Measures which distinguish leaves[1]

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1 Introduction

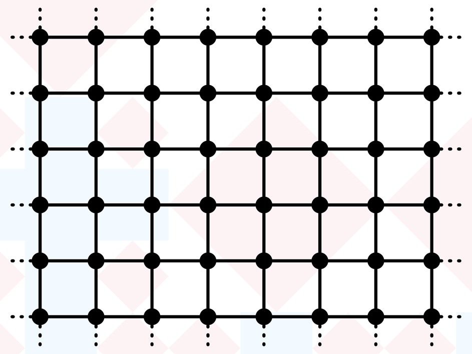
In [3], É. Ghys defines an example of minimal surface lamination with different conformal types on the leaves. In fact all leaves are parabolic except one single leaf which is hyperbolic. To do so, he uses an aperiodic and repetitive subtree of $\mathcal{Z}^2$, the Cayley graph of $\mathbb{Z}^2$, constructed by R. Kenyon. The space of all subtrees of $\mathcal{Z}^2$ rooted at the origin is a compact space foliated by graphs (whose leaves are obtained by translating the root). All the leaves can be thickened at once obtaining a uniquely ergodic lamination [1]. On the other hand, E .Blanc constructed a non-uniquely ergodic example (see [2]) replacing $\mathbb{Z}^2$ by the free group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Here we construct a non-uniquely ergodic minimal set in the Gromov–Hausdorff foliated space considered by É. Ghys. There are two kind of generic leaves having linear or quadratic growth [5].

2 The Gromov–Hausdorff foliated space

Let $\mathcal{T}$ be the set of all infinite subtrees of $\mathcal{Z}^2$ rooted at $(0,0)$. We endow it with the Gromov–Hausdorff metric:

> *«two trees are close if they agree in a big ball around the origin»*



$\mathcal{Z}^2$, the Cayley graph of $\mathbb{Z}^2$

In symbols; for each pair of trees $T, T' \in \mathcal{T}$

$$d(T,T') = e^{-\sup\{n>0 \mid B_T(0,n)=B_{T'}(0,n)\}}$$

or $d(T,T') = 1$ if the supremum does not exists. A classical diagonal argument shows that $\mathcal{T}$ *is a Cantor set*.

Let $\mathcal{R}$ be the equivalence relation defined by

$$T \,\mathcal{R}\, T' \iff T = T' - v \text{ for some } v \in \mathbb{Z}^2.$$

Each $\mathcal{R}$-class is endowed with a natural graph structure: T and $T'-v \in \mathcal{T}$ are joined by an edge if $\|v\| = 1$. Notice that each vertex $v \in T$ determines a tree $T' = T - v \in \mathcal{R}[T]$ and that two vertices v and $v' \in T$ define the same tree if and only if $T = T + v - v'$. Therefore one should think in $\mathcal{R}[T]$ as the vertices set of the graph $T/\operatorname{Iso}(T)$. In particular, if T is *aperiodic* (i.e. it does not agree with any of its translated trees) it is possible to identify T with the geometric realization $\overline{\mathcal{R}}[T]$.

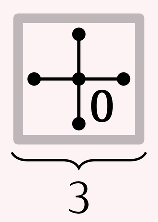
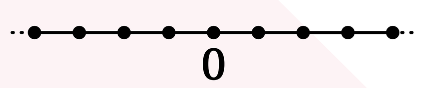
It is shown in [1] that it is possible to construct a graph foliated space $(\mathcal{X}, \mathcal{L})$ such that $\mathcal{R}$ is the induced relation by $\mathcal{L}$ on the complete closed transversal $\mathcal{T}$. Each minimal subsets of $(\mathcal{X}, \mathcal{L})$ correspond to the closure of the leave $\overline{\mathcal{R}}[T]$ through a *repetitive tree* T, that is, a tree that *«agree with itself around any vertex»*. More precisely $\forall r > 0$, $\exists R > 0$ such that

$$\forall y \in T,\ B_T(0,r) + v = B_T(v,r) \subseteq B_T(y,R), \text{ for some } v \in \mathbb{Z}^2.$$

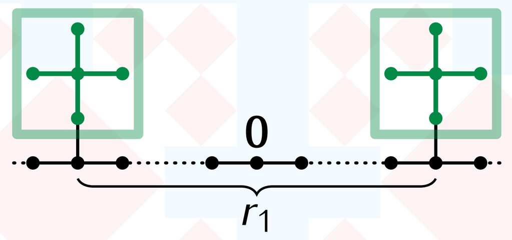
3 Construction

To construct the example we build two aperiodic repetitive trees by means of two increasing families of patches with different appearance ratio in each tree. Both trees belongs to the same minimal lamination, because they are defined with the same patches.

