BASICS OF DIFFERENTIABLE MANIFOLDS

Riemannian geometry – IMPA – Miguel Domínguez Vázquez

1. Differentiable manifolds

Definition 1.1. A *differentiable manifold* or *smooth manifold* of dimension n is a Hausdorff topological space M with countable base endowed with a family

$$\{x_\alpha \colon U_\alpha \subset \mathbb{R}^n \to V_\alpha = x_\alpha(U_\alpha) \subset M\}_{\alpha\in\Lambda}$$

of bijective maps x_α from open sets U_α of $\mathbb{R}^n$ to open sets V_α of M, such that:
(1) x_α is a homeomorphism, for each $\alpha \in \Lambda$.
(2) $M = \bigcup_{\alpha\in\Lambda} V_\alpha$.
(3) If $V_\alpha \cap V_\beta \neq \emptyset$, for $\alpha, \beta \in \Lambda$, then the map

$$x_\beta^{-1} \circ x_\alpha \colon x_\alpha^{-1}(V_\alpha \cap V_\beta) \to x_\beta^{-1}(V_\alpha \cap V_\beta)$$

is a diffeomorphism between open sets of $\mathbb{R}^n$.

In these conditions, (U_α, x_α), or simply x_α, is called a *parametrization* or *coordinate system.* Each inverse map x_α^{-1} is called a *coordinate chart*, and V_α is a *coordinate open set* or *coordinate neighborhood.* A family $\{(U_\alpha, x_\alpha)\}_{\alpha\in\Lambda}$ in the conditions above is a *differentiable structure* or *coordinate atlas.*

Definition 1.2. Let M^m and N^n be smooth manifolds. A map $\varphi \colon M \to N$ is *differentiable* at $p \in M$ if, given parametrizations $x \colon U \subset \mathbb{R}^m \to V \subset M$ around p and $y \colon U' \subset \mathbb{R}^n \to V' \subset N$ around $\varphi(p)$, the map $y^{-1} \circ \varphi \circ x$, which is defined in a neighborhood of $x^{-1}(p)$, is differentiable at $x^{-1}(p)$ as a map between open sets of Euclidean spaces.

The map φ is *differentiable* or *smooth* if it is differentiable at every point of M.

The set of all smooth maps from M to $\mathbb{R}$ will be denoted by $\mathcal{C}^\infty(M)$.

Definition 1.3. Let $\varphi \colon M \to N$ be smooth. Then:
(1) φ is a *diffeomorphism* if φ is bijective and φ^{-1} is smooth.
(2) φ is a *local diffeomorphism* if for each $p \in M$ there is an open neighborhood U of p in M such that $\varphi(U)$ is open in N and $\varphi|_U \colon U \to \varphi(U)$ is a diffeomorphism.

Definition 1.4. A *(smooth) partition of unity* of a smooth manifold M is a collection $\{f_\alpha \colon M \to \mathbb{R}\}_{\alpha\in\Lambda}$ of functions $f_\alpha \in \mathcal{C}^\infty(M)$ such that:
(1) $0 \leq f_\alpha \leq 1$ for all $\alpha \in \Lambda$.
(2) $\{\operatorname{supp} f_\alpha : \alpha \in \Lambda\}$ is a locally finite collection of subsets of M, that is, each point of M has a neighborhood that meets only finitely many subsets in the collection.
(3) $\sum_{\alpha\in\Lambda} f_\alpha = 1$.

The partition is said to be *subordinate* to an open covering $\mathcal{U}$ of M provided that each set $\operatorname{supp} f_\alpha$ is contained in an element of $\mathcal{U}$.

Theorem 1.5. *Let $\mathcal{U}$ be an open covering of a smooth manifold M. Then M admits a smooth partition of unity subordinate to $\mathcal{U}$.*

2. Tangent space

Definition 2.1. Let M be a smooth manifold. Let $\alpha\colon (-\varepsilon, \varepsilon) \to M$ be a differentiable curve in M with $\alpha(0) = p \in M$. The *tangent vector* to the curve α for $t = 0$ is the map

$$\alpha'(0)\colon \mathcal{C}^\infty(M) \to \mathbb{R}, \qquad f \mapsto \alpha'(0)f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}.$$

A tangent vector at $p \in M$ is the tangent vector for $t = 0$ of some differentiable curve $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$.

