\phi \mathbb{L}^4 \to \mathfrak{a} \oplus \mathfrak{n}$ the projection onto the solvable part of the Iwasawa decomposition of $\mathfrak{so}(1,3)$.

Assume firstly that $\sigma(\mathfrak{g}) \subsetneq \mathfrak{a} \oplus \mathfrak{n}$. Then, $\dim(\sigma(\mathfrak{g})) \leq 2$ and we have $\ker(\sigma|_{\mathfrak{g}}) = \mathfrak{g} \cap (\mathfrak{k}_0 \oplus \mathbb{L}^4)$. Since $\mathfrak{g} \cap \mathbb{L}^4 = 0$ and $\mathfrak{k}_0 \cong \mathfrak{so}(2)$ is one-dimensional, it follows that $\dim(\mathfrak{g} \cap (\mathfrak{k}_0 \oplus \mathbb{L}^4)) \leq 1$. Moreover, as $\dim(\mathfrak{g}) = \dim(\sigma(\mathfrak{g})) + \dim(\mathfrak{g} \cap (\mathfrak{k}_0 \oplus \mathbb{L}^4))$ and G acts on $\mathbb{L}^4$ with cohomogeneity one, then $\dim(\sigma(\mathfrak{g})) = 2$ and $\dim(\mathfrak{g} \cap (\mathfrak{k}_0 \oplus \mathbb{L}^4)) = 1$.

Let then $X + (x_0, x_1, x) \in \mathfrak{g} \cap (\mathfrak{k}_0 \oplus \mathbb{L}^4)$, with $X \in \mathfrak{k}_0$ a nonzero element, $x_0, x_1 \in \mathbb{R}$, and $x \in \mathbb{R}^2$.

Assume that $\sigma(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{v}$, where $\mathfrak{v}$ is a one-dimensional subspace of $\mathfrak{n}$, and let $u \in \mathfrak{v}$ be a nonzero vector. In this case, there exist $b_0, b_1, y_0, y_1 \in \mathbb{R}$ and $b, y \in \mathbb{R}^2$ in such a way that, with the notation above, the set

$$\{X + (x_0, x_1, x), 1 + (b_0, b_1, b), u + (y_0, y_1, y)\}$$

is a basis of $\mathfrak{g}$. Now, taking the bracket

$$[X + (x_0, x_1, x), u + (y_0, y_1, y)] = Xu + (-\langle u, x\rangle, -\langle u, x\rangle, Xy - (x_0 - x_1)u) \in \mathfrak{g},$$

we get, in particular, that $Xu \in \mathfrak{v}$. But X is a nonzero skew-symmetric matrix of $\mathfrak{so}(2)$ and $\mathfrak{v} = \mathbb{R}u \subset \mathbb{R}^2$ is one-dimensional, which yields a contradiction. Therefore, $\sigma(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{v}$ is not possible if $\mathfrak{v} \neq \mathfrak{n}$.

Assume now $\sigma(\mathfrak{g}) = \mathfrak{n}$. Consider an orthonormal basis $\{u, v\}$ of $\mathfrak{n}$, and assume $Xu = v$; thus, $Xv = -u$. Then, there exist $y_0, y_1, z_0, z_1 \in \mathbb{R}$ and $y, z \in \mathbb{R}^2$ such that

$$\{X + (x_0, x_1, x), u + (y_0, y_1, y), v + (z_0, z_1, z)\}$$

constitutes a basis of $\mathfrak{g}$. We have the brackets

$$\begin{aligned}
&[X + (x_0, x_1, x), u + (y_0, y_1, y)] = v + (-\langle x, u\rangle, -\langle x, u\rangle, Xy - (x_0 - x_1)u),\\
&[X + (x_0, x_1, x), v + (z_0, z_1, z)] = -u + (-\langle x, v\rangle, -\langle x, v\rangle, Xz - (x_0 - x_1)v),\\
&[u + (y_0, y_1, y), v + (z_0, z_1, z)] = (\langle z, u\rangle - \langle y, v\rangle, \langle z, u\rangle - \langle y, v\rangle, (z_0 - z_1)u - (y_0 - y_1)v).
\end{aligned}$$

Since the right-hand sides are elements of $\mathfrak{g}$, one obtains the equations

$$\begin{aligned}
z_0 &= z_1 = -\langle x, u\rangle, & -y_0 &= -y_1 = -\langle x, v\rangle, & \langle z, u\rangle &= \langle y, v\rangle,\\
z &= Xy - (x_0 - x_1)u, & -y &= Xz - (x_0 - x_1)v.
\end{aligned}$$

Taking inner product with u and v, and using the third equation above, we get $x_0 = x_1$, $\langle z, u\rangle = \langle y, v\rangle = 0$, and $z = Xy$. Thus, the previous basis of $\mathfrak{g}$ becomes

$$\{X + (x_0, x_0, x), u + (\langle v, x\rangle, \langle v, x\rangle, \lambda u), v + (-\langle u, x\rangle, -\langle u, x\rangle, \lambda v)\},$$

where $\lambda = \langle y, u\rangle$. Now, we have

$$\begin{aligned}
&\mathrm{Ad}(I, (\lambda, 0, \langle v, x\rangle u - \langle x, u\rangle v))(X + (x_0, x_0, x)) = X + (x_0, x_0, 0),\\
&\mathrm{Ad}(I, (\lambda, 0, \langle v, x\rangle u - \langle x, u\rangle v))(u + (\langle v, x\rangle, \langle v, x\rangle, \lambda u)) = u,\\
&\mathrm{Ad}(I, (\lambda, 0, \langle v, x\rangle u - \langle x, u\rangle v))(v + (-\langle u, x\rangle, -\langle u, x\rangle, \lambda v)) = v.
\end{aligned}$$

This shows that $\mathfrak{g}$ is conjugate to $\mathbb{R}(X + x_0\,\mathbf{e}) \oplus \mathfrak{n}$.

It remains to prove that $\mathfrak{g}_\lambda = \mathbb{R}(X + \lambda\,\mathbf{e}) \oplus \mathfrak{n}$, with $\lambda \in \mathbb{R}$, gives indeed a cohomogeneity one action. For an arbitrary element $(p_0, p_1, p) \in \mathbb{L}^4$, we have

$$\begin{aligned}
(X + \lambda\,\mathbf{e}) \cdot (p_0, p_1, p) &= (\lambda, \lambda, Xp) = (\lambda, \lambda, -\langle p, v\rangle u + \langle p, u\rangle v),\\
u \cdot (p_0, p_1, p) &= (\langle p, u\rangle, \langle p, u\rangle, (p_0 - p_1)u),\\
v \cdot (p_0, p_1, p) &= (\langle p, v\rangle, \langle p, v\rangle, (p_0 - p_1)v).
\end{aligned}$$

Moreover, since

$$\begin{vmatrix} \lambda & \langle p, u\rangle & \langle p, v\rangle \\ -\langle p, v\rangle & p_0 - p_1 & 0 \\ \langle p, u\rangle & 0 & p_0 - p_1 \end{vmatrix} = \lambda(p_0 - p_1)^2,$$

the tangent space $T_{(p_0,p_1,p)}(G_\lambda \cdot (p_1,p_1,p)) = \mathfrak{g}_\lambda \cdot (p_0,p_1,p)$ is three-dimensional whenever $\lambda \neq 0$ and $p_0 \neq p_1$, where G_λ denotes the connected subgroup of $I^0(\mathbb{L}^4)$ whose Lie algebra is $\mathfrak{g}_\lambda$. Hence, the action of G_λ on $\mathbb{L}^4$ is of cohomogeneity one whenever $\lambda \neq 0$. Moreover, it is easy to see that $\mathfrak{g}_\lambda \cdot (p_0,p_1,p)$ is independent of $\lambda \neq 0$. Hence, by virtue of Proposition 4.1.3, the action of G_λ is equivalent to the action of G_1. With a suitable change of basis, we can also assume $X = E$, and thus this action corresponds to Case (i) in Lemma 4.3.5.

Finally, assume $\sigma(\mathfrak{g}) = \mathbb{R}(a+u) \oplus \mathbb{R}v$ with $a \in \mathfrak{a} \cong \mathbb{R}$, $a \neq 0$, and $u, v \in \mathfrak{n}$ unit vectors in such a way that $\langle u, v\rangle = 0$. We consider

$$g = \operatorname{Exp}\Big(\frac{1}{a}\,u\Big) = \begin{pmatrix} 1+\frac{|u|^2}{2a^2} & -\frac{|u|^2}{2a^2} & \frac{1}{a}u^t \\ \frac{|u|^2}{2a^2} & 1-\frac{|u|^2}{2a^2} & \frac{1}{a}u^t \\ \frac{1}{a}u & -\frac{1}{a}u & I \end{pmatrix} \in N.$$

Since $[\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{a} \oplus \mathfrak{n}$, it follows that $\sigma(\operatorname{Ad}(g,0)(\mathfrak{g})) = \mathfrak{a} \oplus \mathfrak{v}$ for certain one-dimensional subspace $\mathfrak{v} \subset \mathfrak{n}$. This reduces the study to a previous case, from where it follows that G does not act with cohomogeneity one.

We now deal with the case $\sigma(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{n}$. Let then $\{u, v\}$ be an orthonormal basis of $\mathfrak{n}$. It is not restrictive to assume, making a change of basis if necessary, that $Eu = v$. Since $\sigma(\mathfrak{g}) = \mathfrak{a} \oplus \mathfrak{n}$ there exist $b_0, b_1, y_0, y_1, z_0, z_1, \beta, \mu, \nu \in \mathbb{R}$, and $b, y, z \in \mathbb{R}^2$, in such a way that

$$(\beta E + 1) + (b_0, b_1, b), (\mu E + u) + (y_0, y_1, y), (\nu E + v) + (z_0, z_1, z) \in \mathfrak{g}.$$

Taking brackets, we have

$$\begin{aligned}
&[(\beta E + 1) + (b_0, b_1, b), (\mu E + u) + (y_0, y_1, y)] \\
&= (u + \beta v) + (y_1 - \langle b, u\rangle, y_0 - \langle b, u\rangle, \beta E y - (b_0 - b_1)u - \mu E b), \\
&[(\beta E + 1) + (b_0, b_1, b), (\nu E + v) + (z_0, z_1, z)] \\
&= (-\beta u + v) + (z_1 - \langle b, v\rangle, z_0 - \langle b, v\rangle, \beta E z - (b_0 - b_1)v - \nu E b), \\
&[(\mu E + u) + (y_0, y_1, y), (\nu E + v) + (z_0, z_1, z)] \\
&= (-\mu u - \nu v) + (\langle z, u\rangle - \langle y, v\rangle, \langle z, u\rangle - \langle y, v\rangle, (z_0 - z_1)u + \mu E z - (y_0 - y_1)v - \nu E y).
\end{aligned}$$

In the first place we get $\mu + \beta\nu = 0$, $\nu - \beta\mu = 0$, $-\mu^2 - \nu^2 = 0$, which follows from the fact that $\pi(\mathfrak{g})$ is a Lie algebra. Thus, $\mu = \nu = 0$. Using this fact, one also obtains the relations

$$\begin{array}{lll}
y_0 + \beta z_0 = y_1 - \langle b, u\rangle, & z_0 - \beta y_0 = z_1 - \langle b, v\rangle, & \langle z, u\rangle = \langle y, v\rangle, \\
y_1 + \beta z_1 = y_0 - \langle b, u\rangle, & z_1 - \beta y_1 = z_0 - \langle b, v\rangle, & \\
y + \beta z = \beta E y - (b_0 - b_1)u, & z - \beta y = \beta E z - (b_0 - b_1)v, & (z_0 - z_1)u = (y_0 - y_1)v,
\end{array}$$

Since u and v are linearly independent, we easily obtain $y_0 = y_1$ and $z_0 = z_1$. Then, $\langle b, v\rangle = \beta y_0$, $\langle b, u\rangle = -\beta z_0$. Taking inner product with u and v, we get

$$\begin{aligned}
\langle y, u\rangle + 2\beta\langle z, u\rangle &= b_1 - b_0, \\
-\beta\langle y, u\rangle + \langle z, u\rangle + \beta\langle z, v\rangle &= 0, \\
-2\beta\langle z, u\rangle + \langle z, v\rangle &= b_1 - b_0,
\end{aligned}$$

whose solution is $\langle y,u\rangle = \langle z,v\rangle = b_1 - b_0$, $\langle z,u\rangle = \langle y,v\rangle = 0$. In particular, we have $b = -\beta z_0 u + \beta y_0 v$, $z = (b_1 - b_0)v$, and $y = (b_1 - b_0)u$.

Now we conjugate to obtain

$$\begin{aligned}
&\mathrm{Ad}(I,(b_1,b_0,y_0u+z_0v))((\beta E+1)+(b_0,b_1,-\beta z_0u+\beta y_0v)) = \beta E+1,\\
&\mathrm{Ad}(I,(b_1,b_0,y_0u+z_0v))(u+(y_0,y_0,(b_1-b_0)u)) = u,\\
&\mathrm{Ad}(I,(b_1,b_0,y_0u+z_0v))(v+(z_0,z_0,(b_1-b_0)v)) = v.
\end{aligned}$$

Thus, we can assume $\beta E+1, u, v \in \mathfrak{g}$.

Now, if $\dim(\mathfrak{g}) = 3$, the three elements above constitute a basis of $\mathfrak{g}$. We define $\mathfrak{g}_\beta = \mathbb{R}(\beta E+1)\oplus\mathfrak{n}$. For an arbitrary $\mathbf{p} = (p_0,p_1,p)\in\mathbb{L}^4$, we have

$$\begin{aligned}
(\beta E+1)\cdot(p_0,p_1,p) &= (p_1,p_0,\beta\langle p,v\rangle u - \beta\langle p,u\rangle v),\\
u\cdot(p_0,p_1,p) &= (\langle p,u\rangle,\langle p,u\rangle,(p_0-p_1)u),\\
v\cdot(p_0,p_1,p) &= (\langle p,v\rangle,\langle p,v\rangle,(p_0-p_1)v).
\end{aligned}$$

We obtain $\mathfrak{g}_\beta\cdot\mathbf{p} = (\mathbb{R}\mathbf{p})^\perp$, if $p_0\neq p_1$, $\mathfrak{g}_\beta\cdot(p_0,p_0,0) = \mathbb{R}\mathbf{e}$, if $p_0\neq 0$, and $\mathfrak{g}_\beta\cdot\mathbf{0} = 0$, independently of β. In particular, $\mathfrak{g}_\beta\cdot\mathbf{p}$ is three-dimensional whenever $p_0\neq p_1$. Therefore, the corresponding connected Lie subgroup G_β of $I^0(\mathbb{L}^4)$ whose Lie algebra is $\mathfrak{g}_\beta$ acts on $\mathbb{L}^4$ with cohomogeneity one.

If $\beta = 0$, we get exactly $\mathfrak{a}\oplus\mathfrak{n}$, and thus, the action of G is orbit equivalent to the action of AN, as in Case (ii) of Lemma 4.3.5. In this case, $(\mathfrak{a}\oplus\mathfrak{n})\cdot(p_0,p_0,p) = \mathbb{R}\mathbf{e}$ if $p_0\neq 0$ and $p\neq 0$.

If $\beta\neq 0$, then $\mathfrak{g}_\beta\cdot(p_0,p_0,p) = (\mathbb{R}\mathbf{e}\oplus\mathbb{R}\mathbf{p})^\perp$ for $p_0\neq 0$, $p\neq 0$, independently of $\beta\neq 0$. Hence, the action of G_β is not orbit equivalent to the action of AN because $\mathfrak{g}_\beta\cdot(p_0,p_0,p)$ is a two-dimensional subspace. Moreover, if $\beta_1,\beta_2\neq 0$, then the actions of G_{β_1} and G_{β_2} are orbit equivalent. This corresponds to Case (iii) in the statement of Lemma 4.3.5.

