Bony attractors in random dynamical systems and smooth skew products

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The study of possible structures of attractors is very important for dynamical systems theory. There are well-known examples of dynamical systems such that their attractors are either smooth manifolds or locally look like a product of a smooth manifold and a Cantor-like set (Lorentz attractor, Smale–Williams solenoid). The talk is devoted to a new type of attractors so-called "bony" attractors.

Recall that a map $F\colon Y\times Z\to Y\times Z$ is called a *skew product* if $F(y,z)=(f(y),h(y,z))$ for some f, h. We will say that an attractor A of a skew product $F\colon Y\times Z\to Y\times Z$ is «*bony*» if A is a union of a graph of a continuous function defined on some subset of the base Y and an uncountable set of vertical segments ("bones") that belong to the closure of the graph.

Denote by Σ^k the space of bi-infinite sequences of numbers $1, \dots, k$. Define a Bernoulli measure μ on Σ^k using some probabilities $p_0, \dots, p_{k-1}$. Let d be a "k-adic" metric on Σ^k,

$$d(\omega,\widetilde{\omega})=k^{-\min\{\,i|\omega_i\neq\widetilde{\omega}_i \text{ or } \omega_{-i}\neq\widetilde{\omega}_{-i}\,\}}.$$

Let $\sigma\colon \Sigma^k\to\Sigma^k$, $(\sigma\omega)_j=\omega_{j+1}$ be the Bernoulli shift.

Let us consider the space of *step skew products* over the Bernoulli shift with a fiber $I=[0;1]$, i. e. the space of dynamical systems of the form

$$F\colon \Sigma^k\times I\to\Sigma^k\times I,\quad (\omega,x)\mapsto(\sigma\omega,f_{\omega_0}(x)),$$

where $f_1,\dots,f_k\colon I\to I$, f_i are C^1-smooth.

The main result is the following theorem and its smooth analogue.

Theorem 1. *For every $k\geq 2$ there exists an open non-empty subset of the space of C^1-smooth step skew products F over the Bernoulli shift $\sigma\colon\Sigma^k\to\Sigma^k$ with a fiber $I=[0;1]$ such that for every dynamical system that belongs to this subset the following conditions hold:*

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- *the maximal attractor* $A_{max} = \bigcap_{n\geq 0} F^n(\Sigma^k \times I)$ *is a union of a graph* Γ *of a continuous function* $g\colon D \to I$, $D \subset \Sigma^k$ *and a set of vertical segments ("bones"), one bone over each point* $\omega \notin D$*;*

- $\dim_H(\Omega) < \dim_H(\Sigma^k)$ *where* $\Omega = \Sigma^k \setminus D$ *is a set of fibers that contain bones; moreover,* $\mu(\Omega) = 0$*;*

- *the set* Ω *is uncountable and dense in* Σ^k*;*

- *for every subset* $S \subset \Sigma^k$ *of a full measure the maximal attractor of the map* F *coincides with the closure of the set* $A_{max} \bigcap S \times I$*; in particular, the "bones" belong to the closure of the graph;*

- *the maximal attractor coincides with the Milnor attractor.*

Note that the maximal attractor must belong to the set $\Sigma^k \times J$ where

$$J = J(f_0, \dots, f_{k-1}) = [\min_i \min(\operatorname{Fix} f_i); \max_i \max(\operatorname{Fix} f_i)]. \tag{1}$$

The following sufficient conditions play the key role in the proof of Theorem 1

Theorem 2. *Let* $f_0, \dots, f_{k-1}\colon I \to I$ *be strictly monotone maps. Let* J *be the segment* $J(f_0, \dots, f_{k-1})$ *(see* (1)*). Suppose that the following conditions hold:*

1. *there exists a finite set of finite compositions of the maps* f_i *such that the complements to the images of the segment* I *under these compositions cover the segement* I*.*

2. *there exists a finite composition of the maps* f_i *such that one of its fixed points is a repellor;*

3. *there exists a finite set of finite compositions of the maps* f_i *such that each composition contracts on the segment* I *and the images of the segment* J *under these compositions cover the segment* J*.*

Then the conclusions of Theorem 1 *hold for the corresponding step skew product* $F\colon \Sigma^k \times I \to \Sigma^k \times I$*.*