Equivalently, a tangent vector to M at $p \in M$ is an $\mathbb{R}$-linear map $v\colon \mathcal{C}^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule:

$$v(fg) = v(f)g(p) + f(p)v(g), \quad \text{for all } f, g \in \mathcal{C}^\infty(M).$$

Equivalently, a tangent vector to M at $p \in M$ is an equivalence class of differentiable curves $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ under the equivalence relation:

$$\alpha \cong \beta \quad \text{if} \quad \left.\frac{d(x^{-1} \circ \alpha)}{dt}\right|_{t=0} = \left.\frac{d(x^{-1} \circ \beta)}{dt}\right|_{t=0},$$

for some coordinate system x around p.

The set of all tangent vectors to M at $p \in M$ is called the *tangent space* of M at p and denoted by T_pM. It is a vector space of the same dimension as M.

Definition 2.2. Let $x\colon U \to V \subset M$ be a parametrization of a smooth manifold M^n, denote the coordinates in U by $(x^1, \ldots, x^n)$, and let $p \in V$. The *coordinate vectors* (associated to x) at the point p are the tangent vectors $\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \in T_pM$ such that

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \left.\frac{\partial(f \circ x)}{\partial x^i}\right|_{x^{-1}(p)}, \quad \text{for all } f \in \mathcal{C}^\infty(M), \quad i = 1, \ldots n.$$

Sometimes we also write $\partial_i|_p$ for $\left.\frac{\partial}{\partial x^i}\right|_p$, and we usually forget about the subindex indicating the point: ∂_i, $\frac{\partial}{\partial x^i}$. The coordinate vectors form a basis of T_pM.

The real number $\left.\frac{\partial}{\partial x^i}\right|_p (f)$ is the *directional derivative* of f at p in the i-th coordinate direction.

Definition 2.3. The *tangent* and *cotangent bundles* of a smooth manifold M are, respectively:

$$TM = \bigcup_{p \in M} T_pM \equiv \{(p, v) : p \in M,\, v \in T_pM\},$$

$$T^*M = \bigcup_{p \in M} T_p^*M \equiv \{(p, \theta) : p \in M,\, \theta \in T_p^*M\},$$

where T_p^*M is the dual vector space to T_pM.

Both are vector bundles with fibre $\mathbb{R}^n$ and base M. In particular, they admit canonical projection maps, namely $TM \to M$, $(p, v) \mapsto p$, and $T^*M \to M$, $(p, \theta) \to p$.

3. Differential maps, types of smooth maps, and submanifolds

Definition 3.1. Let $\varphi\colon M \to N$ be a smooth map, and $p \in M$. The *differential* of φ at p is the $\mathbb{R}$-linear map φ_{*p} (also denoted by $d\varphi_p$ or $d\varphi(p)$) given by:

$$\varphi_{*p}\colon T_pM \to T_{\varphi(p)}N, \quad v \mapsto \varphi_{*p}(v),$$

where the vector $\varphi_{*p}(v)$ is defined by

$$\varphi_{*p}(v)\colon \mathcal{C}^\infty(N) \to \mathbb{R}, \quad f \mapsto v(f \circ \varphi),$$

or, equivalently (by the definition of tangent vector that uses curves), by

$$\varphi_{*p}(\alpha'(0)) = (\varphi \circ \alpha)'(0),$$

where $\alpha\colon (-\varepsilon, \varepsilon) \to M$ is a differentiable curve with $\alpha(0) = p$.

The map $\varphi_* = d\varphi\colon TM \to TN$, $(p, v) \mapsto (\varphi(p), \varphi_{*p}(v))$, is the *differential* of φ.