Finally, we still have to consider the case $\dim(\mathfrak{g}) = 4$. Then, $\pi(\mathfrak{g}) = \mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$. Thus, there exists $(x_0,x_1,x)\in\mathbb{L}^4$ in such a way that $\{E+(x_0,x_1,x),\beta E+1,u,v\}$ is a basis of $\mathfrak{g}$. Since

$$\begin{aligned}
&[E+(x_0,x_1,x),u] = v+(\langle x,u\rangle,\langle x,u\rangle,(x_0-x_1)u),\\
&[E+(x_0,x_1,x),v] = -u+(\langle x,v\rangle,\langle x,v\rangle,(x_0-x_1)v),\\
&[E+(x_0,x_1,x),\beta E+1] = -\beta Ex-(x_1,x_0,0),
\end{aligned}$$

are elements of $\mathfrak{g}$, we get $x_0 = x_1 = 0$ and $x = 0$. Then, $\mathfrak{g} = \mathfrak{k}_0\oplus\mathfrak{a}\oplus\mathfrak{n}$, and the corresponding connected Lie subgroup $G = K_0AN$ of $I^0(\mathbb{L}^4)$ whose Lie algebra is $\mathfrak{g}$ is known to act on $\mathbb{L}^4$ with cohomogeneity one, as real hyperbolic spaces are orbits of this action (see [14]). This corresponds to Case (iv) of Lemma 4.3.5. $\square$

The results of this section imply part (1) of Theorem 4.3.1. Indeed, let G be a connected Lie subgroup of $I^0(\mathbb{L}^4)$ with Lie algebra $\mathfrak{g}$ acting with cohomogeneity one on $\mathbb{L}^4$. Recall from Lemma 4.3.2 that, if the purely translational part $\mathfrak{v} = \mathfrak{g}\cap\mathbb{L}^4$ is nondegenerate, then the action of G reduces to a cohomogeneity one action on $\mathfrak{v}^\perp$, the orthogonal complement of $\mathfrak{v}$ in $\mathbb{L}^4$. We have the following possibilities:

- If $\mathfrak{v}$ is a Lorentzian subspace, the action of G is orbit equivalent to the action of $SO(k) \times \mathbb{L}^{4-k}$, for $k \in \{1, 2, 3\}$, which corresponds to Case 1.(a) of Theorem 4.3.1.
- If $\mathfrak{v}$ is a Riemannian subspace and $\dim(\mathfrak{v}) \geq 1$, the action of G is orbit equivalent to the action of $SO^0(1, k) \times \mathbb{R}^{3-k}$, for $k \in \{0, 1, 2, 3\}$, or the action of $A\widetilde{N} \times \mathbb{R}$, where $\mathbb{R}$ denotes a spacelike line in $\mathbb{L}^4$ and $A\widetilde{N}$ is the solvable part of the Iwasawa decomposition of $SO^0(1, 2)$. This corresponds to Cases 1.(b) and 1.(c) of Theorem 4.3.1.
- Finally, if $\dim(\mathfrak{v}) = 0$, we distinguish two possibilities, depending on whether the projection of $\mathfrak{g}$ onto $\mathfrak{so}(1, 3)$, $\pi(\mathfrak{g})$, is contained in a reductive subalgebra of $\mathfrak{so}(1, 3)$ or it is contained in a parabolic one. If $\pi(\mathfrak{g})$ is contained in a reductive subalgebra, the action of G on $\mathbb{L}^4$ cannot be of cohomogeneity one, whereas if $\pi(\mathfrak{g})$ is contained in a parabolic subalgebra, then $\mathfrak{g}$ must be one of the Lie subalgebras listed in Lemma 4.3.5, up to orbit equivalence. More specifically, if $\mathfrak{g} \in \{\mathfrak{a} \oplus \mathfrak{n},\ \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}\}$ we get Case 1.(d) of Theorem 4.3.1, if $\mathfrak{g} = \mathbb{R}(E + \mathbf{e}) \oplus \mathfrak{n}$ we get Case 1.(e), and if $\mathfrak{g} = \mathbb{R}(E + 1) \oplus \mathfrak{n}$ we obtain Case 1.(f).

4.3.2 Degenerate translational part

Now we assume that $\mathfrak{v} = \mathfrak{g} \cap \mathbb{L}^{n+1}$ is a degenerate subspace of $\mathbb{L}^{n+1}$.

Lemma 4.3.6. *Let $\mathfrak{v}$ be a degenerate subspace of $\mathbb{L}^{n+1}$. Then, there exists $g \in SO^0(1, n)$ such that $g \cdot \mathfrak{v} = \mathbb{R}\mathbf{e} \oplus \mathfrak{w}$, where $\mathbf{e} = (1, 1, 0, \ldots, 0) \in \mathbb{L}^{n+1}$ and $\mathfrak{w} \subset \mathbb{R}^{n-1} = \{(x_0, x_1, \ldots, x_n) \in \mathbb{L}^{n+1} : x_0 = x_1 = 0\}$. In particular, the metric is positive definite on $\mathfrak{w}$.*

Proof. Since $\mathfrak{v}$ is degenerate, there is a nonzero vector $\mathbf{z} \in \mathfrak{v}$ such that $\langle \mathbf{z}, \mathbf{v} \rangle = 0$, for all $\mathbf{v} \in \mathfrak{v}$. In particular, $\mathbf{z}$ is a lightlike vector. As $SO^0(1, n)$ acts transitively on the set of future-oriented lightlike vectors, by conjugating by an element of $SO^0(1, n)$ one can assume that this vector is, precisely, $\mathbf{z} = \mathbf{e}$. Now we complete this vector to a basis of $\mathfrak{v}$ and get $\mathfrak{v} = \mathbb{R}\mathbf{e} \oplus \operatorname{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. We write $\mathbf{w}_i = (w_{i0}, \ldots, w_{in})$. Since $0 = \langle \mathbf{e}, \mathbf{w}_i \rangle = -w_{i0} + w_{i1}$, we can take $\mathfrak{w} = \operatorname{span}\{\mathbf{w}_1 - w_{10}\mathbf{e}, \ldots, \mathbf{w}_k - w_{k0}\mathbf{e}\}$ and the result follows. □

According to Lemma 4.3.6, we can write $\mathfrak{v} = \mathbb{R}\mathbf{e} \oplus \mathfrak{w}$, with $\mathfrak{w}$ a Riemannian vector subspace orthogonal to $\mathbf{e}$.

Let $X + \mathbf{u} \in \mathfrak{g}$ and $\mathbf{w} \in \mathfrak{w}$ be arbitrary elements. Then,

$$X\mathbf{e} = [X + \mathbf{u}, \mathbf{e}] \in \mathfrak{g} \cap \mathbb{L}^{n+1} = \mathfrak{v}, \qquad X\mathbf{w} = [X + \mathbf{u}, \mathbf{w}] \in \mathfrak{g} \cap \mathbb{L}^{n+1} = \mathfrak{v}.$$

Thus, we can write $X\mathbf{e} = \lambda\mathbf{e} + \mathbf{w}_0$, with $\lambda \in \mathbb{R}$ and $\mathbf{w}_0 \in \mathfrak{w}$. Since $X\mathbf{w}_0 \in \mathfrak{v}$, we have

$$0 = \langle X\mathbf{w}_0, \mathbf{e} \rangle = -\langle \mathbf{w}_0, X\mathbf{e} \rangle = -\langle \mathbf{w}_0, \lambda\mathbf{e} + \mathbf{w}_0 \rangle = -\langle \mathbf{w}_0, \mathbf{w}_0 \rangle$$

and, as $\mathfrak{w}$ is Riemannian, we get $\mathbf{w}_0 = 0$. This implies $X\mathbf{e} \in \mathbb{R}\mathbf{e}$, and therefore, $X \in \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

Now we write $X \equiv Y + a + v$, with $Y \in \mathfrak{k}_0$, $a \in \mathfrak{a}$, $v \in \mathbb{R}^{n-1}$, and $\mathbf{w} = (0, 0, w)$ for some $w \in \mathbb{R}^{n-1}$. Then, $(\langle v, w\rangle, \langle v, w\rangle, Yw) = X\mathbf{w} \in \mathfrak{v} = \mathbb{R}\mathbf{e} \oplus \mathfrak{w}$, and since $\mathbf{w} \in \mathfrak{w}$ is arbitrary, we get $Y\mathfrak{w} \subset \mathfrak{w}$. Nondegeneracy of $\mathfrak{w}$ means that $Y\mathfrak{w}^\perp \subset \mathfrak{w}^\perp$, where $\mathfrak{w}^\perp$ denotes the orthogonal complement of $\mathfrak{w}$ in $\mathbb{R}^{n-1}$.

This gives a further decomposition of several spaces. On the one hand, we have $Y \in \mathfrak{so}(\mathfrak{w}^\perp) \oplus \mathfrak{so}(\mathfrak{w})$. We can also write $\mathfrak{n} = \mathfrak{w}^\perp \oplus \mathfrak{w}$. Thus, $Y = Y^\perp + Y^\top$, with $Y^\perp \in \mathfrak{so}(\mathfrak{w}^\perp)$, $Y^\top \in \mathfrak{so}(\mathfrak{w})$, and $v = v^\perp + v^\top$, with $v^\perp \in \mathfrak{w}^\perp$, $v^\top \in \mathfrak{w}$. Since $\mathfrak{w} \subset \mathfrak{g}$, if $X + \mathbf{u}$ is an arbitrary element of $\mathfrak{g}$, one can also assume $\mathbf{u} = (u_0, u_1, u^\perp)$, with $u^\perp \in \mathfrak{w}^\perp$. In particular, $\pi(\mathfrak{g}) \subset \mathfrak{so}(\mathfrak{w}^\perp) \oplus \mathfrak{so}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{n} \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$.

We define the projection

$$\sigma\colon (\mathfrak{so}(\mathfrak{w}^\perp) \oplus \mathfrak{so}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{n}) \oplus_\phi \mathbb{L}^{n+1} \to (\mathfrak{so}(\mathfrak{w}^\perp) \oplus \mathfrak{a} \oplus \mathfrak{w}^\perp) \oplus_\phi (\mathbb{L}^{n+1} \ominus \mathfrak{w}),$$

which is a Lie algebra homomorphism. Consider $\mathfrak{h} = \sigma(\mathfrak{g})$, and let H be the connected Lie subgroup of $I^0(\mathbb{L}^{n+1})$ whose Lie algebra is $\mathfrak{h}$. Notice that $\mathbb{R}\mathbf{e} \subset \mathfrak{h}$.

Lemma 4.3.7. *Under these conditions, the actions of $H \times \mathfrak{w}$ and G are orbit equivalent.*

Proof. We continue using the notation described above. Consider then the element $\mathbf{p} = (p_0, p_1, p^\perp, p^\top) \in \mathbb{L}^{n+1}$, with p_0, $p_1 \in \mathbb{R}$, $p^\perp \in \mathfrak{w}^\perp$ and $p^\top \in \mathfrak{w}$.

We firstly show that $(\mathfrak{h} \oplus \mathfrak{w}) \cdot \mathbf{p} \subset \mathfrak{g} \cdot \mathbf{p}$. Let $Y^\perp \in \mathfrak{so}(\mathfrak{w}^\perp)$, $v^\perp \in \mathfrak{w}^\perp \subset \mathfrak{n}$, $u^\perp \in \mathfrak{w}^\perp$, $w \in \mathfrak{w}$ and $a, u_0, u_1 \in \mathbb{R}$, such that $Y^\perp + a + v^\perp + (u_0, u_1, u^\perp) \in \mathfrak{h}$. Then, by definition of $\mathfrak{h}$, there exists $Y^\top \in \mathfrak{so}(\mathfrak{w})$ and $v^\top \in \mathfrak{w} \subset \mathfrak{n}$ in such a way that $Y^\perp + Y^\top + a + v^\perp + v^\top + (u_0, u_1, u^\perp) \in \mathfrak{g}$. Since $-\langle v^\top, p^\top\rangle\mathbf{e} - (p_0 - p_1)v^\top - Y^\top p^\top \in \mathfrak{v}$ and

$$(Y^\perp + a + v^\perp + (u_0, u_1, u^\perp) + w) \cdot \mathbf{p} = \begin{pmatrix} ap_1 + \langle v^\perp, p^\perp\rangle + u_0 \\ ap_0 + \langle v^\perp, p^\perp\rangle + u_1 \\ (p_0 - p_1)v^\perp + Y^\perp p^\perp + u^\perp \\ w \end{pmatrix}$$

$$= \big(Y^\perp + Y^\top + a + v^\perp + v^\top + (u_0, u_1, u^\perp) - \langle v^\top, p^\top\rangle\mathbf{e} - (p_0 - p_1)v^\top - Y^\top p^\top + w\big) \cdot \mathbf{p}$$

lies in $\mathfrak{g} \cdot \mathbf{p}$, we obtain $(\mathfrak{h} \oplus \mathfrak{w}) \cdot \mathbf{p} \subset \mathfrak{g} \cdot \mathbf{p}$.

Conversely, let $Y^\perp \in \mathfrak{so}(\mathfrak{v}^\perp)$, $Y^\top \in \mathfrak{so}(\mathfrak{v})$, $v^\perp \in \mathfrak{w}^\perp \subset \mathfrak{n}$, $v^\top \in \mathfrak{w} \subset \mathfrak{n}$, $u^\perp \in \mathfrak{w}^\perp$, $w \in \mathfrak{w}$ and $a, \lambda, u_0, u_1 \in \mathbb{R}$, such that $Y^\perp + Y^\top + a + v^\perp + v^\top + (u_0, u_1, u^\perp) \in \mathfrak{g}$ and $\lambda\mathbf{e} + w \in \mathfrak{v}$. Then, $Y^\perp + a + v^\perp + (u_0, u_1, u^\perp) \in \mathfrak{h}$, $(\langle v^\top, p^\top\rangle + \lambda)\mathbf{e} \in \mathbb{R}\mathbf{e} \subset \mathfrak{h}$, $Y^\top p^\top \in \mathfrak{w}$, and thus

$$\begin{aligned}
(Y^\perp + Y^\top &+ a + v^\perp + v^\top + (u_0, u_1, u^\perp) + \lambda\mathbf{e} + w) \cdot \mathbf{p} \\
&= \begin{pmatrix} ap_1 + \langle v^\perp, p^\perp\rangle + \langle v^\top, p^\top\rangle + \lambda + u_0 \\ ap_0 + \langle v^\perp, p^\perp\rangle + \langle v^\top, p^\top\rangle + \lambda + u_1 \\ (p_0 - p_1)v^\perp + Y^\perp p^\perp + u^\perp \\ (p_0 - p_1)v^\top + Y^\top p^\top + w \end{pmatrix} \\
&= (Y^\perp + a + v^\perp + (u_0, u_1, u^\perp) + (\langle v^\top, p^\top\rangle + \lambda)\mathbf{e} + w + Y^\top p^\top) \cdot \mathbf{p},
\end{aligned}$$

which belongs to $(\mathfrak{h} \oplus \mathfrak{w}) \cdot \mathbf{p}$.

Therefore the orbits of G and $H \times \mathfrak{w}$ coincide by virtue of Proposition 4.1.3. $\square$

We now turn our attention to the particular case $n = 3$ in order to continue the proof of Theorem 4.3.1. In view of Lemma 4.3.7, the action of G reduces to a cohomogeneity one action on $\mathbb{L}^4 \ominus \mathfrak{w}$ whose purely translational part is $\mathbb{R}\mathbf{e}$. For dimension reasons, we clearly have $0 \leq \dim(\mathfrak{w}) \leq 2$.

