A geometrical approach to Bowen-Series coding of the geodesic flow on hyperbolic surfaces of finite volume

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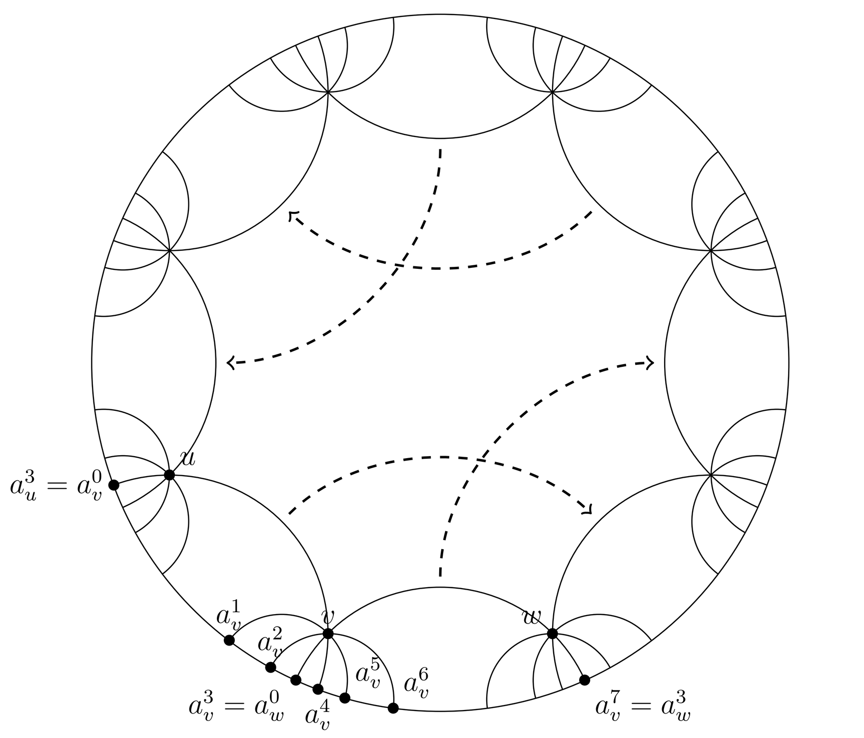
1. Bowen-Series coding

This coding is a combinatorial object that was introduced in [2]. Its dynamical properties hold strong relations with the geodesic flow on hyperbolic surfaces.

1.1 Construction

We note $\mathbb{D}$ the Poincare disk and $\partial\mathbb{D} \simeq \mathbb{S}^1$ its boundary. Let Γ be a cofinite fuchsian group of the first kind that isn't a triangle group, and that admits a fundamental domain $\mathcal{D}$ satisfying these properties :
– $\mathcal{D}$ is geodesically convex ;
– its boundary is a finitely-sided geodesic polygon with sides E and vertices $V \subset \mathbb{D} \cup \partial\mathbb{D}$;
– for every $e \in E$, there exists $\gamma_e \in \Gamma \setminus \{\text{id}\}$ such that $\gamma_e(e) \in E$; and $f = \gamma_e(e)$ if and only if $\gamma_f(f) = e$;
– $(\gamma_e)_{e\in E}$ generates Γ.
For example, we can take a Dirichlet domain based at any $P \in \mathbb{D}$ that isn't the center of an elliptic isometry of Γ.
Each side $e \in E$ can be extended to an unique geodesic $\tilde{e}$. Let $\mathcal{N}$ be the network of geodesics $\gamma(\tilde{e})$ where $\gamma \in \Gamma$, $e \in E$ and $\gamma(\tilde{e})$ either passes by an inner vertex or by two distinct vertices of $\mathcal{D}$. We make the following *even corners* assumption : **no geodesic in $\mathcal{N}$ crosses the interior of the fundamental domain**.
The trace of $\mathcal{N}$ on $\partial\mathbb{D}$ is a finite set that delimits a finite partition in intervals. For $v \in V$, we name the endpoints of geodesics of $\mathcal{N}_v = \{g \in \mathcal{N} \mid v \in g\}$:
– when v is an inner vertex, $\mathcal{N}_v$ contains $m_v = 2n_v$ elements. We start by calling a_v^0 the furthest endpoint of the left edge that goes through v, and likewise $a_v^{m_v-1}$ for the right edge. By the even corners hypothesis, all other endpoints are in $]a_v^0; a_v^{m_v-1}[$; so we order them a_v^k for $0 < k < m_v - 1$.
– when v is on the boundary, we artificially set $n_v = 3$, so that $a_v^1 = a_v^2 = a_v^3 = a_v^4 = v$.