Definition 3.2. Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds. Then:

(1) φ is an *immersion* at $p \in M$ if $\varphi_{*p}\colon T_p \to T_{\varphi(p)}N$ is injective. If it is immersion at all points of M, we say that φ is an immersion.

(2) φ is an *submersion* at $p \in M$ if $\varphi_{*p}\colon T_p \to T_{\varphi(p)}N$ is onto. If it is submersion at all points of M, we say that φ is a submersion.

(3) φ is a *(smooth) embedding* if φ is an immersion and φ is a homeomorphism onto $\varphi(M)$, where $\varphi(M)$ has the topology induced by N.

(4) If $M \subset N$, φ is the inclusion map, and φ is an immersion, then M is an *immersed submanifold* of N.

(5) If $M \subset N$, φ is the inclusion map, and φ is an embedding, then M is a *regular* or *embedded submanifold* of N.

Theorem 3.3. (Rank theorem.) *Let M^m and N^n be smooth manifolds and $\varphi\colon M \to N$ a smooth map with constant rank k (i.e. $\dim \operatorname{Im} \varphi_{*p} = k$, for all $p \in M$). Then for each $p \in M$ there exist a parametrization x on M with coordinates $(x^1, \ldots, x^m)$ centered at p (i.e. $x(0) = p$), and a parametrization on N with coordinates $(v^1, \ldots, v^n)$ centered at $\varphi(p)$, with respect to which φ has the following coordinate representation:*

$$\varphi(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

Proposition 3.4. *Let $\pi\colon M \to N$ be a submersion. Then π is an open map and, if π is onto, then π is a quotient map.*

Theorem 3.5. (Inverse function theorem.) *Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds. If φ_{*p} is a linear isomorphism, then there is an open neighborhood U of p in M such that $\varphi(U)$ is open in N and $\varphi|_U\colon U \to \varphi(U)$ is a diffeomorphism.*

In particular, φ is a local diffeomorphism if and only if φ_{*p} is a linear isomorphism for all $p \in M$.

Theorem 3.6. *Let $\varphi\colon M \to N$ be a smooth map between two smooth manifolds. Then φ is an immersion if and only if for each $p \in M$ there exists an open neighborhood U around p in M such that $\varphi|_U\colon U \to N$ is an embedding.*

Theorem 3.7. *A subset M of a smooth manifold N^n is an embedded submanifold of dimension m if and only if for each $p \in M$ there exists a coordinate system of N around p adapted to M, that is, a coordinate system of N centered at p of the form*

$$x\colon U = [-1,1]^n \subset \mathbb{R}^n \to N,$$

such that $x(0,\dots,0) = p$, and $x([-1,1]^m \times \{(0,\overset{n-m}{\cdots},0)\}) = x(U) \cap M$.

Theorem 3.8. *Let $\varphi\colon M \to N$ be an injective immersion which is proper, that is, the inverse image of every compact set is compact (this holds if M is compact). Then φ is an embedding and $\varphi(M)$ is an embedded submanifold of M.*

Theorem 3.9. *Let $\varphi\colon M^m \to N^n$ be a smooth map with constant rank k. Then every level set $\varphi^{-1}(q)$, with $q \in N$, is a closed embedded submanifold of codimension k in M.*

Theorem 3.10. (Regular level set theorem.) *Let $\varphi\colon M^m \to N^n$ be a smooth map. Let $q \in N$ be a regular value for φ, that is, for every $p \in \varphi^{-1}(q)$, the differential $\varphi_{*p}\colon T_pM \to T_qN$ is surjective. Then the regular level set $\varphi^{-1}(q)$ is a closed embedded submanifold of codimension n in M.*

4. Vector fields

Definition 4.1. A *vector field* X in a smooth manifold M is a correspondence that maps each $p \in M$ to a tangent vector $X_p = X(p) \in T_pM$.

A vector field on M is *differentiable* (or *smooth*) if $X\colon M \to TM$ is a smooth map or, equivalently, if $X(f)\colon M \to \mathbb{R}$, $p \mapsto X_p(f)$, is a smooth map for all $f \in \mathcal{C}^\infty(M)$. An equivalent characterization is that, for every parametrization with coordinates $(x^1,\dots,x^n)$, the coefficients a_i, $i = 1,\dots,n$, in the linear combination $X = a_1\partial_1 + \dots a_n\partial_n$, are smooth functions.