If $\dim(\mathfrak{w}) = 2$, we have to determine cohomogeneity one actions on $\mathbb{L}^2$ by a group H such that $\mathfrak{h} \cap \mathbb{L}^2 = \mathbb{R}\mathbf{e}$. It follows from Theorem 4.1.1 that the only such action, up to orbit equivalence, is the action of $\mathbb{R}\mathbf{e}$ itself. Thus, the action of G is orbit equivalent to the action of $\mathbb{W}^3$, which corresponds to Case 2.(a) of Theorem 4.3.1.

If $\dim(\mathfrak{w}) = 1$, we have to determine cohomogeneity one actions on $\mathbb{L}^3$ by a Lie group H such that $\mathfrak{h} \cap \mathbb{L}^3 = \mathbb{R}\mathbf{e}$. It follows from Theorem 4.1.2 that we have the following actions:

(i) The action of $\widetilde{N} \times \mathbb{R}(1,1,0)$, where $\widetilde{N}$ is the nilpotent part of the Iwasawa decomposition of $SO^0(1,2)$. Hence, in $\mathbb{L}^4$, we get the action of the Lie group G whose Lie algebra is $\mathfrak{g} = \mathbb{R}(1,0) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0,0,0,1)\big)$, where $(1,0) \in \mathfrak{n} \cong \mathbb{R}^2$. This corresponds, after rearranging components, to Case 2.(b) of Theorem 4.3.1.

(ii) The action of the Lie group whose Lie algebra is $\mathbb{R}(u + (1,0,0)) \oplus \mathbb{R}(1,1,0)$, where u is a unit vector of $\tilde{\mathfrak{n}}$, the nilpotent part of the Iwasawa decomposition of $\mathfrak{so}(1,2)$. This induces the action of the Lie group whose Lie algebra is $\mathfrak{g} = \mathbb{R}((1,0) + (1,0,0,0)) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0,0,0,1)\big)$, where $(1,0) \in \mathfrak{n} \cong \mathbb{R}^2$. This corresponds to Case 2.(c) of Theorem 4.3.1.

(iii) The action of the Lie group whose Lie algebra is $\mathbb{R}(1 + (0,0,\lambda)) \oplus \mathbb{R}(1,1,0)$. This induces the action on $\mathbb{L}^4$ of the Lie group whose Lie algebra is given by $\mathfrak{g} = \mathbb{R}(1 + (0,0,\lambda,0)) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0,0,0,1)\big)$. This corresponds to Case 2.(d) of Theorem 4.3.1.

Finally, we have to study the case $\mathfrak{w} = 0$. To tackle this problem, we have to determine cohomogeneity one actions on $\mathbb{L}^4$ by a connected Lie subgroup G of $I^0(\mathbb{L}^4)$, with Lie algebra $\mathfrak{g}$, in such a way that $\mathfrak{g} \cap \mathbb{L}^4 = \mathbb{R}\mathbf{e}$.

Since $\pi(\mathfrak{g}) \subset \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, we will firstly determine the conjugacy classes of the Lie subalgebras of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. Patrangenaru has given a classification of the Lie subalgebras of $\mathfrak{so}(1,3)$, up to conjugacy in $SO^0(1,3)$, in [65]. This classification result includes the subalgebras of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. However, in our proof it is essential for this classification to be up to conjugacy in K_0AN, so that $\mathbb{R}\mathbf{e}$ remains invariant. Hence, we include the result for $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ below.

Lemma 4.3.8. *If $\mathfrak{h}$ is a nontrivial proper Lie subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$, then $\mathfrak{h}$ is conjugate, by an element of K_0AN, to one of the following:*

(i) *If* $\dim(\mathfrak{h}) = 1$*:* $\mathfrak{k}_0$, $\mathfrak{a}$, $\mathbb{R}(\alpha E + 1)$, $\alpha \neq 0$, *or a one-dimensional subspace of* $\mathfrak{n}$.

(ii) *If* $\dim(\mathfrak{h}) = 2$*:* $\mathfrak{k}_0 \oplus \mathfrak{a}$, $\mathfrak{a} \oplus \mathfrak{v}$, *where* $\mathfrak{v}$ *is a one-dimensional subspace of* $\mathfrak{n}$, *or* $\mathfrak{n}$.

(iii) *If* $\dim(\mathfrak{h}) = 3$*:* $\mathbb{R}(\alpha E + a) \oplus \mathfrak{n}$, *where* $\alpha E + a \neq 0$.

Proof. Assume firstly that $\mathfrak{h}$ is one-dimensional. It is clear that $\mathfrak{k}_0$, $\mathfrak{a}$, $\mathbb{R}(\alpha E+1)$, and $\mathfrak{n}$ are not conjugate because the Jordan canonical forms of their matrix representations are different (pure imaginary diagonal, real diagonal, complex diagonal, and nilpotent, respectively). Assume $\mathfrak{h} = \mathbb{R}(\alpha E + a + u)$. If $\alpha = 0$ and $a = 0$, then we get that $\mathfrak{h}$ is a one-dimensional subspace of $\mathfrak{n}$. If $\alpha = 0$ but $a \neq 0$, conjugating by $\operatorname{Exp}(\frac{1}{a}u)$ we get $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{a}$. Finally, suppose $\alpha \neq 0$. It is clear that $\alpha E + a \operatorname{Id}$ is an isomorphism of $\mathfrak{n} \cong \mathbb{R}^2$. Thus, there exists $v \in \mathfrak{n}$ such that $(\alpha E + a \operatorname{Id})v = -u$. Conjugating $\mathfrak{h}$ by $g = \operatorname{Exp}(v)$, we obtain $\operatorname{Ad}(g)\mathfrak{h} = \mathbb{R}(\alpha E + a)$. If $a = 0$, then $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{k}_0$, whereas if $a \neq 0$, $\operatorname{Ad}(g)\mathfrak{h} = \mathbb{R}(\alpha E/a + 1)$.

Now we assume $\dim(\mathfrak{h}) = 2$. The nonconjugacy of the examples given in the statement is again a consequence of the different Jordan canonical forms of their corresponding matrix representations. If $\mathfrak{h}$ has trivial projection onto $\mathfrak{n}$, we get $\mathfrak{k}_0 \oplus \mathfrak{a}$. Assume that the dimension of the orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{n}$ is one. In such a case, there is a basis $\{\alpha E + a + u, \beta E + b\}$ of $\mathfrak{h}$. We have $[\alpha E + a + u, \beta E + b] = -bu - \beta E u$. Since E acts as a skew-symmetric transformation of $\mathfrak{n}$, Eu is orthogonal to u, which implies $\beta = 0$. We can then take $b = 1$ and $a = 0$. However, the right-hand side of the previous equation is in $\mathfrak{h}$ if, and only if, $\alpha = 0$. Thus, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{v}$, where $\mathfrak{v}$ denotes a one-dimensional subspace of $\mathfrak{n}$.

We deal now with the case $\dim(\mathfrak{h}) = 2$, and where the orthogonal projection of $\mathfrak{h}$ onto $\mathfrak{n}$ is $\mathfrak{n}$. In this case, a basis of $\mathfrak{h}$ can be taken as $\{\alpha E + a + u, \beta E + b + v\}$, where $\{u, v\}$ is an orthonormal basis of $\mathfrak{n}$ in such a way that $Eu = v$, $Ev = -u$. Taking brackets, we obtain $[\alpha E + a + u, \beta E + b + v] = -(b + \alpha)u + (a - \beta)v$. The fact that the right-hand side of this equation is in $\mathfrak{h}$ implies

$$(a - \beta)\beta - \alpha(b + \alpha) = 0, \qquad (a - \beta)b - (b + \alpha)a = 0.$$

If $\beta = \alpha = 0$, we get $\{a + u, b + v\}$ as a basis of $\mathfrak{h}$. In this case, if $a \neq 0$, we conjugate by $g = \operatorname{Exp}(\frac{1}{a}u)$ and we get $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{a} \oplus \mathbb{R}(v - bu/a)$. Similarly, if $b \neq 0$, conjugating $g = \operatorname{Exp}(-\frac{1}{b}v)$ we obtain $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{a} \oplus \mathbb{R}(u + av/b)$. This two cases are of the form $\mathfrak{a} \oplus \mathfrak{v}$, with $\mathfrak{v}$ a one-dimensional subspace of $\mathfrak{n}$. If $a = b = 0$, we get $\mathfrak{h} = \mathfrak{n}$.

If $\beta = 0$ but $\alpha \neq 0$, then $a = 0$ and $b = -\alpha$. We get $\mathfrak{h} = \mathbb{R}(\alpha E + u) \oplus \mathbb{R}(-\alpha + v)$. Conjugating by $g = \operatorname{Exp}(-\frac{1}{\alpha}v)$ gives $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{a}$.

Now assume $\beta \neq 0$. On the one hand, the second equation gives $b = -\alpha a/\beta$, and hence, the first equation transforms into $(a - \beta)(\alpha^2 + \beta^2) = 0$. Since $\beta \neq 0$, we must have $a = \beta$, and thus, $b = -\alpha$. Then, $\mathfrak{h} = \mathbb{R}(\alpha E + \beta + u) \oplus (\beta E - \alpha + v)$. In this case, conjugation by the element

$$g = \operatorname{Exp}\Big(\frac{\beta}{\alpha^2 + \beta^2}u - \frac{\alpha}{\alpha^2 + \beta^2}v\Big)$$

yields, after some elementary calculations, $\operatorname{Ad}(g)\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{a}$ again. This settles the two-dimensional case.

Finally, we assume that $\mathfrak{h}$ is a three-dimensional Lie subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. Then, there exists a vector $\xi \in \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ that is orthogonal to $\mathfrak{h}$. Let us write such a vector as $\xi = \alpha E + a + u$. We prove that $u = 0$. Indeed, there exist $v \in (\mathfrak{n} \ominus \mathbb{R}u) \cap \mathfrak{h}$ and $\beta E + \gamma u \in \mathfrak{h}$. Then, $[\beta E + \gamma u, v] = \beta[E, v] \in \mathfrak{h}$ is not orthogonal to ξ unless $u = 0$. This implies $\mathfrak{h} = \mathbb{R}(\alpha E + a) \oplus \mathfrak{n}$, and finishes the proof of the lemma. $\square$

Now we turn our attention again to the study of cohomogeneity one actions on $\mathbb{L}^4$ with one-dimensional degenerate translational part. This is what we study in the following result.

Lemma 4.3.9. *Let G be a connected Lie subgroup of $I^0(\mathbb{L}^4) = SO^0(1,3) \times_\Phi \mathbb{L}^4$ with Lie algebra $\mathfrak{g}$. Assume that G acts isometrically on $\mathbb{L}^4$ in such a way that $\mathfrak{g} \cap \mathbb{L}^4 = \mathbb{R}\mathbf{e}$. Then, the action of G is orbit equivalent to the action of the connected Lie subgroup of $I^0(\mathbb{L}^4)$ whose Lie algebra is given by one of the following possibilities:*

(i) $\mathbb{W}^3$;

(ii) $(\mathfrak{k}_0 \oplus \mathfrak{n}) \oplus_\phi \mathbb{R}\mathbf{e}$;

(iii) $(\mathfrak{k}_0 \oplus \mathfrak{a}) \oplus_\phi \mathbb{R}\mathbf{e}$;

(iv) $(\mathfrak{a} \oplus \mathfrak{v}) \oplus_\phi \mathbb{R}\mathbf{e}$, *where $\mathfrak{v}$ is a one-dimensional subspace of $\mathfrak{n}$;*

(v) $\big(\mathbb{R}(1 + (0, 0, b)) \oplus \mathfrak{v}\big) \oplus_\phi \mathbb{R}\mathbf{e}$, *where $\mathfrak{v}$ denotes a one-dimensional subspace of $\mathfrak{n}$ and $b \in \mathbb{R}^2$;*

(vi) $\mathfrak{n} \oplus_\phi \mathbb{R}\mathbf{e}$;

(vii) $\big(\mathbb{R}(u + (0, 0, x)) \oplus \mathbb{R}(v + (0, 0, y))\big) \oplus_\phi \mathbb{R}\mathbf{e}$, *where $\{u, v\}$ is an orthonormal basis of $\mathfrak{n}$, and x, $y \in \mathbb{R}^2$.*

Proof. As we have seen, $\pi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{g} \cap \mathbb{L}^4 = \mathbb{R}\mathbf{e}$. If there is an orbit through $\mathbf{p} \in \mathbb{L}^4$ with $\dim(G \cdot \mathbf{p}) = 3$, then $\dim(G) \geq 3$, which implies $\dim(\pi(\mathfrak{g})) \geq 2$. The conjugacy classes of subalgebras of $\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ by elements of K_0AN are a consequence of Lemma 4.3.8.

We start with the case $\pi(\mathfrak{g}) = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$. As before, let us consider an orthonormal basis $\{u, v\}$ of $\mathfrak{n}$ in such a way that $Eu = v$, $Ev = -u$. Then, a basis of $\mathfrak{g}$ can be written as

$$\{E + (a_0, a_1, a), 1 + (b_0, b_1, b), u + (x_0, x_1, x), v + (y_0, y_1, y), \mathbf{e}\},$$

where a_0, a_1, b_0, b_1, x_0, x_1, y_0, $y_1 \in \mathbb{R}$, and a, b, x, $y \in \mathbb{R}^2$. In order to shorten the notation, let us write $a_2 = \langle a, u\rangle$, $a_3 = \langle a, v\rangle$, $b_2 = \langle b, u\rangle$, $b_3 = \langle b, v\rangle$, $x_2 = \langle x, u\rangle$, $x_3 = \langle x, v\rangle$, $y_2 = \langle y, u\rangle$, and $y_3 = \langle y, v\rangle$ The relevant Lie brackets that we need here are

$$\begin{aligned}
&[E + (a_0, a_1, a), 1 + (b_0, b_1, b)] = (-a_1, -a_0, -b_3 u + b_2 v),\\
&[u + (x_0, x_1, x), v + (y_0, y_1, y)] = (y_2 - x_3, y_2 - x_3, (y_0 - y_1)u - (x_0 - x_1)v),\\
&[E + (a_0, a_1, a), u + (x_0, x_1, x)] = v + (-a_2, -a_2, -(x_3 + (a_0 - a_1))u + x_2 v),\\
&[1 + (b_0, b_1, b), u + (x_0, x_1, x)] = u + (x_1 - b_2, x_0 - b_2, (b_1 - b_0)u),\\
&[1 + (b_0, b_1, b), v + (y_0, y_1, y)] = v + (y_1 - b_3, y_0 - b_3, (b_1 - b_0)v).
\end{aligned}$$

The fact that the right-hand sides of the previous equations are elements of $\mathfrak{g}$ has the following implications. From the first equation, we get $\langle b, u\rangle = \langle b, v\rangle = 0$, which means $b = 0$. Moreover, $(-a_1, -a_0)$ must be proportional to $(1, 1)$, so $a_0 = a_1$, but since $\mathbf{e} \in \mathfrak{g}$, we may subtract $a_0\mathbf{e}$, and thus, we can assume $a_0 = a_1 = 0$. The second equation gives $x_0 = x_1$, $y_0 = y_1$, and since $\mathbf{e} \in \mathfrak{g}$, we may set $x_0 = x_1 = y_0 = y_1 = 0$. Now, the right-hand side of the third equation must be the sum of $v + (0, 0, y)$ and a multiple of $\mathbf{e}$. Then, $\langle y, u\rangle = -\langle x, v\rangle$ and $\langle y, v\rangle = \langle x, u\rangle$. From the last two equations we readily have $\langle x, u\rangle = b_1 - b_0$ and $\langle x, v\rangle = 0$. Thus, our original basis reads now

$$\{E + (0, 0, a), 1 + (b_0, b_1, 0), u + (0, 0, (b_1 - b_0)u), v + (0, 0, (b_1 - b_0)v), \mathbf{e}\}.$$

Consider the element $g = (I, (b_1, b_0, a_3u - a_2v))$. Then, we have

$$\begin{aligned} &\mathrm{Ad}(g)(E + (0, 0, a)) = E, && \mathrm{Ad}(g)(u + (0, 0, (b_1 - b_0)u)) = u - \langle a, v\rangle\mathbf{e}, \\ &\mathrm{Ad}(g)(1 + (b_0, b_1, 0)) = 1, && \mathrm{Ad}(g)(v + (0, 0, (b_1 - b_0)v)) = v + \langle a, u\rangle\mathbf{e}, \\ &\mathrm{Ad}(g)(\mathbf{e}) = \mathbf{e}, \end{aligned}$$

which implies $\mathrm{Ad}(g)\mathfrak{g} = (\mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}) \oplus_\phi \mathbb{R}\mathbf{e}$. Since one of the orbits of K_0AN is a real hyperbolic space and $\mathbb{R}\mathbf{e}$ is transversal to it, it follows that the action of its corresponding Lie subgroup is of cohomogeneity zero.