For $v \in V$, let γ_v be the generator associated with the edge of the domain at the right of v, $I_{v,L} = [a_v^{n_v}; a_w^{n_w}[$ and $I_{v,R} =]a_v^{n_v-1}; a_w^{n_w-1}]$. We define the *left and right Bowen-Series transformations* by :

$$\begin{aligned} T_L \ : \ & \mathbb{S}^1 \to \mathbb{S}^1 \\ & x \in I_{v,L} \mapsto \gamma_v(x) \\ T_R \ : \ & \mathbb{S}^1 \to \mathbb{S}^1 \\ & x \in I_{v,R} \mapsto \gamma_v(x) \end{aligned}$$

T_L and T_R are Markov on the partition of $\mathbb{S}^1$ delimited by all the a_v^k.
For all $k \geq 0$, we can also recursively define the *left and right words in the Bowen-Series coding* by :

$$\begin{aligned} \gamma_L^k \ : \ & \mathbb{S}^1 \to \Gamma \\ & x \in I_{v,L} \mapsto \gamma_L^k[x] = \gamma_L^{k-1}[T_L(x)]\gamma_v \\ \gamma_R^k \ : \ & \mathbb{S}^1 \to \Gamma \\ & x \in I_{v,R} \mapsto \gamma_R^k[x] = \gamma_R^{k-1}[T_R(x)]\gamma_v \end{aligned}$$

1.2 Fundamental lemma

In all the following, T denote either T_L or T_R.
The lemma we're going to state encompasses all the dynamical properties of T. Most of the subsequent results are corrolaries of this one.
Note $\mathbb{I} = \{]e^{ia}; e^{ib}] \mid 0 \leq a \leq b \leq 2\pi\}$. Let X be a set on which Γ acts on the left. Γ acts also naturally on the left on $\mathbb{I}$.
We says that $F : \mathbb{I} \times X \to \mathbb{C}$ verifies $\mathcal{I}(I, \gamma)$ when :

$$\forall x \in X, F(I, x) = F(\gamma(I), \gamma(x)).$$

Lemma 1.2.1. *Let $F : \mathbb{I} \times X \to \mathbb{C}$ such that*
(i) if F verifies $\mathcal{I}(I, \gamma)$, then it verifies $\mathcal{I}(J, \gamma)$ for all $J \subset I$, $J \in \mathbb{I}$ (inclusion) ;
(ii) if $I, J, I \sqcup J \in \mathbb{I}$, then $F(I \sqcup J, x) = F(I, x) + F(J, x)$ for all $x \in X$ (additivity for contiguous intervals) ;
(iii) if $(b_n) \to b$, then $(F(]a; b_n], x)) \to F(]a; b], x)$ for all $x \in X$ (continuity).
(iv) F verifies $\mathcal{I}(I_v, \gamma_v)$ for every $v \in V$.
Then F verifies $\mathcal{I}(I, \gamma)$ for every $I \in \mathbb{I}$ and $\gamma \in \Gamma$.
Basically, it allows us to transport the combinatorics of intervals under the action of T to relations in X.

1.3 Periodic points and hyperbolic isometries of Γ

The lemma can easily prove this famous result :
Theorem 1.3.1 (Series, [2])**.** *T is orbit-equivalent with the group Γ, i.e. for all $x, y \in \mathbb{S}^1$,*

$$\exists \gamma \in \Gamma, y = \gamma(x) \Leftrightarrow \exists p, q \geq 0, T^p(x) = T^q(y).$$

Since Γ is a group of the first kind, this implies :
Theorem 1.3.2 (Pre-periodic points)**.**
(i) If $y \in Per(T)$, then $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) = y\}$ is dense in $\mathbb{S}^1$.
(ii) $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) \in Per(T)\} \subset Fix(\Gamma)$.
The fact that pre-images of a periodic points are dense forbids us to be able to find a trivial word in the coding :
Theorem 1.3.3. *For all $x \in \mathbb{S}^1$ and $k > 0$, $\gamma^k[x] \neq id$.*
The fundamental lemma can actually prove stronger variant of the orbit-equivalence theorem :
Theorem 1.3.4 (Series, revisited)**.** *$(x, k) \to \gamma^k[x]$ is orbit-equivalent with the group Γ, i.e.*

$$\forall x \in \mathbb{S}^1, \forall \gamma \in \Gamma, \exists p, q \geq 0, \gamma^p[x] = \gamma^q[\gamma(x)]\gamma.$$