We denote the set of smooth vector fields on a smooth manifold M by $\mathfrak{X}(M)$. This set is a $\mathcal{C}^\infty(M)$-module of rank n, and an $\mathbb{R}$-vector space of infinite dimension.

Definition 4.2. Let $X, Y \in \mathfrak{X}(M)$. The *Lie bracket* of X and Y is the smooth vector field $[X,Y] \in \mathfrak{X}(M)$ such that

$$[X,Y](f) = X(Y(f)) - Y(X(f)), \quad \text{for all } f \in \mathcal{C}^\infty(M).$$

The Lie bracket satisfies the following properties: anticommutativity, $\mathbb{R}$-bilinearity, and Jacobi identity. In particular, $\mathfrak{X}(M)$ endowed with the Lie bracket is a Lie algebra of infinite dimension.

Definition 4.3. Let $\varphi\colon M \to N$ be a smooth map, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Y are *φ-related* if $\varphi_{*p}(X_p) = Y_{\varphi(p)}$ for all $p \in M$.

Proposition 4.4. *Let $\varphi\colon M \to N$ be a smooth map, $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$. If X_1 is φ-related to Y_1, and X_2 is φ-related to Y_2, then $[X_1,X_2]$ is φ-related to $[Y_1,Y_2]$.*

Definition 4.5. Let $X \in \mathfrak{X}(M)$. A differentiable curve α in M is an *integral curve* of X if $\alpha'(t) = X_{\alpha(t)}$, for all t in the domain of α.

Theorem 4.6. *Let $X \in \mathfrak{X}(M)$, $p \in M$. Then there exist an open neighborhood U of p in M, an interval $(-\delta, \delta)$, $\delta > 0$, and a smooth map*

$$\varphi\colon (-\delta, \delta) \times U \to M$$

such that the curve $t \in (-\delta, \delta) \mapsto \varphi(t, q) \in M$, is the unique integral curve of X that goes through $q \in U$ for $t = 0$.

It is customary to write φ_t such that $\varphi_t(q) = \varphi(t, q)$ and call $\varphi_t\colon U \to M$ the local flow *of X. It is a* local 1-parameter group *that satisfies $\varphi_0 = \mathrm{id}$, $\varphi_t^{-1} = \varphi_{-t}$, and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ when defined.*

Definition 4.7. A smooth vector field $X \in \mathfrak{X}(M)$ is *complete* if its integral curves exist for all time. In this case, the flow is defined in $\mathbb{R} \times M$.

Proposition 4.8. *If a smooth manifold M is compact, and $X \in \mathfrak{X}(M)$, then X is complete.*

Proposition 4.9. *Let M be an embedded submanifold of N. If $X \in \mathfrak{X}(N)$ is tangent to M (i.e. for all $p \in M$, $X_p \in T_pM$), then the restriction $X|_M$ of X to M is a smooth vector field. If $Y \in \mathfrak{X}(N)$ is also tangent, then $[X, Y]$ is tangent and $[X, Y]|_M = [X|_M, Y|_M]$.*

Definition 4.10. Let M^n be a smooth manifold and let $k \in \{1, \dots, n\}$. A *distribution* $\mathcal{D}$ of rank k on M is a choice of a k-dimensional vector subspace $\mathcal{D}_p$ of T_pM, for each $p \in M$.

The distribution $\mathcal{D}$ is smooth if for each $p \in M$ there exists an open neighborhood U of p and smooth vector fields $X_1, \dots, X_k \in \mathfrak{X}(U)$ that generate $\mathcal{D}_q$ for every $q \in U$. We denote by $\Gamma(\mathcal{D})$ the set of smooth vector fields $X \in \mathfrak{X}(M)$ such that $X_p \in \mathcal{D}_p$ for all $p \in M$.