Assume now that $\pi(\mathfrak{g}) = \mathbb{R}(\alpha E + a) \oplus \mathfrak{n}$ and consider the basis

$$\{(\alpha E + a) + (b_0, b_1, b), u + (x_0, x_1, x), v + (y_0, y_1, y), \mathbf{e}\},$$

where α, a, b_0, b_1, x_0, x_1, y_0, $y_1 \in \mathbb{R}$, b, x, $y \in \mathbb{R}^2$, and $\{u, v\}$ is an orthonormal basis of $\mathfrak{n}$ in such a way that $Eu = v$, $Ev = -u$. In order to shorten the notation, let us write $b_2 = \langle b, u\rangle$, $b_3 = \langle b, v\rangle$, $x_2 = \langle x, u\rangle$, $x_3 = \langle x, v\rangle$, $y_2 = \langle y, u\rangle$, and $y_3 = \langle y, v\rangle$. Firstly, we have

$$[u + (x_0, x_1, x), v + (y_0, y_1, y)] = (y_2 - x_3, y_2 - x_3, (y_0 - y_1)u - (x_0 - x_1)v).$$

Since the right-hand side of the previous equation lies in $\mathfrak{g}$, we have $x_0 = x_1$, $y_0 = y_1$. Since $\mathbf{e} \in \mathfrak{g}$, we can take $x_0 = x_1 = y_0 = y_1 = 0$. Considering these facts, we have

$$\begin{aligned} [(\alpha E + a) + (b_0, b_1, b), u + (0, 0, x)] &= (au + \alpha v) - (b_2, b_2, (b_0 - b_1 + \alpha x_3)u - \alpha x_2 v), \\ [(\alpha E + a) + (b_0, b_1, b), v + (0, 0, y)] &= (av - \alpha u) - (b_3, b_3, \alpha y_3 u - (b_1 - b_0 + \alpha y_2)v), \end{aligned}$$

which implies

$$\begin{aligned} b_1 - b_0 - \alpha x_3 &= ax_2 + \alpha y_2, & \alpha x_2 &= ax_3 + \alpha y_3, \\ -\alpha y_3 &= -\alpha x_2 + ay_2, & b_1 - b_0 + \alpha y_2 &= -\alpha x_3 + ay_3. \end{aligned} \tag{4.1}$$

If $a \neq 0$, we can assume $a = 1$ (just changing the first element of the basis by itself divided by a). We get $x_2 = b_1 - b_0$, $x_3 = 0$, $y_2 = 0$, and $y_3 = b_1 - b_0$. Considering the element $g = (I, (b_1, b_0, 0))$, we obtain $\mathrm{Ad}(g)\mathfrak{g} = \mathbb{R}(\alpha E + 1) + (0, 0, b)) \oplus \mathfrak{n} \oplus_\phi \mathbb{R}\mathbf{e}$. For a given $\mathbf{p} = (p_0, p_1, p) \in \mathbb{L}^4$ we have

$$\begin{aligned} ((\alpha E + 1) + (0, 0, b)) \cdot \mathbf{p} &= (p_1, p_0, (b_2 - \alpha\langle p, v\rangle)u + (b_3 + \alpha\langle p, u\rangle)v), \\ u \cdot \mathbf{p} &= (\langle p, u\rangle, \langle p, u\rangle, (p_0 - p_1)u), \\ v \cdot \mathbf{p} &= (\langle p, v\rangle, \langle p, v\rangle, (p_0 - p_1)v). \end{aligned}$$

Recall that $\mathrm{Ad}(g)\mathfrak{g}\cdot\mathbf{p}$ can be identified with the tangent space of the orbit of gGg^{-1} through $\mathbf{p}$. Then, since

$$\begin{vmatrix} p_1 & \langle p,u\rangle & \langle p,v\rangle & 1 \\ p_0 & \langle p,u\rangle & \langle p,v\rangle & 1 \\ b_2-\alpha\langle p,v\rangle & p_0-p_1 & 0 & 0 \\ b_3+\alpha\langle p,u\rangle & 0 & p_0-p_1 & 0 \end{vmatrix} = -(p_0-p_1)^3,$$

we conclude that there are orbits of dimension 4 whenever $p_0 \neq p_1$, which implies that G acts with cohomogeneity zero.

It remains to deal with the case $a=0$. We can set $\alpha=1$. Solving (4.1) yields $y_2=-x_3$, $y_3=x_2$, and $b_1=b_0$. Conjugating by $g=(I,(x_2,0,b_3u-b_2v))$, we get

$$\begin{aligned} &\mathrm{Ad}(g)(E+(0,0,b))=E, && \mathrm{Ad}(g)(\mathbf{e})=\mathbf{e}, \\ &\mathrm{Ad}(g)(u+(0,0,x))=x_3v-b_3\mathbf{e}, && \mathrm{Ad}(g)(v+(0,0,-x_3u+x_2v)=-x_3u+b_2\mathbf{e}. \end{aligned}$$

Thus, $\{E, u+(0,0,\lambda v), v+(0,0,-\lambda u), \mathbf{e}\}$, with $\lambda\in\mathbb{R}$, is a basis of $\mathfrak{g}$. In order to determine the tangent space at a point $\mathbf{p}=(p_0,p_1,p)\in\mathbb{L}^4$, we calculate

$$\begin{aligned} E\cdot\mathbf{p} &= -\langle p,v\rangle u+\langle p,u\rangle v, \\ (u+(0,0,\lambda v))\cdot\mathbf{p} &= \langle p,u\rangle\mathbf{e}+(p_0-p_1)u+\lambda v, \\ (v+(0,0,-\lambda u))\cdot\mathbf{p} &= \langle p,v\rangle\mathbf{e}-\lambda u+(p_0-p_1)v. \end{aligned}$$

If $\lambda=0$, it follows from the previous equations that $\mathfrak{g}\cdot\mathbf{p}=\mathbb{R}\mathbf{e}\oplus\mathbb{R}u\oplus\mathbb{R}v$ whenever $p_0\neq p_1$, but $\mathfrak{g}\cdot(p_0,p_0,p)=\mathbb{R}\mathbf{e}\oplus\mathbb{R}(-\langle p,v\rangle u+\langle p,u\rangle v)$. However, if $\lambda\neq 0$, the tangent space is always $\mathfrak{g}\cdot\mathbf{p}=\mathbb{R}\mathbf{e}\oplus\mathbb{R}u\oplus\mathbb{R}v\cong\mathbb{R}\mathbf{e}\oplus\mathbb{R}^2=\mathbb{W}^3$. This means that, using Proposition 4.1.3, for $\lambda\neq 0$, the action of G is orbit equivalent to the action of $\mathbb{W}^3$ by translations, which corresponds to Case (i) of Lemma 4.3.9. If $\lambda=0$ we get, up to orbit equivalence, the action of $K_0N\times_\phi\mathbb{R}\mathbf{e}$, as stated in Case (ii) of Lemma 4.3.9.

Assume now $\pi(\mathfrak{g})=\mathfrak{k}_0\oplus\mathfrak{a}$ and consider the following basis of $\mathfrak{g}$:

$$\{E+(a_0,a_1,a_2,a_3), 1+(b_0,b_1,b_2,b_3), \mathbf{e}\},$$

where a_i, $b_i\in\mathbb{R}$, for $i\in\{0,1,2,3\}$. Since

$$[E+(a_0,a_1,a_2,a_3), 1+(b_0,b_1,b_2,b_3)] = (-a_1,-a_0,-b_3,b_2)\in\mathfrak{g},$$

it follows that $a_0=a_1$ and $b_2=b_3=0$. Moreover, since $\mathbf{e}\in\mathfrak{g}$, we can assume $a_0=a_1=0$. A simple calculation shows that $\mathrm{Ad}(I,(b_1,b_0,a_3,-a_2))\mathfrak{g}=(\mathfrak{k}_0\oplus\mathfrak{a})\oplus_\phi\mathbb{R}\mathbf{e}$. Thus, we can assume $\mathfrak{g}=(\mathfrak{k}_0\oplus\mathfrak{a})\oplus_\phi\mathbb{R}\mathbf{e}$.

We calculate the tangent space of an orbit through $\mathbf{p}=(p_0,p_1,p)$. If $p_0=p_1$, then $\mathfrak{g}\cdot\mathbf{p}=\mathbb{R}\mathbf{e}\oplus(\mathbb{R}^2\ominus\mathbb{R}p)$, whereas if $p_0\neq p_1$ we have $\mathfrak{g}\cdot\mathbf{p}=\mathbb{L}^2\oplus(\mathbb{R}^2\ominus\mathbb{R}p)$. Thus, this action is of cohomogeneity one and corresponds to Case (iii) of Lemma 4.3.9.

We deal now with the case $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{v}$, where $\mathfrak{v}$ is a one-dimensional subspace of $\mathfrak{n}$. Let $u\in\mathfrak{n}$ be a unit vector in such a way that $\mathfrak{v}=\mathbb{R}u$. We can take a basis of $\mathfrak{g}$ of the form

$$\{1+(b_0,b_1,b),u+(x_0,x_1,x),\mathbf{e}\}.$$

Taking brackets, we have

$$[1+(b_0,b_1,b),u+(x_0,x_1,x)] = u+(x_1-\langle b,u\rangle, x_0-\langle b,u\rangle,(b_1-b_0)u) \in \mathfrak{g}.$$

Since this element must be the sum of $u+(x_0,x_1,x)$ and a multiple of $\mathbf{e}$, we have $x_0 = x_1$, $\langle x,u\rangle = b_1-b_0$, and $\langle x,v\rangle = 0$. Since $\mathbf{e}\in\mathfrak{g}$, we can set $x_0 = x_1 = 0$. Conjugating by $(I,(b_1,b_0,0))$ yields the new basis $\{1+(0,0,b),u,\mathbf{e}\}$, with $b\in\mathbb{R}^2$.

We calculate the tangent space $\mathfrak{g}\cdot\mathbf{p}$, where $\mathbf{p} = (p_0,p_1,p)\in\mathbb{L}^4$. Assume $b=0$. If $p_0 = p_1$, then $\mathfrak{g}\cdot\mathbf{p} = \mathbb{R}\mathbf{e}$, whereas if $p_0\neq p_1$ then $\mathfrak{g}\cdot\mathbf{p} = \mathbb{L}^2\oplus\mathbb{R}u$, which is three-dimensional. Now, assume $b\neq 0$. If $p_0 = p_1$, then $\mathfrak{g}\cdot\mathbf{p} = \mathbb{R}\mathbf{e}\oplus\mathbb{R}b$, whereas if $p_0\neq p_1$ then $\mathfrak{g}\cdot\mathbf{p}$ is again three-dimensional. Therefore, the actions with $b=0$ and $b\neq 0$ cannot be orbit equivalent because in the latter case there are no one-dimensional orbits. They correspond to Cases (iv) and (v) of Lemma 4.3.9.

Finally, assume $\pi(\mathfrak{g}) = \mathfrak{n}$. Let $\{u,v\}$ be an orthonormal basis of $\mathfrak{n}$ and consider the following basis of $\mathfrak{g}$:

$$\{u+(x_0,x_1,x), v+(y_0,y_1,y),\mathbf{e}\},$$

where x_0, x_1, y_0, $y_1\in\mathbb{R}$, and x, $y\in\mathbb{R}^2$. Since

$$[u+(x_0,x_1,x),v+(y_0,y_1,y)] = (\langle y,u\rangle-\langle x,v\rangle,\langle y,u\rangle-\langle x,v\rangle,(y_0-y_1)u+(x_1-x_0)v)\in\mathfrak{g},$$

we can set $x_0 = x_1 = y_0 = y_1 = 0$. For $\mathbf{p} = (p_0,p_1,p)\in\mathbb{L}^4$, we have

$$(u+(0,0,x))\cdot\mathbf{p} = \langle p,u\rangle\mathbf{e}+(p_0-p_1)u+x,$$
$$(v+(0,0,y))\cdot\mathbf{p} = \langle p,v\rangle\mathbf{e}+(p_0-p_1)v+y.$$

Thus, $\mathfrak{g}\cdot\mathbf{p} = \operatorname{span}\{\mathbf{e},(p_0-p_1)u+x,(p_0-p_1)v+y\}$. This implies that the dimension of an orbit of G depends on the rank of the matrix

$$\begin{pmatrix}\langle x,u\rangle-(p_1-p_0) & \langle y,u\rangle\\ \langle x,v\rangle & \langle y,v\rangle-(p_1-p_0)\end{pmatrix}.$$

Assume $\langle x,v\rangle = \langle y,u\rangle = 0$ and $\lambda := \langle x,u\rangle = \langle y,v\rangle$. Conjugating by the element $g = (I,(\lambda,0,0,0))$, one obtains $\operatorname{Ad}(g)\mathfrak{g} = \mathfrak{n}\oplus_\phi\mathbb{R}\mathbf{e}$. For a point $\mathbf{p} = (p_0,p_1,p)\in\mathbb{L}^4$, we have $\operatorname{Ad}(g)(\mathfrak{g})\cdot\mathbf{p} = \mathbb{R}\mathbf{e}$ if $p_0 = p_1$, and $\operatorname{Ad}(g)(\mathfrak{g})\cdot\mathbf{p} = \mathbb{R}\mathbf{e}\oplus\mathbb{R}^2$ if $p_1\neq p_0$. Hence, the corresponding action is of cohomogeneity one and corresponds to Case (vi) of Lemma 4.3.9.

