(Closed) Elastic Curves And Rods

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Abstract

Lecture 1 The Bernoulli-Euler Elastica

We derive the Euler-Lagrange equations for the *Elastica*, or *Elastic Curve* in Euclidean space. Using Nöether's theorem, conservation laws lead to first integrals and then to *Killing Fields*. We explicitly solve the equations, giving rise to a parametrized space of solutions. We solve the boundary value problem for closed elastic curves and determine all solutions. We look briefly at the question of stability of solutions.

Lecture 2 Elastic curves in Riemannian manifolds

Using the standard machinery of Riemannian geometry, including the covariant derivative and Riemann tensor, we extend the notion of elastic curve to Riemannian manifolds. In the case of space forms, we demonstrate the existence of Killing fields and cylindrical coordinate systems, leading to integration of the equations of the elastica. We look particularly at the case of the hyperbolic plane.

Lecture 3 Elastic rods, knots, and curve straightening.

Returning to the Euclidean elastica, we examine the phenomenon of knotted solutions. We generalize the notion of elastica to that of the centerline of a uniform, or symmetric, Kirchhoff elastic rod. This can be reformulated as a variational problem on curves involving a new quantity, the total torsion. We investigate the closed curve solutions to this problem and discover new knotted solutions.

Lecture 4 Liouville Integrability and Hamiltonian systems

The Bernoulli-Euler elastic curve and the uniform Kirchhoff elastic rod can be viewed as solutions of *geometric variational problems on frame bundles.* The cotangent bundle T^*E of the frame bundle of a manifold has a natural symplectic structure. Using ideas from Control Theory, especially the *Pontrjagin Maximum Principle*, we derive the Hamiltonian equations of the elastica on T^*E. Using the Poisson bracket on functions as a calculational tool, we see why the equations of the elastica turn out to be integrable on space forms. We briefly consider other geometric variational problems, including the elastica in complex projective space and the affine and centro-affine elastic curves.

Lecture 5 Miscellaneous topics

We look at some of the ways the elastic curve appears in geometry. The free elastica in the hyperbolic plane gives rise to a *Willmore Torus* in $\mathbb{R}^3$. the elastic helices in complex projective space give rise to Willmore tori in the five sphere. Closed elastic curves and rod centerlines in Euclidean space give rise to soliton solutions of the Localized Induction Equation. We also briefly look at some unsolved problems.