The smooth distribution $\mathcal{D}$ is called *involutive* if $[X, Y] \in \Gamma(\mathcal{D})$, for all $X, Y \in \Gamma(\mathcal{D})$.

An immersed submanifold N of M is an *integral manifold* of the smooth distribution $\mathcal{D}$ if $T_pN = \mathcal{D}_p$ for all $p \in N$.

Theorem 4.11. (Frobenius theorem.) *A smooth distribution $\mathcal{D}$ is involutive if and only if there is an integral manifold of $\mathcal{D}$ through each point in M.*

The collection of all maximal connected integral manifolds of an involutive distribution $\mathcal{D}$ on M forms a foliation of M.

5. Tensors

Definition 5.1. Let V be a finite-dimensional $\mathbb{R}$-vector space, and V^* its dual space. A *tensor of type* (r, s) on V, also called *s-covariant, r-contravariant tensor*, is a multilinear map

$$V^* \times \overset{r}{\cdots} \times V^* \times V \times \overset{s}{\cdots} \times V \to \mathbb{R}.$$

We denote by $T_{r,s}(V)$ the space of all tensors on V of type (r, s).

Proposition 5.2. *Let V be a finite-dimensional vector space. There is a natural (i.e. basis-independent) isomorphism between $T_{r+1,s}(V)$ and the space of multilinear maps*

$$V^* \times \overset{r}{\cdots} \times V^* \times V \times \overset{s}{\cdots} \times V \to V.$$

In particular, we have the identification $T_{1,1}(V) = \mathrm{End}(V)$*, and also* $T_{1,0}(V) = V^{**} = V$ *and* $T_{0,0} = \mathbb{R}$*.*

Definition 5.3. A *tensor field of type* (r, s) on a smooth manifold M is a $\mathcal{C}^\infty(M)$-multilinear map of the form

$$\Omega^1(M) \times \overset{r}{\cdots} \times \Omega^1(M) \times \mathfrak{X}(M) \times \overset{s}{\cdots} \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M),$$

where $\Omega^1(M)$ is the set of all 1-forms on M (i.e. of all smooth maps $\theta\colon M \to T^*M$ with $\theta_p \in T_p^*M$, $p \in M$).

We denote the space of all tensor fields of type (r, s) on M by $\mathcal{T}_{r,s}(M)$.

Proposition 5.4. *Let* M *be a smooth manifold. There is a natural (i.e. basis-independent) isomorphism between* $T_{r+1,s}(M)$ *and the space of* $\mathcal{C}^\infty(M)$*-multilinear maps*

$$\Omega^1(M) \times \overset{r}{\cdots} \times \Omega^1(M) \times \mathfrak{X}(M) \times \overset{s}{\cdots} \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

In particular, we have the identification $\mathcal{T}_{1,1}(M) = \mathrm{End}(TM)$*. Here* $\mathrm{End}(TM)$ *is the set of all smooth maps* $F\colon TM \to TM$ *such that* $F(T_pM) \subset T_pM$ *for all* $p \in M$*, and* $F|_{T_pM}\colon T_pM \to T_pM$ *is* $\mathbb{R}$*-linear. We also have* $\mathcal{T}_{1,0}(M) = \mathfrak{X}(M)$*,* $\mathcal{T}_{0,1} = \Omega^1(M)$*, and* $\mathcal{T}_{0,0} = \mathcal{C}^\infty(M)$*.*

Definition 5.5. Let $(E_1, \dots, E_n)$ be a *local frame* on an open set U of a smooth manifold M^n, that is, n smooth vector fields defined on some open set U such that $(E_1|_p, \dots, E_n|_p)$ is a basis for T_pM at each point $p \in U$. Consider the corresponding *dual coframe* $(\varphi^1, \dots, \varphi^n)$, that is, those 1-forms satisfying $\varphi^i(E_j) = \delta^i_j$, for $i, j = 1, \dots, n$.