Otherwise, the matrix above can never be identically zero. Its rank is 2 whenever

$$(p_1-p_0)^2-(\langle x,u\rangle+\langle y,v\rangle)(p_1-p_0)+\langle x,u\rangle\langle y,v\rangle-\langle x,v\rangle\langle y,u\rangle\neq 0,$$

that is, when p_1-p_0 is not an eigenvalue of the matrix $(x|y)$ whose columns are the vectors x and y. Thus, the action of G is in this case of cohomogeneity one. This corresponds to Case (vii) of Lemma 4.3.9. □

The results of this section imply part (2) of Theorem 4.3.1. Indeed, let G be a connected Lie subgroup of $I^0(\mathbb{L}^4)$ acting with cohomogeneity one on $\mathbb{L}^4$. Let $\mathfrak{g}$

denote the Lie algebra of G and consider $\mathfrak{v} = \mathfrak{g} \cap \mathbb{L}^4$ its pure transaltional part. Recall from Lemma 4.3.6 that, if $\mathfrak{v}$ is a degenerate subspace of $\mathbb{L}^4$, then it can be written as $\mathfrak{v} = \mathbb{R}\mathbf{e} \oplus \mathfrak{w}$, where $\mathbf{e} = (1, 1, 0, 0)$ and $\mathfrak{w}$ is a Riemannian subspace. Moreover, under these conditions, Lemma 4.3.7 ensures that the action of G reduces to a cohomogeneity one action on $\mathbb{L}^4 \ominus \mathfrak{w}$ whose pure translational part is $\mathbb{R}\mathbf{e}$. We have the following possibilities:

- If $\dim(\mathfrak{w}) = 2$, the action of G is orbit equivalent to the action of $\mathbb{W}^3$ by translations, which corresponds to Case 2.(a) of Theorem 4.3.1.
- If $\dim(\mathfrak{w}) = 1$, the action of G is orbit equivalent to the action of the connected Lie subgroup of $I^0(\mathbb{L}^4)$ whose Lie algebra is one of the following:
  - $\mathbb{R}(1, 0) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0, 0, 0, 1)\big)$, which corresponds to Case 2.(b) of Theorem 4.3.1;
  - $\mathbb{R}\big((1, 0)+(1, 0, 0, 0)\big) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0, 0, 0, 1)\big)$, which corresponds to Case 2.(c) of Theorem 4.3.1;
  - $\mathbb{R}\big(1 + (0, 0, \lambda, 0)\big) \oplus_\phi \big(\mathbb{R}\mathbf{e} \oplus \mathbb{R}(0, 0, 0, 1)\big)$, which corresponds to Case 2.(d) of Theorem 4.3.1.
- Finally, if $\dim(\mathfrak{w}) = 0$, the action of G is orbit equivalent to the action of the connected Lie subgroup of $I^0(\mathbb{L}^4)$ whose Lie algebra is one of the subalgebras given in Lemma 4.3.9. More specifically, if $\mathfrak{g} \in \{(\mathfrak{k}_0 \oplus \mathfrak{n}) \oplus_\phi \mathbb{R}\mathbf{e},\ \mathfrak{n} \oplus_\phi \mathbb{R}\mathbf{e}\}$ we obtain Case 2.(e) of Theorem 4.3.1, whereas Cases (i), (iii), (iv), (v) and (vii) of Lemma 4.3.9 correspond to Cases 2.(a), 2.(f), 2.(g), 2.(h) and 2.(i) of Theorem 4.3.1, respectively.

Conclusions and open problems

The first contribution of this thesis consists of several classification results of ruled real hypersurfaces satisfying some additional geometric properties in nonflat complex space forms, namely complex projective and hyperbolic spaces (see Chapter 2). More specifically, we have obtained the following results:

- We have proved that any ruled real hypersurface having constant mean curvature in a nonflat complex space form must be minimal. Then, the classification of ruled real hypersurfaces with constant mean curvature in nonflat complex space forms follows from a known result due to Lohnherr and Reckziegel.

- We have obtained a complete classification of ruled real hypersurfaces whose shape operators have constant norm in nonflat complex space forms. In particular, this classification result contains a new inhomogeneous example in the complex hyperbolic space, whereas there are no examples in complex projective spaces.

- We have proved that every biharmonic ruled real hypersurface in a nonflat complex space form must be minimal. Their classification follows from a known result due to Lohnherr and Reckziegel.

In Chapter 3 we have focused on the study of homogeneous CR submanifolds in complex hyperbolic spaces. In particular, we have derived the following results:

- We have proved a result that allows us to characterize homogeneous CR submanifolds in Hermitian symmetric spaces of noncompact type in terms of the Lie algebras of the Lie subgroups determining such submanifolds.

- We have obtained the classification of homogeneous CR submanifolds in complex hyperbolic spaces that arise as orbits of connected Lie subgroups of the solvable part of the Iwasawa decomposition of the isometry group of the ambient space. We have also studied the congruence classes of the examples in this classification.

Finally, in Chapter 4, we have studied cohomogeneity one actions on Minkowski spacetimes. These are the main achievements we have obtained:

- We have given an alternative proof for the classification of cohomogeneity one actions on Euclidean spaces using Lie group theory.

- We have obtained some structural results in relation to the study of cohomogeneity one actions on Minkowski spacetimes $\mathbb{L}^{n+1}$.
- We have derived the classification of cohomogeneity one actions on the four-dimensional Minkowski spacetime $\mathbb{L}^4$, up to orbit equivalence, finding the existence of examples of actions with degenerate orbits.

In view of these results, there are still some open problems and questions to be solved in relation to the geometric objects investigated in this thesis. We comment on the following ones:

- Obtain new examples of ruled real hypersurfaces satisfying additional geometric properties of interest.
- Characterize ruled real hypersurfaces in nonflat complex space forms whose principal curvatures are related by means of a general quadratic function.
- Study biharmonic Hopf hypersurfaces in nonflat complex space forms. See [67] for some partial result in this line.
- Complete the classification of homogeneous CR submanifolds in complex hyperbolic spaces. This seems to be a very involved problem, so any progress may be of interest. The particular case of homogeneous Lagrangian submanifolds appears to be of special relevance.
- Investigate homogeneous CR submanifolds in complex projective spaces.
- Extend the results obtained in Chapter 3 to the entire context of Hermitian symmetric spaces of noncompact type.
- Apply the methods and techniques developed in Chapter 4 to the study of not necessarily proper cohomogeneity one actions on Minkowski spacetimes of arbitrary dimension.
- Develop new methods in order to classify cohomogeneity one actions on the setting of semi-Riemannian space forms.

Resumo en galego

Intuitivamente, a simetría é a propiedade dos obxectos que fai que resulten semellantes cando os observamos desde diferentes perspectivas. Trátase dunha cualidade importante desde o punto de vista de disciplinas coma a bioloxía, a química ou a arte. O concepto de simetría tamén ten cabida no campo das matemáticas. Neste contexto, convén sinalar que a simetría non é unha propiedade unicamente aplicable a figuras xeométricas, senón que outros obxectos máis abstractos tamén poden ter simetrías. O estudo de ditos obxectos deu lugar a importantes resultados en diversas áreas das matemáticas. Por exemplo, a teoría de Galois asegura que se unha ecuación polinómica non ten as simetrías axeitadas, entón non é resoluble por radicais. Outro exemplo constitúeno os coñecidos teoremas de Noether, que afirman que as simetrías dun sistema físico se traducen en leis de conservación.

O concepto de simetría pode definirse de xeito rigoroso empregando a terminoloxía da teoría de grupos. Así, dado un grupo G, dise que un obxecto é G-simétrico se é invariante baixo a acción de G. En particular, a noción de simetría sempre está vinculada á acción dun grupo que, de feito, especifica o tipo de simetría que un determinado obxecto ten.

No marco da xeometría semi-riemanniana, é natural considerar como obxecto de estudo o grupo de isometrías da variedade, é dicir, o grupo constituído polas transformacións da variedade que preservan a súa métrica. A acción dun subgrupo do grupo de isometrías nunha determinada variedade semi-riemanniana denomínase *acción isométrica*. Unha variedade semi-riemanniana dise homoxénea se o seu grupo de isometrías actúa transitivamente sobre ela. Unha subvariedade é *(extrinsecamente) homoxénea* se coincide coa órbita da acción dun subgrupo do grupo de isometrías da variedade ambiente.

A experiencia demostrou que o problema de clasificación de accións isométricas ou, equivalentemente, de subvariedades homoxéneas, nunha determinada variedade semi-riemanniana pode resultar moi complicado. Por esta razón, é usual restrinxir a nosa atención e limitarnos ao estudo de certos tipos específicos de accións isométricas que sexan máis manexables. Por exemplo, o estudo de accións transitivas, isto é, aquelas nas que a propia variedade é a única órbita da acción, foi abordado en diversos contextos.

As *accións de cohomoxeneidade un* constitúen outra clase particular de accións isométricas en variedades semi-riemannianas que teñen atraído considerable atención. Unha acción isométrica dise de cohomoxeneidade un se as súas órbitas teñen codimensión un. No contexto particular da xeometría lorentziana non existen demasiados resultados de clasificación en relación a este tipo de accións isométricas. Un dos principais obxectivos desta tese é o de estudar e clasificar accións de cohomoxeneidade un en variedades de Lorentz, salvo equivalencia de órbitas. Máis concretamente, abordamos este problema tomando como variedade ambiente o *espazo-tempo de Minkowski*, é dicir, o análogo ao espazo euclidiano con signatura lorentziana.

Non obstante, no marco da xeometría de Riemann si se obtiveron numerosos e importantes resultados en relación ao problema de clasificación de accións de cohomoxeneidade un ou, equivalentemente, de hipersuperficies homoxéneas. Por exemplo, a clasificación de accións de cohomoxeneidade un nos espazos de curvatura constante, é dicir, os espazos euclidianos $\mathbb{R}^n$, as esferas S^n e os espazos hiperbólicos reais $\mathbb{R}H^n$, foi abordado e, finalmente, resolto por varios autores. Máis concretamente, a clasificación de hipersuperficies homoxéneas nos espazos euclidianos dedúcese de dous traballos de Levi-Civita [49] e Segre [68] dedicados ao estudo das hipersuperficies isoparamétricas de ditos espazos. A clasificación en esferas, que resultou ser un problema moito máis complexo, séguese dun traballo de Hsiang e Lawson [44] relacionado con subvariedades minimais neste tipo de variedades. As hipersuperficies homoxéneas nos espazos hiperbólicos reais foran clasificadas bastante antes por Cartan en [20].

A complexidade do problema de clasificación de hipersuperficies homoxéneas nunha determinada variedade riemanniana aumenta considerablemente cando dita variedade está dotada dunha estrutura de Kähler. Os exemplos máis sinxelos de variedades de Kähler son os *espazos forma complexos*, constituídos por tres importantes familias de variedades: os espazos euclidianos complexos $\mathbb{C}^n$, os espazos proxectivos complexos $\mathbb{C}P^n$ e os espazos hiperbólicos complexos $\mathbb{C}H^n$. O problema de clasificación de hipersuperficies homoxéneas neste tipo de variedades foi abordado e concluído por diversos autores. En particular, a clasificación no caso proxectivo resolveuna Takagi en [70], mentres que as hipersuperficies homoxéneas nos espazos hiperbólicos complexos foron clasificadas por Berndt e Tamaru en [15].

Un dos principais obxectivos desta tese é o estudo da xeometría de subvariedades no contexto dos espazos forma complexos non chans, é dicir, nos espazos proxectivo e hiperbólico complexos. Por suposto, a estrutura complexa subxacente xoga un papel fundamental á hora de resolver problemas neste marco de traballo. En xeral, no contexto das variedades de Kähler e, máis concretamente, no dos espazos forma complexos, é posible definir os conceptos de subvariedades complexas e totalmente reais, que dependen precisamente da estrutura complexa da variedade ambiente. A noción de *subvariedade CR* constitúe unha xeneralización destes dous tipos de subvariedades. Nesta tese estudamos subvariedades CR homoxéneas no espazo hiperbólico complexo empregando, para tal fin, a estrutura de grupo de Lie do grupo de isometrías de dita variedade ambiente. Existen varios exemplos coñecidos de subvariedades CR homoxéneas no espazo hiperbólico complexo que motivan este problema, tales como as *subvariedades de Berndt-Brück* [11] e, en particular, as *hipersuperficies de Lohnherr* [50].

A hipersuperficie de Lohnherr de $\mathbb{C}H^n$ satisfai numerosas propiedades importantes que a convirten nun atractivo obxecto de estudo. Por exemplo, pode ser caracterizada como a única hipersuperficie minimal e homoxénea no espazo hiperbólico complexo [13]. Caracterízase tamén por ser a única hipersuperficie en $\mathbb{C}H^n$ con curvaturas principais constantes que é, ademais, *regrada* [51]. O concepto de hipersuperficie regrada nun espazo forma complexo garda unha estreita relación coa estrutura complexa da variedade ambiente. Nos últimos anos obtivéronse diversos traballos sobre o estudo de hipersuperficies reais regradas nos espazos forma complexos. Nesta tese centrámonos tamén na clasificación de certas hipersuperficies regradas que satisfán determinadas propiedades xeométricas adicionais nos espazos proxectivo e hiperbólico complexos.

A continuación, presentamos os resultados orixinais desta tese.

Hipersuperficies reais regradas en espazos forma complexos

Unha *hipersuperficie real regrada* nun espazo proxectivo ou hiperbólico complexo $\mathbb{C}P^n$ ou $\mathbb{C}H^n$ é unha subvariedade de codimensión real un que está foliada localmente por hipersuperficies complexas totalmente xeodésicas da variedade ambiente, máis concretamente, $\mathbb{C}P^{n-1}$ ou $\mathbb{C}H^{n-1}$, respectivamente.

As hipersuperficies reais regradas nos espazos forma complexos non chans constitúen unha clase moi ampla de hipersuperficies reais. Por este motivo, é natural centrarse no estudo deste tipo de hipersuperficies impoñendo algunha condición xeométrica adicional. Por exemplo, Lohnherr e Reckziegel obtiveron en [51] a clasificación das hipersuperficies regradas minimais nos espazos proxectivo e hiperbólico complexos. Máis concretamente, se M denota unha hipersuperficie minimal regrada nun espazo forma complexo non chan, entón M debe ser unha parte aberta dalgunha das seguintes hipersuperficies:

(i) unha hipersuperficie de tipo Kimura (cf. [47]) en $\mathbb{C}P^n$ o $\mathbb{C}H^n$,

(ii) un bisector en $\mathbb{C}H^n$, ou

(iii) unha hipersuperficie de Lohnherr en $\mathbb{C}H^n$.

No Capítulo 2 desta tese abordamos o problema de clasificación de hipersuperficies reais regradas nos espazos forma complexos non chans que satisfán, adicionalmente, certas propiedades relacionadas coa constancia das curvaturas medias de orde superior. En xeral, defínense as curvaturas medias de orde superior dunha hipersuperficie como os polinomios simétricos elementais nas variables dadas polas curvaturas principais de dita hipersuperficie. É un feito coñecido que toda hipersuperficie regrada nun espazo forma complexo non chan ten exactamente dúas curvaturas principais non nulas, α e β. Polo tanto, existirán unicamente dous polinomios simétricos elementais non triviais: a curvatura media (de orde un), $\alpha+\beta$, e a curvatura media de orde dous, $\alpha\beta$. Dedicamos a Sección 2.3 á clasificación de hipersuperficies regradas con curvatura media constante nos espazos proxectivo e hiperbólico complexos. Máis concretamente, demostramos o seguinte resultado, que deu lugar ao artigo [36].

Teorema 1. *Sexa M unha hipersuperficie real regrada con curvatura media constante nun espazo proxectivo ou hiperbólico complexo. Entón, M é minimal.*

A clasificación de hipersuperficies reais regradas con curvatura media constante obtense a partir deste teorema, empregando a clasificación de hipersuperficies regradas minimais de Lohnherr e Reckziegel [51] previamente citada.

Por outra banda, as hipersuperficies reais regradas en $\mathbb{C}P^n$ e $\mathbb{C}H^n$ cuxa curvatura media de orde dous é constante foron recentemente caracterizadas por Kimura, Maeda e Tanabe en [48].