This allows us to prove the other inclusion :
Theorem 1.3.5 (Pre-periodic points, revisited)**.**
(i) $\{x \in \mathbb{S}^1 \mid \exists p \geq 0, T^p(x) \in Per(T)\} = Fix(\Gamma)$.
(ii) If $x \in Per(T)$ has period k, then $\gamma^k[x]$ is primitive.
With this, we're now in position to identify periodic T-orbits with conjugacy classes of primitive elements of Γ under the extra hypothesis $|T'| \geq 1$. This is verified in particular when edges of the fundamental domain are the isometric circles of the associated generators, thus for a Dirichlet domain centered at 0.
Theorem 1.3.6. *Suppose that $|T'| \geq 1$. There is a bijection between periodic hyperbolic orbits of T and conjugacy classes of primitive hyperbolic elements of Γ.*
Corollary 1.3.7. *Suppose that $|T'| \geq 1$. There is a bijection between periodic orbits of T that don't pass through a cusp of Γ and conjugacy classes of primitive hyperbolic elements of Γ.*
Morita gives a similar result in [4] but a finite number of periodic orbits are missing from the counting.

2. Natural extension

One can wonder what is the relation between the Bowen-Series coding and the geodesic flow on $\mathbb{D}/\Gamma$. We link them through the *geodesic billiard* of $\mathcal{D}$.

2.1 Construction

T_L and T_R can be seen as the factors of
Let Δ be the diagonal of the two-dimensional torus $\mathbb{T}^2$.
Theorem 2.1.1. *There exists $C \subset \mathbb{T}^2 \setminus \Delta$ and $T_C : C \to C$ such that :*
(i) T_C is a bijection ;
(ii) For every $(x, y) \in C$,

$$T_C(x, y) = (\gamma_R[y](x), \gamma_R[y](y)) = (S_L(x, y), T_R(y))$$
$$T_C^{-1}(x, y) = (\gamma_L[x](x), \gamma_L[x](y)) = (T_L(x), S_R(x, y));$$

(iii) $T_C^p(x, y) = (x, y) \Leftrightarrow T_L^p(x) = x \Leftrightarrow T_R^p(y) = y$.
A first construction of an extension of the coding was given in [1], but only in the cocompact case and for a specific fundamental domain.

2.2 Extension and geodesic billiard

Let B be the set of all geodesics of $\mathbb{D}$ that either
– cross the interior of $\mathcal{D}$;
– pass through a vertice $v \in V$ while keeping the fundamental domain on their right ;
– are in $\mathcal{N}$ and keep the domain on their left.
We set $T_B(x, y) = (\gamma_e(x), \gamma_e(y))$ whenever $(x, y) \in B$ leaves the fundamental domain by the edge e. T_B is a bijection of B and (B, T_B) is called the *geodesic billiard*.
The geodesic flow on $\mathbb{D}/\Gamma$ can be obtained as the suspension of (B, T_B) by the transit time, and inversely (B, T_B) can be seen as a Poincare section of the geodesic flow.
Theorem 2.2.1. *(B, T_B) and (C, T_C) are conjugated. More precisely, we can construct a bijection $\varphi : B \to C$ such that :*
(i) $\varphi = id$ on $B \cap C$;
(ii) there exists $p > 0$, $X_1 \ldots X_p \subset \mathbb{T}^2 \setminus \Delta$ and $\gamma_1 \ldots \gamma_p \in \Gamma$ for which :
– $B \setminus C = \sqcup_{k=1}^p X_i$;
– $C \setminus B = \sqcup_{k=1}^p Y_i$ where $Y_i = \gamma_i(X_i)$;
– $\forall i, \forall (x, y) \in X_i, \varphi(x, y) = (\gamma_i(x), \gamma_i(y))$.
(iii) $\varphi T_B = T_C \varphi$.
The most interesting fact is that the conjugacy is determined by a finite partition.

2.3 Bordering geodesics

All the precedent results allow us to give a precise description of what the geodesics that edge the fundamental domain really are on the surface.
Theorem 2.3.1. *Let g be a geodesic of $\mathbb{D}$ that borders the fundamental domain $\mathcal{D}$. Then :*
(i) either g projects itself onto a closed geodesic of $\mathbb{D}/\Gamma$;
(ii) or both endpoints of g are projected onto (possibly different) cusps of $\mathbb{D}/\Gamma$.