In terms of such local frame, a tensor field F of type (r, s) on M can be written in the form

$$F = \sum_{i_1,\dots,i_s=1}^{n} \sum_{j_1,\dots,j_r=1}^{n} F^{j_1,\dots,j_r}_{i_1,\dots,i_s} E_{j_1} \otimes \cdots \otimes E_{j_r} \otimes \varphi^{i_1} \otimes \cdots \otimes \varphi^{i_s},$$

where the $\mathcal{C}^\infty(U)$-functions

$$F^{j_1,\dots,j_r}_{i_1,\dots,i_s} = F(\varphi^{j_1}, \dots, \varphi^{j_r}, E_{i_1}, \dots, E_{i_s}).$$

are called the *components* of the tensor F in the local frame fixed above.

In particular, in terms of a coordinate frame $\{\partial_1, \dots, \partial_n\}$ and its dual coframe $\{dx^1, \dots, dx^n\}$, F has the coordinate expression

$$F = \sum_{i_1,\dots,i_s=1}^{n} \sum_{j_1,\dots,j_r=1}^{n} F^{j_1,\dots,j_r}_{i_1,\dots,i_s} \partial_{j_1} \otimes \cdots \otimes \partial_{j_r} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_s}.$$

Definition 5.6. Let F be a tensor field of type $(r+1, s+1)$ on a smooth manifold M. The *contraction* of F over the k-th covariant index and the l-contravariant index is the tensor field $\operatorname{tr} F$ of type (r, s), where $(\operatorname{tr} F)(\omega^1, \dots, \omega^r, X_1, \dots, X_s)$ is defined as the trace of the endomorphism

$$F(\omega^1, \dots, \omega^{l-1}, \cdot, \omega^{l+1}, \dots, \omega^r, X_1, \dots, X_{k-1}, \cdot, X_{k+1}, \dots, X_s) \in \mathcal{T}_{1,1}(M).$$

In terms of a local frame, the components of $\operatorname{tr} F$ are

$$(\operatorname{tr} F)^{j_1,\dots,j_r}_{i_1,\dots,i_s} = \sum_{m=1}^{n} F^{j_1,\dots,j_{k-1},m,j_{k+1},\dots,j_r}_{i_1,\dots,i_{l-1},m,i_{l+1},\dots,i_s}.$$

In particular, the contraction of a tensor field of type $(1,1)$, that is, of an endomorphism of TM, is given by its trace.

Definition 5.7. Let $\varphi\colon M \to N$ be a smooth map, and F a covariant tensor field on N of type $(0,s)$. The *pullback* of F by φ is the covariant tensor field $\varphi^* F$ of type $(0,s)$ on M given by

$$(\varphi^* F)(X_1,\dots,X_s) = F(\varphi_* X_1,\dots,\varphi_* X_s), \quad X_1,\dots,X_s \in \mathfrak{X}(M).$$

6. Lie derivative

Definition 6.1. Let M be a smooth manifold, X a smooth vector field on M, and φ_t its flow. Let F be a covariant tensor field of type $(0,s)$ on M. The *Lie derivative* of F with respect to X is the smooth tensor field $\mathcal{L}_X F$ of type $(0,s)$ on M defined by

$$(\mathcal{L}_X F)_p = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^* F)_p$$

In particular, $\mathcal{L}_X f = X(f)$ for any $f \in \mathcal{C}^\infty(M)$. We also define the Lie derivative of $Y \in \mathfrak{X}(M)$ with respect to X by $\mathcal{L}_X Y = [X,Y]$.

Proposition 6.2. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Let $f \in \mathcal{C}^\infty(M)$ and F a covariant tensor field of type $(0,s)$ on M. Then we have:*

(1) $\mathcal{L}_X(fF) = X(f)F + f\mathcal{L}_X F$.

(2) *If $Y_1,\dots,Y_s \in \mathfrak{X}(M)$, then*

$$\mathcal{L}_X(F(Y_1,\dots,Y_s)) = (\mathcal{L}_X F)(Y_1,\dots,Y_s) + F(\mathcal{L}_X Y_1,\dots,Y_s) + \cdots + F(Y_1,\dots,\mathcal{L}_X Y_s).$$

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