A norma ao cadrado do operador de configuración, $\alpha^2 + \beta^2$, que pode expresarse dun xeito sinxelo en termos das curvaturas medias de primeira e segunda orde, constitúe un terceiro invariante xeométrico clásico no estudo das hipersuperficies. É natural interesarse, polo tanto, por aquelas hipersuperficies reais regradas nos espazos forma complexos non chans tales que a norma do seu operador de configuración é constante. Tratamos este problema na Sección 2.4, na cal obtemos unha clasificación completa que inclúe un novo exemplo non homoxéneo. Máis concretamente, probamos o seguinte resultado. A súa demostración deu lugar aos artigos [37] e [66].

Teorema 2. *Sexa M unha hipersuperficie real regrada nun espazo proxectivo ou hiperbólico complexo. Entón, a norma do operador de configuración de M é constante se, e só se, M é unha parte aberta dalgunha das seguintes hipersuperficies:*

1. *unha hipersuperficie de Lohnherr en $\mathbb{C}H^n$, ou*

2. *a hipersuperficie real regrada construída adxuntando $\mathbb{C}H^{n-1}$ totalmente xeodésicos perpendicularmente a un círculo de curvatura $\kappa = \sqrt{-c/2}$ nunha recta hiperbólica complexa totalmente xeodésica $\mathbb{C}H^1$, onde $c < 0$ denota a curvatura seccional holomorfa do espazo ambiente $\mathbb{C}H^n$.*

En particular, todos os exemplos proporcionados por este teorema de clasificación satisfán a propiedade de ser hipersuperficies fortemente 2-Hopf. A noción de hipersuperficie fortemente 2-Hopf foi introducida en [28] e resulta ser unha propiedade importante á hora de construír o exemplo non homoxéneo de hipersuperficie real regrada cuxo operador de configuración ten norma constante proporcionado neste teorema.

Para finalizar, centramos a nosa atención nas hipersuperficies reais regradas nos espazos forma complexos non chans que son, ademais, biharmónicas. O concepto de hipersuperficie biharmónica xeneraliza o de hipersuperficie minimal. Con máis precisión, unha hipersuperficie é biharmónica se a inmersión isométrica que a define é un punto crítico do funcional de bienerxía ou, equivalentemente, se o seu campo de bitensión asociado é identicamente nulo.

Motivados por un recente teorema de Sasahara [67] no que presenta unha clasificación das hipersuperficies reais regradas biharmónicas no espazo proxectivo complexo, dedicamos a Sección 2.5 a estender este resultado ao contexto xeral dos espazos forma complexos non chans. Máis concretamente, probamos o seguinte teorema, que pode atoparse no artigo [66].

Teorema 3. *Sexa M unha hipersuperficie real regrada nun espazo proxectivo ou hiperbólico complexo. Entón, M é biharmónica se, e só se, é minimal.*

Novamente, a clasificación das hipersuperficies reais regradas biharmónicas nos espazos forma complexos non chans obtense ao combinar este teorema co resultado de clasificación de hipersuperficies regradas minimais de Lohnherr e Reckziegel [51].

Subvariedades CR homoxéneas no espazo hiperbólico complexo

Dise que unha subvariedade dunha variedade hermitiana é unha *subvariedade CR (subvariedade de Cauchy-Riemann* ou *complexa-real)* se os seus subespazos tanxentes holomorfos maximais definen unha distribución e, ademais, a distribución complementaria é totalmente real. Dito doutro xeito, unha subvariedade dunha variedade hermitiana é CR se satisfai que o espazo tanxente en cada un dos seus puntos pode ser descomposto como unha suma directa ortogonal dun subespazo complexo e un subespazo totalmente real. Este concepto foi introducido por Bejancu en [10] e xeneraliza as nocións de subvariedades complexas e totalmente reais.

Nesta tese estamos particularmente interesados na clasificación das subvariedades CR homoxéneas no espazo hiperbólico complexo. Este tipo de subvariedades inclúe diversos exemplos de interese no contexto dos espazos simétricos hermitianos, como as hipersuperficies reais, as subvariedades de Kähler ou as subvariedades lagrangianas, entre outras.

Por exemplo, as hipersuperficies reais homoxéneas ou, equivalentemente, as accións isométricas de cohomoxeneidade un, no espazo hiperbólico complexo foron clasificadas por Berndt e Tamaru en [15]. As subvariedades de Kähler homoxéneas en $\mathbb{C}H^n$ foron tamén clasificadas en [26] por Di Scala, Ishi e Loi. Neste caso, os únicos exemplos do resultado de clasificación son subespazos hiperbólicos complexos totalmente xeodésicos $\mathbb{C}H^k$.

As subvariedades totalmente reais de máxima dimensión dunha variedade hermitiana denomínanse subvariedades lagrangianas. O problema de clasificación de subvariedades lagrangianas homoxéneas no espazo hiperbólico complexo resulta bastante complicado debido, fundamentalmente, ao feito de que o grupo de isometrías desta variedade ambiente non é compacto. A pesar de que este problema segue aberto, recentemente obtivéronse diversos resultados parciais relacionados coa clasificación de subvariedades lagrangianas homoxéneas en $\mathbb{C}H^n$. Máis concretamente, Hashinaga e Kajigaya obtiveron en [43] a clasificación das subvariedades lagrangianas homoxéneas do espazo hiperbólico complexo que xorden como órbitas de subgrupos da parte resoluble da descomposición de Iwasawa do grupo de isometrías de $\mathbb{C}H^n$.

En vista destes resultados, centramos a nosa atención na clasificación das subvariedades CR homoxéneas no espazo hiperbólico complexo obtidas como órbitas de subgrupos da parte resoluble da descomposición de Iwasawa do grupo de isometrías da variedade ambiente, AN. Dedicamos o Capítulo 3 a abordar este problema. Os resultados obtidos en dito capítulo están recollidos no traballo [32]. Para comezar, probamos o seguinte resultado.

Teorema 4. *Unha órbita da acción dun subgrupo conexo de AN é unha subvariedade CR de $\mathbb{C}H^n$ se, e só se, é congruente á órbita $H \cdot g(o)$, onde H é un subgrupo de Lie conexo de AN con álxebra de Lie $\mathfrak{h}$ e $g \in AN$, para algún dos seguintes casos:*

1. *$\mathfrak{h} = \mathfrak{r}$ e $g \in AN$; neste caso, todas as H-órbitas son subvariedades totalmente reais que constitúen unha subfoliación homoxénea da foliación de $\mathbb{C}H^n$ por horoesferas.*

2. *$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$ e $g \in \mathrm{Exp}((\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathfrak{g}_{2\alpha})$; neste caso, as H-órbitas CR son subvariedades totalmente reais equidistantes a un $\mathbb{R}H^k$ totalmente xeodésico, con $k \in \{1, \ldots, n\}$.*

3. *$\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ e $g \in AN$; neste caso, todas as H-órbitas son subvariedades CR congruentes entre si e, ademais, constitúen unha subfoliación da foliación por horoesferas de $\mathbb{C}H^n$.*

4. *$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ e $g \in \mathrm{Exp}(J\mathfrak{r})$; neste caso, as H-órbitas CR son as follas dunha foliación polar homoxénea cuxa única folla minimal é unha subvariedade de Berndt-Brück nun $\mathbb{C}H^k$ totalmente xeodésico en $\mathbb{C}H^n$, con $k \in \{2, \ldots, n\}$.*

Aquí, $\mathfrak{c}$ e $\mathfrak{r}$ denotan un subespazo complexo e outro totalmente real de $\mathfrak{g}_\alpha$, e $o \in \mathbb{C}H^n$ é o punto fixo da parte compacta da descomposición de Iwasawa baixo consideración.

Tamén estudamos, neste capítulo, as clases de congruencia deste resultado de clasificación. Convén sinalar que o noso teorema inclúe unha cantidade non numerable de clases de congruencia de exemplos, algúns deles de especial relevancia, como algunhas subvariedades de Berndt-Brück ou determinadas órbitas de accións polares.

Accións de cohomoxeneidade un no espazo-tempo de Minkowski

Un dos obxectivos fundamentais deste traballo é o de estudar e comprender as accións isométricas en variedades dotadas dunha métrica de Lorentz. No contexto da xeometría de Lorentz, o espazo-tempo de Minkowski $\mathbb{L}^{n+1}$, isto é, o análogo ao espazo euclidiano con signatura lorentziana, constitúe o exemplo de variedade máis sinxelo. Dende o punto de vista da Física, o espazo-tempo de Minkowski de dimensión catro é un exemplo particularmente interesante, pois é a variedade sobre a que se modela a Teoría da Relatividade Especial.

Diversos resultados relacionados co estudo de accións isométricas foron obtidos no marco da xeometría de Lorentz. Por exemplo, Adams e Stuck investigaron accións transitivas en variedades de Lorentz en [2] e [3]. Nesta tese estamos interesados nun caso particular de accións non transitivas: as accións de cohomoxeneidade un no espazo-tempo de Minkowski.

No contexto da xeometría de Riemann, é común asumir que as accións isométricas a clasificar son propias, pois estas satisfán importantes propiedades que fan que o seu estudo sexa moito máis sinxelo có das accións isométricas arbitrarias. Se unha acción é propia, entón os seus grupos de isotropía son compactos, as súas órbitas son subvariedades pechadas mergulladas e o espazo de órbitas da acción é Hausdorff. O estudo de accións propias de cohomoxeneidade un xa foi abordado no contexto das variedades de Lorentz e, en particular, no espazo-tempo de Minkowski, por Ahmadi e Kashani [4].

Non obstante, existen exemplos que motivan o estudo das accións de cohomoxeneidade un, non necesariamente propias, en variedades de Lorentz. En efecto, a acción natural do grupo $SO^0(1,n)$ no espazo-tempo de Minkowski $(n+1)$-dimensional, $\mathbb{L}^{n+1}$, non é propia xa que, en caso de selo, todas as súas órbitas serían subvariedades pechadas mergulladas. Porén, os conos de luz pasado e futuro son órbitas non pechadas de dita acción. Nesta tese non asumimos o carácter propio das accións e dicimos, nestas condicións, que unha acción isométrica non necesariamente propia é de *cohomoxeneidade un* se a menor codimensión das súas órbitas é un.

Motivados por un traballo recente no que Berndt, Díaz-Ramos e Vanaei obtiveron a clasificación das accións de cohomoxeneidade un nos espazo-tempos de Minkowski de dimensións dous e tres [14], dedicamos o Capítulo 4 desta tese a abordar o correspondente problema de clasificación no caso de dimensión arbitraria, empregando, para tal fin, a estructura de grupo de Lie do grupo de isometrías da variedade ambiente.

Para comezar, introducimos un resultado de factorización. Este resultado afirma que se G é un subgrupo de Lie conexo do grupo de isometrías do espazo-tempo de Minkowski $I^0(\mathbb{L}^{n+1})$ actuando con cohomoxeneidade un en $\mathbb{L}^{n+1}$, entón ten as mesmas órbitas que a acción dun grupo de Lie da forma $H\times\mathfrak{v}$. Aquí, H denota un subgrupo de G que actúa con cohomoxeneidade un en $\mathbb{L}^{n+1}\ominus\mathfrak{v}$ e $\mathfrak{v}$ é un subespazo non dexenerado de $\mathbb{L}^{n+1}$. Este resultado está recollido en [33].

Por último, centramos a nosa atención no caso particular $n=3$ e presentamos unha clasificación das accións de cohomoxeneidade un, non necesariamente propias, no espazo-tempo de Minkowski catro-dimensional $\mathbb{L}^4$, salvo equivalencia de órbitas. A continuación, presentamos o enunciado de dito teorema de clasificación, que tamén está recollido en [33].

Teorema 5. *Sexa G un subgrupo de Lie conexo de $I^0(\mathbb{L}^4)=SO^0(1,3)\times_\Phi\mathbb{L}^4$ con álxebra de Lie $\mathfrak{g}$ e supoñamos que G actúa con cohomoxeneidade un en $\mathbb{L}^4$. Consideremos a descomposición de Iwasawa $SO^0(1,3)=KAN$, e tamén a correspondente descomposición a nivel de álxebras de Lie, $\mathfrak{so}(1,3)=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$. Entón, a acción de G é, salvo equivalencia de órbitas, unha das seguintes:*

1. *Accións cuxa parte traslacional é non dexenerada:*

   (a) *a acción de $SO(k)\times\mathbb{L}^{4-k}$, con $k\in\{1,2,3\}$;*

   (b) *a acción de $SO^0(1,k)\times\mathbb{R}^{3-k}$, onde $k\in\{0,1,2,3\}$;*

   (c) *a acción de $\widetilde{AN}\times\mathbb{R}$, onde $\mathbb{R}$ denota unha recta espacial en $\mathbb{L}^4$ e $\widetilde{AN}$ é a parte resoluble da descomposición de Iwasawa de $SO^0(1,2)$;*

   (d) *a acción de QAN, onde $Q\in\{\{I\},K_0\}$;*

   (e) *a acción do subgrupo de Lie conexo cuxa álxebra de Lie é $\mathbb{R}(E+\mathbf{e})\oplus\mathfrak{n}$, onde*

$$\mathbb{R}(E+\mathbf{e})\oplus\mathfrak{n}=\mathbb{R}\left(\begin{pmatrix}0&0&0^t\\0&0&0^t\\0&0&E\end{pmatrix}+\mathbf{e}\right)\oplus\mathfrak{n};$$

(f) *a acción do subgrupo de Lie conexo cuxa álxebra de Lie é* $\mathbb{R}(E+1)\oplus\mathfrak{n}$, *onde*

$$\mathbb{R}(E+1)\oplus\mathfrak{n} = \mathbb{R}\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & E \end{pmatrix}\oplus\mathfrak{n};$$

2. *Accións cuxa parte traslacional é dexenerada:*

(a) *a acción de* $\mathbb{W}^3$*;*

(b) *a acción de* $\operatorname{Exp}(\mathfrak{v})\times\mathbb{W}^2$, *onde* $\mathfrak{v}$ *é o subespazo de* $\mathfrak{n}$ *xerado polo elemento* $(0,1)\in\mathfrak{n}\cong\mathbb{R}^2$*;*

(c) *a acción do grupo de Lie cuxa álxebra de Lie é* $\mathfrak{g}=\mathbb{R}(v+(1,0,0,0))\oplus\mathbb{W}^2$, *onde* $v=(0,1)\in\mathfrak{n}\cong\mathbb{R}^2$ *e*

$$\mathbb{R}\left(v+(1,0,0,0)\right)\oplus\mathbb{W}^2 = \mathbb{R}\left(\begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)\oplus\mathbb{W}^2;$$

(d) *a acción do grupo de Lie cuxa álxebra de Lie é* $\mathfrak{g}=\mathbb{R}(1+(0,0,0,\lambda))\oplus\mathbb{W}^2$, *onde* $\lambda>0$ *e*

$$\mathbb{R}\left(1+(0,0,0,\lambda)\right)\oplus\mathbb{W}^2 = \mathbb{R}\left(\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix}0\\0\\0\\\lambda\end{pmatrix}\right)\oplus\mathbb{W}^2;$$

(e) *a acción de* $QN\times\mathbb{W}^1$, *onde* $Q\in\{\{I\},K_0\}$*;*

(f) *a acción de* $K_0A\times\mathbb{W}^1$*;*

(g) *a acción de* $A\operatorname{Exp}(\mathfrak{v})\times\mathbb{W}^1$, *onde* $\mathfrak{v}$ *denota un subespazo de* $\mathfrak{n}$ *de dimensión un;*