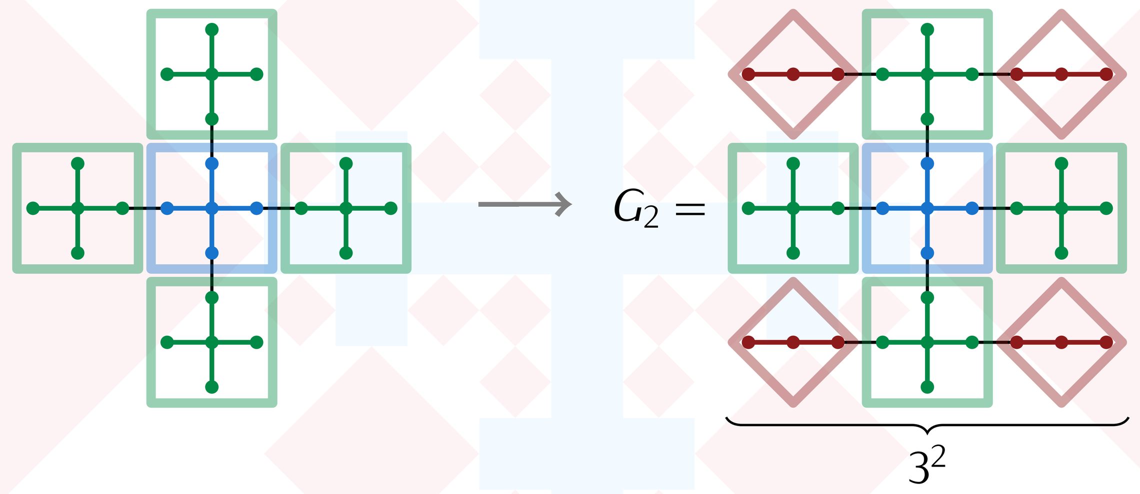
Let's start with two trees, G_1 and P_1

$G_1 =$  $P_1 =$ 

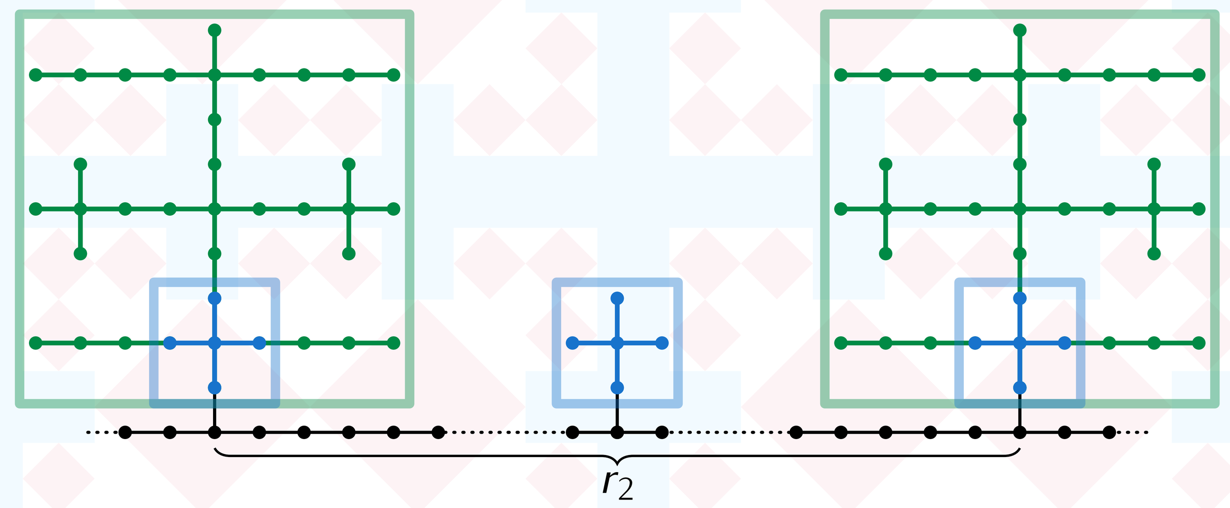
We choose $r_1 \in \mathbb{N}$ such that ${}^{\#G_1}/_{r_1} \leqslant {}^1/_3$. Then we glue to P_1 ***infinitely many copies of G_1*** at intervals of length r_1 as is shown in the following figure