3. Transfer operator and eigenfunctions of the hyperbolic laplacian

3.1 Helgason boundary values

Consider :
(i) $\mathcal{E}_\lambda$ the space of the eigenfunctions of the hyperbolic laplacian on $\mathbb{D}$ for the eigenvalue λ ;
(ii) $\mathcal{E}_\lambda^e$ those that are at most of exponential growth in the hyperbolic radius ;
(iii) $\mathcal{D}'(\mathbb{S}^1)$ the space of distributions of $\mathbb{S}^1$.
A well-known result says that you can represent every function f of $\mathcal{E}_\lambda^e$ by a couple of distributions $\mathcal{D}_{f,s}$ and $\mathcal{D}_{f,1-s}$, the *Helgason boundary values* of f.
Theorem 3.1.1 (Helgason)**.**

$$\begin{aligned} \mathcal{P}^s \ : \ \mathcal{D}'(\mathbb{S}^1) & \to \mathcal{E}_{-s(1-s)}^e \\ T & \mapsto z \mapsto \langle T, P^s(z, .)\rangle \end{aligned}$$

is a continuous isomorphism of reciprocal $f \to \mathcal{D}_{f,s}$.
When the eigenfunctions are bounded, Otal refined this result in [5]. First, we define the *derivative* of a continus function F defined over $[0; 2\pi[$ by the linear functional

$$\begin{aligned} F' \ : \ \mathcal{C}^1(\mathbb{S}^1) & \to \mathbb{C} \\ \varphi & \mapsto (F(2\pi) - F(0))\varphi(0) - \int_0^{2\pi} \varphi'(t)F(t)dt \end{aligned}.$$

Then take :
(i) $\mathcal{E}_\lambda^b$ the space of bounded functions of $\mathcal{E}_\lambda$;
(ii) Λ_α the space of α-Hölder functions over $[0; 2\pi[$ that vanish at 0 ;
(iii) Λ_α^1 the space of derivates of such functions.
Theorem 3.1.2 (Otal)**.**

$$\begin{aligned} \mathcal{P}^s \ : \ \Lambda_{\Re(s)}^1 & \to \mathcal{E}_{-s(1-s)}^b \\ T & \mapsto z \mapsto \langle T, P^s(z, .)\rangle \end{aligned}$$

is a continuous isomorphism of reciprocal $f \to \mathcal{D}_{f,s}$. More precisely, if $f = \mathcal{P}^s(D')$ with $D \in \Lambda_\alpha$, then

$$\forall z \in \mathbb{D}, |f(z)| \leq C(s)\|D\|_\alpha e^{-(\alpha - \Re(s))d(0,z)}.$$

This implies that if f is a bounded solution of $\Delta f = -s(1-s)f$ that is also automorphic for a cofinite group Γ, then $\Delta_{f,s}$ is the derivative of a $\Re(s)$-Hölder function **and nothing more**.

3.2 Eigenfunctions and eigendistributions of the transfer operator

The *transfer operator* of $T = T_L$ or T_R is given by :

$$\begin{aligned} \mathcal{L}_s \ : \ & E \to E \\ & f \mapsto \mathcal{L}_s(f) : y \in \mathbb{S}^1 \mapsto \sum_{T(x)=y} \frac{f(x)}{|T'(x)|^s} \end{aligned}$$

The eigendistributions of this operator for the eigenvalue 1 are exactly the Helgason boundary values of eigenfunctions of the hyperbolic laplacian :
Theorem 3.2.1. *Let $\nu \in \Lambda_{\Re(s)}^1$. Then $\mathcal{L}_{s,R}^\star \nu = \nu$ if and only if $\mathcal{P}^s(\nu) \in \mathcal{E}_{-s(1-s)}^b$ is Γ-automorphic.*
This result was hinted in [6] for a different setting.
It's natural to focus now on the 1-eigenfunctions of the transfer operator. Let $d(x, y) = \frac{|x-y|}{2}$ be the Gromov distance on $\mathbb{S}^1$ and $k^s(x, y) = d(x, y)^{-2s}$. The only thing we know about the eigenfunctions is that there are more of them than eigendistributions :
Theorem 3.2.2 (Lopes-Thieullen, [3])**.** *Suppose Γ cocompact. Take $\mathcal{D}_{f,s}$ the boundary value of $f \in \mathcal{E}_{-s(1-s)}^b$ Γ-invariant. Let $\psi_{f,s}(x) = \langle \mathcal{D}_{f,s}, k^s(x, .)\mathbb{1}_C(x, .)\rangle$ where C is the support of the natural extension (C, T_C) of T_L and T_R. Then $\mathcal{L}_{s,L}\psi_{f,s} = \psi_{f,s}$ and $\mathcal{D}_{f,s} \to \psi_{f,s}$ is injective.*

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