(h) *a acción do subgrupo de Lie conexo cuxa álxebra de Lie é* $\mathbb{R}(1+(0,0,b))\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1$, *onde* $\mathfrak{v}$ *é un subespazo de* $\mathfrak{n}$ *de dimensión un,* $b\in\mathbb{R}^2$ *e*

$$\mathbb{R}\left(1+(0,0,b)\right)\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1 = \mathbb{R}\left(\begin{pmatrix} 0 & 1 & 0^t \\ 1 & 0 & 0^t \\ 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix}0\\0\\b\end{pmatrix}\right)\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1;$$

(i) *a acción do subgrupo de Lie conexo cuxa álxebra de Lie está dada por* $\big(\mathbb{R}(u+(0,0,x))\oplus\mathbb{R}(v+(0,0,y))\big)\oplus_\phi\mathbb{W}^1$, *onde* $\{u,v\}$ *é unha base ortonormal de* $\mathfrak{n}$, x, $y\in\mathbb{R}^2$ *e*

$$\begin{aligned}&\big(\mathbb{R}(u+(0,0,x))\oplus\mathbb{R}(v+(0,0,y))\big)\oplus_\phi\mathbb{W}^1\\ &=\mathbb{R}\left(\begin{pmatrix} 0 & 0 & u^t \\ 0 & 0 & u^t \\ u & -u & 0 \end{pmatrix}+\begin{pmatrix}0\\0\\x\end{pmatrix}\right)\oplus\mathbb{R}\left(\begin{pmatrix} 0 & 0 & v^t \\ 0 & 0 & v^t \\ v & -v & 0 \end{pmatrix}+\begin{pmatrix}0\\0\\y\end{pmatrix}\right)\oplus_\phi\mathbb{W}^1.\end{aligned}$$

Neste enunciado, denotamos por $\mathbf{e}$ o vector lumínico $\mathbf{e} = (1,1,0,0) \in \mathbb{L}^4$ e por $\mathbb{W}^1$ a recta xerada por $\mathbf{e}$. Ademais, $\mathbb{W}^2$ e $\mathbb{W}^3$ denotan os subespazos dexenerados de $\mathbb{L}^4$ dados por

$$\mathbb{W}^2 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2, \qquad\qquad \mathbb{W}^3 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3,$$

onde $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_3\}$ é unha base ortonormal de $\mathbb{L}^4$ tal que $\langle \mathbf{e}_0, \mathbf{e}_0 \rangle = -1$ e $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$, para $i \in \{1,2,3\}$. Por último, E é o xenerador de $\mathfrak{k}_0 = N_{\mathfrak{k}}(\mathfrak{a}) \cong \mathfrak{so}(2)$ dado por

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Resumen en castellano

Intuitivamente, la simetría es la propiedad de los objetos que hace que resulten semejantes cuando los observamos desde diferentes perspectivas. Se trata de una cualidad importante desde el punto de vista de disciplinas como la biología, la química o el arte. El concepto de simetría también tiene cabida en el campo de las matemáticas. En este contexto, cabe señalar que la simetría no es una propiedad únicamente aplicable a figuras geométricas, sino que otros objetos más abstractos también pueden tener simetrías. El estudio de dichos objetos ha dado lugar a importantes resultados en diversas áreas de las matemáticas. Por ejemplo, la teoría de Galois asegura que si una ecuación polinómica no tiene las simetrías adecuadas, entonces no es resoluble por radicales. Otro ejemplo lo constituyen los conocidos teoremas de Noether, que afirman que las simetrías de un sistema físico se traducen en leyes de conservación.

El concepto de simetría puede ser definido de modo riguroso empleando la terminología de la teoría de grupos. Así, dado un grupo G, se dice que un objeto es G-simétrico si es invariante bajo la acción de G. En particular, el concepto de simetría siempre está vinculado a la acción de un grupo que, de hecho, especifica el tipo de simetría que un determinado objeto tiene.

En el marco de la geometría semi-riemanniana, es natural considerar como objeto de estudio el grupo de isometrías de la variedad, es decir, el grupo constituido por las transformaciones de la variedad que preservan su métrica. La acción de un subgrupo del grupo de isometrías en una determinada variedad semi-riemanniana se denomina *acción isométrica*. Una variedad semi-riemanniana se dice homogénea si su grupo de isometrías actúa transitivamente sobre ella. Una subvariedad es *(extrínsecamente) homogénea* si coincide con la órbita de la acción de un subgrupo del grupo de isometrías de la variedad ambiente.

La experiencia ha demostrado que el problema de clasificación de acciones isométricas o, equivalentemente, de subvariedades homogéneas, en una determinada variedad semi-riemanniana puede resultar de lo más complicado. Por este motivo, es usual restringir nuestra atención y limitarnos al estudio de determinados tipos específicos de acciones isométricas que sean más manejables. Por ejemplo, el estudio de acciones transitivas, esto es, aquellas en las que la propia variedad es la única órbita de la acción, ha sido abordado en diversos contextos.

Las *acciones de cohomogeneidad uno* constituyen otra clase particular de acciones isométricas en variedades semi-riemannianas que han atraído considerable atención. Una acción isométrica se dice de cohomogeneidad uno si sus órbitas tienen codimensión uno. En el contexto particular de la geometría lorentziana no existen demasiados resultados de clasificación en relación a este tipo de acciones isométricas. Uno de los principales objetivos del presente trabajo es el de estudiar y clasificar acciones de cohomogeneidad uno en variedades de Lorentz, salvo equivalencia de órbitas. Más concretamente, abordamos este problema tomando como variedad ambiente el *espacio-tiempo de Minkowski*, es decir, el análogo al espacio euclídeo con signatura lorentziana.

No obstante, en el marco de la geometría de Riemann sí se han obtenido numerosos e importantes resultados en relación al problema de clasificación de acciones de cohomogeneidad uno o, equivalentemente, de hipersuperficies homogéneas. Por ejemplo, la clasificación de acciones de cohomogeneidad uno en los espacios de curvatura constante, es decir, los espacios euclídeos $\mathbb{R}^n$, las esferas S^n y los espacios hiperbólicos reales $\mathbb{R}H^n$, ha sido abordado y, finalmente, resuelto por varios autores. Más concretamente, la clasificación de hipersuperficies homogéneas en los espacios euclídeos puede deducirse de dos trabajos de Levi-Civita [49] y Segre [68] dedicados al estudio de las hipersuperficies isoparamétricas de dichos espacios. La clasificación en esferas, que ha resultado ser un problema mucho más complicado, se sigue de un trabajo de Hsiang y Lawson [44] relacionado con subvariedades minimales en este tipo de variedades. Las hipersuperficies homogéneas en los espacios hiperbólicos reales ya habían sido clasificadas bastante antes por Cartan en [20].

La complejidad del problema de clasificación de hipersuperficies homogéneas en una determinada variedad riemanniana aumenta considerablemente cuando dicha variedad está dotada de una estructura de Kähler. Los ejemplos más sencillos de variedades de Kähler son los *espacios forma complejos*, constituidos por tres importantes familias de variedades: los espacios euclídeos complejos $\mathbb{C}^n$, los espacios proyectivos complejos $\mathbb{C}P^n$ y los espacios hiperbólicos complejos $\mathbb{C}H^n$. El problema de clasificación de hipersuperficies homogéneas en este tipo de variedades ha sido abordado y concluido por diversos autores. En particular, la clasificación en el caso proyectivo ha sido resuelta por Takagi en [70], mientras que las hipersuperficies homogéneas en los espacios hiperbólicos complejos han sido clasificadas por Berndt y Tamaru en [15].

Uno de los principales objetivos de esta tesis es el estudio de la geometría de subvariedades en el contexto de los espacios forma complejos no llanos, es decir, en los espacios proyectivo e hiperbólico complejos. Por supuesto, la estructura compleja subyacente juega un papel fundamental a la hora de resolver problemas en este marco de trabajo. En general, en el contexto de las variedades de Kähler y, más concretamente, en el de los espacios forma complejos, es posible definir los conceptos de subvariedades complejas y totalmente reales, que dependen precisamente de la estructura compleja de la variedad ambiente. La noción de *subvariedad CR* constituye una generalización de estos dos tipos de subvariedades. En esta tesis estudiamos subvariedades CR homogéneas en el espacio hiperbólico complejo haciendo uso, para tal fin, de la estructura de grupo de Lie del grupo de isometrías de dicha variedad ambiente. Existen varios ejemplos conocidos de subvariedades CR homogéneas en el espacio hiperbólico complejo que motivan este problema, tales como las *subvariedades de Berndt-Brück* [11] y, en particular, las *hipersuperficies de Lohnherr* [50].

La hipersuperficie de Lohnherr de $\mathbb{C}H^n$ satisface numerosas propiedades importantes que la convierten en un atractivo objeto de estudio. Por ejemplo, puede ser caracterizada como la única hipersuperficie minimal y homogénea en el espacio hiperbólico complejo [13]. También se caracteriza por ser la única hipersuperficie en $\mathbb{C}H^n$ con curvaturas principales constantes que es, además, *reglada* [51]. El concepto de hipersuperficie reglada en un espacio forma complejo guarda una estrecha relación con la estructura compleja de la variedad ambiente. Diversos trabajos sobre el estudio de hipersuperficies reales regladas en los espacios forma complejos han sido obtenidos en los últimos años. En esta tesis también nos centramos en la clasificación de ciertas hipersuperficies regladas que satisfacen determinadas propiedades geométricas adicionales en los espacios proyectivo e hiperbólico complejos.

A continuación, presentamos los resultados originales de esta tesis.

Hipersuperficies reales regladas en espacios forma complejos

Una *hipersuperficie real reglada* en un espacio proyectivo o hiperbólico complejo $\mathbb{C}P^n$ o $\mathbb{C}H^n$ es una subvariedad de codimensión real uno que está foliada localmente por hipersuperficies complejas totalmente geodésicas de la variedad ambiente, más concretamente, $\mathbb{C}P^{n-1}$ o $\mathbb{C}H^{n-1}$, respectivamente.

Las hipersuperficies reales regladas en los espacios forma complejos no llanos constituyen una clase muy amplia de hipersuperficies reales. Por este motivo, es natural centrarse en el estudio de este tipo de hipersuperficies imponiendo alguna condición geométrica adicional. Por ejemplo, Lohnherr y Reckziegel obtuvieron en [51] la clasificación de las hipersuperficies regladas minimales en los espacios proyectivo e hiperbólico complejos. Más concretamente, si M denota una hipersuperficie minimal reglada en un espacio forma complejo no llano, entonces ha de ser una parte abierta de alguna de las hipersuperficies siguientes:

(i) una hipersuperficie de tipo Kimura (cf. [47]) en $\mathbb{C}P^n$ o $\mathbb{C}H^n$,

(ii) un bisector en $\mathbb{C}H^n$, o

(iii) una hipersuperficie de Lohnherr en $\mathbb{C}H^n$.

En el Capítulo 2 de esta tesis abordamos el problema de clasificación de hipersuperficies reales regladas en los espacios forma complejos no llanos que satisfacen, adicionalmente, ciertas propiedades relacionadas con la constancia de las curvaturas medias de orden superior. En general, las curvaturas medias de orden superior de una hipersuperficie se definen como los polinomios simétricos elementales en las variables dadas por las curvaturas principales de dicha hipersuperficie. Es un hecho conocido que toda hipersuperficie reglada en un espacio forma complejo no llano tiene exactamente dos curvaturas principales no nulas, α y β. Por ello, existirán únicamente dos polinomios simétricos elementales no triviales: la curvatura media (de orden uno), $\alpha + \beta$, y la curvatura media de orden dos, $\alpha\beta$. La Sección 2.3 está dedicada a la clasificación de hipersuperficies regladas con curvatura media constante en los espacios proyectivo e hiperbólico complejos. Más concretamente, demostramos el siguiente resultado, que ha dado lugar al artículo [36].

Teorema 1. *Sea M una hipersuperficie real reglada con curvatura media constante en un espacio proyectivo o hiperbólico complejo. Entonces, M es minimal.*

La clasificación de hipersuperficies reales regladas con curvatura media constante se obtiene a partir de este teorema, haciendo uso de la clasificación de hipersuperficies regladas minimales de Lohnherr y Reckziegel [51] previamente citada.

Por otra parte, las hipersuperficies reales regladas en $\mathbb{C}P^n$ y $\mathbb{C}H^n$ cuya curvatura media de orden dos es constante han sido recientemente caracterizadas por Kimura, Maeda y Tanabe en [48].

La norma al cuadrado del operador de configuración, $\alpha^2 + \beta^2$, que puede ser expresada de una manera muy sencilla en términos de las curvaturas medias de primer y segundo orden, constituye un tercer invariante geométrico clásico en el estudio de las hipersuperficies. Es natural interesarse, por tanto, en aquellas hipersuperficies reales regladas en los espacios forma complejos no llanos tales que la norma de su operador de configuración es constante. Tratamos este problema en la Sección 2.4, en la cual obtenemos una clasificación completa que incluye un nuevo ejemplo no homogéneo. Más concretamente, probamos el siguiente resultado. Su demostración ha dado lugar a los artículos [37] y [66].

Teorema 2. *Sea M una hipersuperficie real reglada en un espacio proyectivo o hiperbólico complejo. Entonces, la norma del operador de configuración de M es constante si, y solo si, M es una parte abierta de alguna de las siguientes hipersuperficies:*

1. *una hipersuperficie de Lohnherr en $\mathbb{C}H^n$, o*
2. *la hipersuperficie real reglada construida adjuntando $\mathbb{C}H^{n-1}$ totalmente geodésicos perpendicularmente a un círculo de curvatura $\kappa = \sqrt{-c/2}$ en una recta hiperbólica compleja totalmente geodésica $\mathbb{C}H^1$, donde $c < 0$ denota la curvatura seccional holomorfa del espacio ambiente $\mathbb{C}H^n$.*

En particular, todos los ejemplos proporcionados por este teorema de clasificación satisfacen la propiedad de ser hipersuperficies fuertemente 2-Hopf. La noción de hipersuperficie fuertemente 2-Hopf ha sido introducida en [28] y resulta ser una propiedad importante a la hora de construir el ejemplo no homogéneo de hipersuperficie real reglada cuyo operador de configuración tiene norma constante proporcionado en este teorema.

Para finalizar, centramos nuestra atención en aquellas hipersuperficies reales regladas en los espacios forma complejos no llanos que son, además, biarmónicas. El concepto de hipersuperficie biarmónica generaliza el de hipersuperficie minimal. Con más precisión, una hipersuperficie es biarmónica si la inmersión isométrica que la define es un punto crítico del funcional de bienergía o, equivalentemente, si su campo de bitensión asociado es idénticamente nulo.

Motivados por un reciente teorema de Sasahara [67] en el que presenta una clasificación de las hipersuperficies reales regladas biarmónicas en el espacio proyectivo complejo, dedicamos la Sección 2.5 a extender este resultado al contexto general de los espacios forma complejos no llanos. Más concretamente, probamos el siguiente teorema, que puede encontrarse en el artículo [66].

Teorema 3. *Sea M una hipersuperficie real reglada en un espacio proyectivo o hiperbólico complejo. Entonces, M es biarmónica si, y solo si, es minimal.*

De nuevo, la clasificación de las hipersuperficies reales regladas biarmónicas en los espacios forma complejos no llanos se obtiene combinando este teorema con el resultado de clasificación de hipersuperficies regladas minimales de Lohnherr y Reckziegel [51].