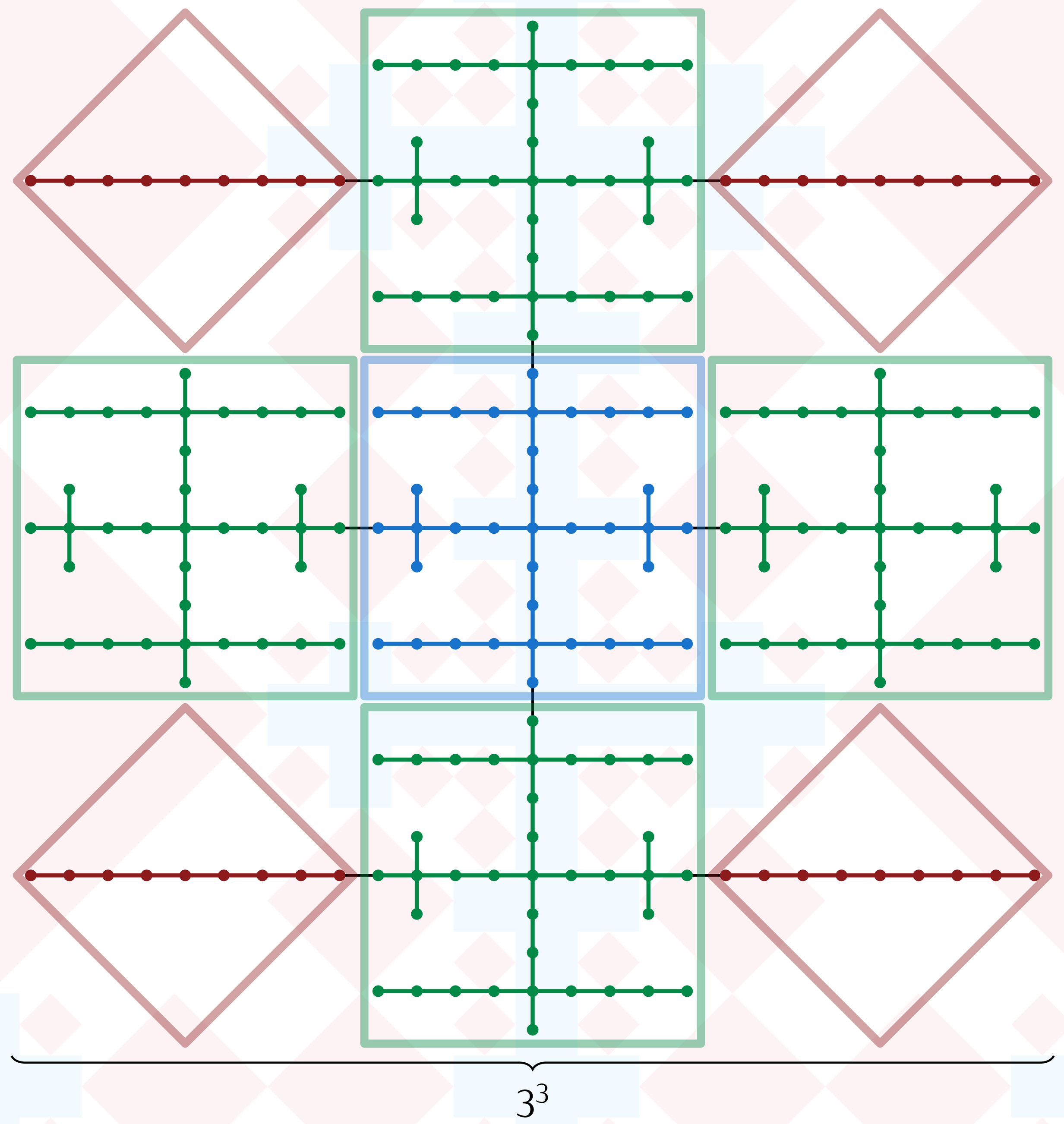
Four copies of G_1 are set around ***G_1 itself***, forming a cross-shaped tree. We add to each "vertical" arm, at each side, ***a copy of $B_{P_1}(0,1)$***:



We continue in this way to construct P_3 and G_3: we choose $r_2 \in r_1\mathbb{N}$ such that ${}^{\#G_2}/_{r_2} \leqslant {}^1/_{3^2}$. We add ***copies of G_2*** to P_2 an intervals of length r_2 replacing some ***copies of G_1***:



Now, we put ***four copies of G_2*** around ***G_2*** forming a cross and we add ***four copies of $B_{P_2}(0, \frac{3^2-1}{2})$*** as in the construction of G_2.



Continuing in this way, we construct two aperiodic repetitive trees P_∞ and G_∞. Obviously, each patch in P_∞ can be found in G_∞ and vice versa, in other words, $X = \overline{\mathcal{R}[P_\infty]} = \overline{\mathcal{R}[G_\infty]}$.

4 Ergodic properties

It is possible to construct probability measures over $\mathcal{T}$ using sequences of balls in the leaves: given $T \in X$

$$\mu_T\big(B(T', e^{-r})\big) = \lim_{n\to\infty} \frac{\#\big(B_{T'}(0,r) \cap B_T(0,n)\big)}{\#B_T(0,n)} = \text{Frequency of } B_{T'}(0,n) \text{ in } T$$

The polynomial growth of $\mathbb{Z}^2$ implies that the measures μ_T are $\mathcal{R}$-invariant, that is, $\mu_T(B) = \mu_T(B - v)$ for each $v \in \mathbb{Z}^2$ and each Borel set $B \subset \mathcal{T}$.

The sequences $\{G_n\}_n$ and $\{B_{P_\infty}(0,n)\}_n$ define two $\mathcal{R}$-invariant probability measures μ_G and μ_P respectively. Lets see they are different; define

$$\begin{aligned} X_{G_k} &= \big\{T \in X \text{ such that } G_k - v \subseteq T,\ v \in G_k\big\} \\ &= \big\{\text{trees in } X \text{ with a copy of } G_k \text{ around } (0,0)\big\}. \end{aligned}$$

As we have put copies of G_k in P_∞ at intervals of length r_k, and r_k is big enough to ensure that ${}^{\#G_k}/_{r_k} \leqslant {}^1/_{3^k}$, the appearance ratio of G_k in P_∞ is *small*. In fact

$$\mu_P(X_{G_k}) \leqslant \frac{1}{3^{k-1}} \implies \mu_P\Big(\bigcap_k X_{G_k}\Big) = 0.$$

On the other hand in G_∞ there exist *a lot of copies* of G_k. One can conclude that

$$\mu_G(X_{G_k}) \geqslant \frac{5}{12} \implies \mu_G\Big(\bigcap_k X_{G_k}\Big) > 0.$$

Hence, using an standard method, one can construct two ergodic invariant measures ν_G and ν_P such that

- ν_G-almost all leaves of $\mathcal{L}$ have the same growth as the function $f(x) = x^{\ln 5/\ln 3}$ (which is the growth of G_∞);
- ν_P-almost all leaves of $\mathcal{L}$ have two ends and linear growth (with is the growth of P_∞).

References

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[2] E. Blanc, Examples of mixed minimal foliated spaces. http://www.umpa.ens-lyon.fr/~eblanc/preprints/mixed.ps.

[3] E. Ghys, Laminations par surfaces de Riemann. *Panoramas & Synthèses* **8** (1999), 49–95.

[4] Á. Lozano Rojo. The Cayley foliated space of a graphed pseudogroup. In *Proceedings of the XIV Fall Workshop on Geometry and Physics*, Publ. de la RSME, Vol. 10 (2006), 267–272.

[5] Á. Lozano Rojo. An example of non uniquely ergodic lamination. To appear in *Ergodic Th. & Dynam. Systems*.

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