Subvariedades CR homogéneas en el espacio hiperbólico complejo

Se dice que una subvariedad de una variedad hermitiana es una *subvariedad CR (subvariedad de Cauchy-Riemann* o *compleja-real)* si sus subespacios tangentes holomorfos maximales definen una distribución y, además, la distribución complementaria es totalmente real. Dicho de otro modo, una subvariedad de una variedad hermitiana es CR si satisface que el espacio tangente en cada uno de sus puntos se descompone como una suma directa ortogonal de un subespacio complejo y un subespacio totalmente real. Este concepto fue introducido por Bejancu en [10] y generaliza las nociones de subvariedades complejas y totalmente reales.

En esta tesis estamos particularmente interesados en la clasificación de las subvariedades CR homogéneas en el espacio hiperbólico complejo. Este tipo de subvariedades incluye diversos ejemplos de interés en el contexto de los espacios simétricos hermitianos, como las hipersuperficies reales, las subvariedades de Kähler o las subvariedades lagrangianas, entre otras.

Por ejemplo, las hipersuperficies reales homogéneas o, equivalentemente, las acciones isométricas de cohomogeneidad uno, en el espacio hiperbólico complejo han sido clasificadas por Berndt y Tamaru en [15]. Las subvariedades de Kähler homogéneas en $\mathbb{C}H^n$ también han sido clasificadas en [26] por Di Scala, Ishi y Loi. En este caso, los únicos ejemplos del resultado de clasificación son subespacios hiperbólicos complejos totalmente geodésicos $\mathbb{C}H^k$.

Las subvariedades totalmente reales de máxima dimensión de una variedad hermitiana se denominan subvariedades lagrangianas. El problema de clasificación de subvariedades lagrangianas homogéneas en el espacio hiperbólico complejo resulta bastante complicado debido, fundamentalmente, al hecho de que el grupo de isometrías de esta variedad ambiente no es compacto. A pesar de que este problema sigue abierto, recientemente se han obtenido diversos resultados parciales relacionados con la clasificación de subvariedades lagrangianas homogéneas en $\mathbb{C}H^n$. Más concretamente, Hashinaga y Kajigaya han obtenido en [43] la clasificación de las subvariedades lagrangianas homogéneas del espacio hiperbólico complejo que surgen como órbitas de subgrupos de la parte resoluble de la descomposición de Iwasawa del grupo de isometrías de $\mathbb{C}H^n$.

En vista de estos resultados, centramos nuestra atención en la clasificación de subvariedades CR homogéneas en el espacio hiperbólico complejo obtenidas como órbitas de subgrupos de la parte resoluble de la descomposición de Iwasawa del grupo de isometrías de la variedad ambiente, AN. Dedicamos el Capítulo 3 a abordar este problema. Los resultados obtenidos en dicho capítulo están recogidos en el trabajo [32]. Para comenzar, probamos el siguiente resultado.

Teorema 4. *Una órbita de la acción de un subgrupo conexo de AN es una subvariedad CR de $\mathbb{C}H^n$ si, y solo si, es congruente a la órbita $H \cdot g(o)$, donde H es un subgrupo de Lie conexo de AN con álgebra de Lie $\mathfrak{h}$ y $g \in AN$, para alguno de los casos siguientes:*

1. *$\mathfrak{h} = \mathfrak{r}$ y $g \in AN$; en este caso, todas las H-órbitas son subvariedades totalmente reales que constituyen una subfoliación homogénea de la foliación de $\mathbb{C}H^n$ por horoesferas.*

2. *$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{r}$ y $g \in \mathrm{Exp}((\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{r}) \oplus \mathfrak{g}_{2\alpha})$; en este caso, las H-órbitas CR son subvariedades totalmente reales equidistantes a un $\mathbb{R}H^k$ totalmente geodésico, con $k \in \{1, \ldots, n\}$.*

3. *$\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ y $g \in AN$; en este caso, todas las H-órbitas son subvariedades CR congruentes entre sí y, además, constituyen una subfoliación de la foliación por horoesferas de $\mathbb{C}H^n$.*

4. *$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{c} \oplus \mathfrak{r} \oplus \mathfrak{g}_{2\alpha}$ y $g \in \mathrm{Exp}(J\mathfrak{r})$; en este caso, las H-órbitas CR son las hojas de una foliación polar homogénea cuya única hoja minimal es una subvariedad de Berndt-Brück en un $\mathbb{C}H^k$ totalmente geodésico en $\mathbb{C}H^n$, con $k \in \{2, \ldots, n\}$.*

Aquí, $\mathfrak{c}$ y $\mathfrak{r}$ denotan un subespacio complejo y otro totalmente real de $\mathfrak{g}_\alpha$, y $o \in \mathbb{C}H^n$ es el punto fijo de la parte compacta de la descomposición de Iwasawa que hemos considerado.

También estudiaremos, en dicho capítulo, las clases de congruencia de este resultado de clasificación. Cabe señalar que nuestro teorema incluye una cantidad no numerable de clases de congruencia de ejemplos, algunos de ellos de especial relevancia, como algunas subvariedades de Berndt-Brück o determinadas órbitas de acciones polares.

Acciones de cohomogeneidad uno en el espacio-tiempo de Minkowski

Uno de los objetivos fundamentales de este trabajo es el de estudiar y comprender las acciones isométricas en variedades dotadas de una métrica de Lorentz. En el contexto de la geometría de Lorentz, el espacio-tiempo de Minkowski $\mathbb{L}^{n+1}$, esto es, el análogo al espacio euclídeo con signatura lorentziana, constituye el ejemplo de variedad más sencillo. Desde el punto de vista de la Física, el espacio-tiempo de Minkowski de dimensión cuatro es un ejemplo particularmente interesante, puesto que es la variedad sobre la que se modela la Teoría de la Relatividad Especial.

Diversos resultados relacionados con el estudio de acciones isométricas se han obtenido en el marco de la geometría de Lorentz. Por ejemplo, Adams y Stuck han investigado acciones transitivas en variedades de Lorentz en [2] y [3]. En este trabajo estamos interesados en un caso particular de acciones no transitivas: las acciones de cohomogeneidad uno en el espacio-tiempo de Minkowski.

En el contexto de la geometría de Riemann, es común asumir que las acciones isométricas a clasificar son propias, puesto que satisfacen importantes propiedades que hacen que su estudio sea mucho más sencillo que el de las acciones isométricas arbitrarias. Si una acción es propia, entonces sus grupos de isotropía son compactos, sus órbitas son subvariedades cerradas embebidas y el espacio de órbitas de la acción es Hausdorff. El estudio de acciones propias de cohomogeneidad uno ya ha sido abordado en el contexto de las variedades de Lorentz y, en particular, en el espacio-tiempo de Minkowski, por Ahmadi y Kashani [4].

Sin embargo, existen ejemplos que motivan el estudio de acciones de cohomogeneidad uno, no necesariamente propias, en variedades de Lorentz. En efecto, la acción natural del grupo $SO^0(1,n)$ en el espacio-tiempo de Minkowski $(n+1)$-dimensional, $\mathbb{L}^{n+1}$, no es propia ya que, si lo fuera, todas sus órbitas serían subvariedades cerradas embebidas. No obstante, los conos de luz pasado y futuro son órbitas no cerradas de dicha acción. En esta tesis no asumimos el carácter propio de las acciones y decimos, en estas condiciones, que una acción isométrica no necesariamente propia es de *cohomogeneidad uno* si la menor codimensión de sus órbitas es uno.

Motivados por un trabajo reciente en el que Berndt, Díaz-Ramos y Vanaei han obtenido la clasificación de las acciones de cohomogeneidad uno en los espacio-tiempos de Minkowski de dimensiones dos y tres [14], dedicamos el Capítulo 4 de esta tesis a abordar el correspondiente problema de clasificación en el caso de dimensión arbitraria, haciendo uso, para tal fin, de la estructura de grupo de Lie del grupo de isometrías de la variedad ambiente.

Para comenzar, introducimos un resultado de factorización. Dicho resultado afirma que si G es un subgrupo de Lie conexo del grupo de isometrías del espacio-tiempo de Minkowski $I^0(\mathbb{L}^{n+1})$ actuando con cohomogeneidad uno en $\mathbb{L}^{n+1}$, entonces tiene las mismas órbitas que la acción de un grupo de Lie de la forma $H \times \mathfrak{v}$. Aquí, H denota un subgrupo de G que actúa con cohomogeneidad uno en $\mathbb{L}^{n+1} \ominus \mathfrak{v}$ y $\mathfrak{v}$ es un subespacio no degenerado de $\mathbb{L}^{n+1}$. Este resultado está recogido en [33].

Por último, centramos nuestra atención en el caso particular $n = 3$ y presentamos una clasificación de las acciones de cohomogeneidad uno, no necesariamente propias, en el espacio-tiempo de Minkowski cuatro-dimensional $\mathbb{L}^4$, salvo equivalencia de órbitas. A continuación, presentamos el enunciado de dicho teorema de clasificación, que también está recogido en [33].

Teorema 5. *Sea G un subgrupo de Lie conexo de $I^0(\mathbb{L}^4) = SO^0(1,3) \times_\Phi \mathbb{L}^4$ con álgebra de Lie $\mathfrak{g}$ y supongamos que G actúa con cohomogeneidad uno en $\mathbb{L}^4$. Consideremos la descomposición de Iwasawa $SO^0(1,3) = KAN$, y también la correspondiente descomposición a nivel de álgebras de Lie, $\mathfrak{so}(1,3) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Entonces, la acción de G es, salvo equivalencia de órbitas, una de las siguientes:*

1. *Acciones cuya parte traslacional es no degenerada:*
   - *(a) la acción de $SO(k) \times \mathbb{L}^{4-k}$, con $k \in \{1,2,3\}$;*
   - *(b) la acción de $SO^0(1,k) \times \mathbb{R}^{3-k}$, donde $k \in \{0,1,2,3\}$;*
   - *(c) la acción de $A\widetilde{N} \times \mathbb{R}$, donde $\mathbb{R}$ denota una recta espacial en $\mathbb{L}^4$ y $A\widetilde{N}$ es la parte resoluble de la descomposición de Iwasawa de $SO^0(1,2)$;*
   - *(d) la acción de QAN, donde $Q \in \{\{I\}, K_0\}$;*

(e) *la acción del subgrupo de Lie conexo cuya álgebra de Lie es* $\mathbb{R}(E+\mathbf{e})\oplus\mathfrak{n}$, *donde*

$$\mathbb{R}(E+\mathbf{e})\oplus\mathfrak{n}=\mathbb{R}\left(\begin{pmatrix}0&0&0^t\\0&0&0^t\\0&0&E\end{pmatrix}+\mathbf{e}\right)\oplus\mathfrak{n};$$

(f) *la acción del subgrupo de Lie conexo cuya álgebra de Lie es* $\mathbb{R}(E+1)\oplus\mathfrak{n}$, *donde*

$$\mathbb{R}(E+1)\oplus\mathfrak{n}=\mathbb{R}\begin{pmatrix}0&1&0^t\\1&0&0^t\\0&0&E\end{pmatrix}\oplus\mathfrak{n};$$

2. *Acciones cuya parte traslacional es degenerada:*

(a) *la acción de* $\mathbb{W}^3$*;*

(b) *la acción de* $\operatorname{Exp}(\mathfrak{v})\times\mathbb{W}^2$, *donde* $\mathfrak{v}$ *el subespacio de* $\mathfrak{n}$ *generado por el elemento* $(0,1)\in\mathfrak{n}\cong\mathbb{R}^2$*;*

(c) *la acción del grupo de Lie cuya álgebra de Lie es* $\mathfrak{g}=\mathbb{R}(v+(1,0,0,0))\oplus\mathbb{W}^2$, *donde* $v=(0,1)\in\mathfrak{n}\cong\mathbb{R}^2$ *y*

$$\mathbb{R}\left(v+(1,0,0,0)\right)\oplus\mathbb{W}^2=\mathbb{R}\left(\begin{pmatrix}0&0&v^t\\0&0&v^t\\v&-v&0\end{pmatrix}+\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)\oplus\mathbb{W}^2;$$

(d) *la acción del grupo de Lie cuya álgebra de Lie es* $\mathfrak{g}=\mathbb{R}(1+(0,0,0,\lambda))\oplus\mathbb{W}^2$, *donde* $\lambda>0$ *y*

$$\mathbb{R}\left(1+(0,0,0,\lambda)\right)\oplus\mathbb{W}^2=\mathbb{R}\left(\begin{pmatrix}0&1&0^t\\1&0&0^t\\0&0&0\end{pmatrix}+\begin{pmatrix}0\\0\\0\\\lambda\end{pmatrix}\right)\oplus\mathbb{W}^2;$$

(e) *la acción de* $QN\times\mathbb{W}^1$, *donde* $Q\in\{\{I\},K_0\}$*;*

(f) *la acción de* $K_0A\times\mathbb{W}^1$*;*

(g) *la acción de* $A\operatorname{Exp}(\mathfrak{v})\times\mathbb{W}^1$, *donde* $\mathfrak{v}$ *denota un subespacio de* $\mathfrak{n}$ *de dimensión uno;*

(h) *la acción del subgrupo de Lie conexo cuya álgebra de Lie es* $\mathbb{R}(1+(0,0,b))\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1$, *donde* $\mathfrak{v}$ *es un subespacio de* $\mathfrak{n}$ *de dimensión uno,* $b\in\mathbb{R}^2$ *y*

$$\mathbb{R}\left(1+(0,0,b)\right)\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1=\mathbb{R}\left(\begin{pmatrix}0&1&0^t\\1&0&0^t\\0&0&0\end{pmatrix}+\begin{pmatrix}0\\0\\b\end{pmatrix}\right)\oplus\mathfrak{v}\oplus_\phi\mathbb{W}^1;$$

(i) la acción del subgrupo de Lie conexo cuya álgebra de Lie está dada por $\big(\mathbb{R}(u+(0,0,x)) \oplus \mathbb{R}(v+(0,0,y))\big) \oplus_\phi \mathbb{W}^1$*, donde* $\{u,v\}$ *es una base ortonormal de* $\mathfrak{n}$*,* x*,* $y \in \mathbb{R}^2$ *y*

$$\begin{aligned}&\big(\mathbb{R}(u+(0,0,x)) \oplus \mathbb{R}(v+(0,0,y))\big) \oplus_\phi \mathbb{W}^1\\ &= \mathbb{R}\left(\begin{pmatrix}0 & 0 & u^t\\ 0 & 0 & u^t\\ u & -u & 0\end{pmatrix} + \begin{pmatrix}0\\0\\x\end{pmatrix}\right) \oplus \mathbb{R}\left(\begin{pmatrix}0 & 0 & v^t\\ 0 & 0 & v^t\\ v & -v & 0\end{pmatrix} + \begin{pmatrix}0\\0\\y\end{pmatrix}\right) \oplus_\phi \mathbb{W}^1.\end{aligned}$$

En este enunciado, denotamos por $\mathbf{e}$ el vector luminoso $\mathbf{e} = (1,1,0,0) \in \mathbb{L}^4$ y por $\mathbb{W}^1$ la recta generada por $\mathbf{e}$. Además, $\mathbb{W}^2$ y $\mathbb{W}^3$ denotan los subespacios degenerados de $\mathbb{L}^4$ dados por

$$\mathbb{W}^2 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2, \qquad \mathbb{W}^3 = \mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3,$$

donde $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_3\}$ es una base ortonormal de $\mathbb{L}^4$ tal que $\langle \mathbf{e}_0, \mathbf{e}_0\rangle = -1$ y $\langle \mathbf{e}_i, \mathbf{e}_i\rangle = 1$, para $i \in \{1,2,3\}$. Por último, E es el generador de $\mathfrak{k}_0 = N_{\mathfrak{k}}(\mathfrak{a}) \cong \mathfrak{so}(2)$ dado por

$$E = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

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