Practical Session

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1 Spline-Elastrip

From the classical Calculus of Variations [6], let X be some subset of sufficiently smooth functions defined on some closed interval $[a, b]$, consider functionals $F\colon X \longrightarrow \mathbb{R}$ of the form

$$F(x) = \int_a^b f(t, x, \dot{x})\, dt$$

where $\dot{x} = dx/dt$ and $f\colon \mathbf{R}^3 \longrightarrow \mathbf{R}$ is a given function. In searching for $x_f \in X$ with the property that $F(x_f) \leq F(x)$ for all $x \in X$, to expose potential candidates for x_f the conventional approach utilizes the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}}(t, x, \dot{x}) = \frac{\partial f}{\partial x}(t, x, \dot{x}) \tag{1}$$

with the boundary conditions. In the simplest problem of the calculus of variations, If there is sufficient smoothness, then (1) is a second order differential equation. To deal with isoperimetric constraints of the form

$$G(x) = \int g(t, x(t), \dot{x}(t))\, dt = 0,$$

the classical theory establishes some version of the Lagrange multiplier theorem. Superficially, replace f by $f + \lambda g$ in (1). Typically, the domain X is not "flat", which leads to considerable complications.

Geometric methods are made accessible once an inner product $\langle , \rangle$ is defined on the functions space H. The gradient and the directional derivative of a functional F are paired dually by $\langle \nabla F(x), \vec{v} \rangle = DF(x)\,\vec{v}$. The x_f solution, called *critical point*, is determined by the equation $DF(x_f) = 0$, that also includes the boundary conditions. This equation, independent of the inner product, is equivalent to $\nabla F(x_f) = 0$. This statement incorporates the Euler-Lagrange equation as well as necessary conditions given by the natural boundary conditions. It follows that the last equation is in general more selective than the Euler-Lagrange equation. Although the choice of metric is not important when finding solutions of this last equation, from the numerical point of view, a "good choice" of metric on H makes "numerically friendly" gradient formulas in terms of certain Euler operators. [9, 10, 11].

Far from the dark side of the theory, it is known that the vector $-\nabla F(x)$ points in the direction of minimal increase. So, with slight abuse of notation, choosing a starting point (curve, function) $x_o \in H$, the new one $x_1 = x_o - h\nabla F(x_o)$, for h small enough, satisfies $F(x_1) < F(x_o)$. This is nothing but the steepest descent method, that may be used to solve the general isoperimetric problem by following the trajectory in the negative gradient direction. The curve sequence $\{x_n\}$ created this way evolves searching for a minimum x_f of the variational problem.

This experimental point of view may be applied in known problems of calculus of variations like these:

1. Looking for the smallest length path joining two points in a bi-dimensional surface patch.
2. The Bachistochrona problems.
3. The Catenoid, least area surface of revolution.
4. Constant mean curvature surfaces of revolution.
5. The Elasticae and "hyper-elastic" curves in the plane:

   $>$From Euler-Bernoulli, let γ be a smooth curve in the Euclidean plane and let κ be its signed curvature. The quantity $F(\gamma) = \int_\gamma \kappa^2 ds$ is known as the total squared curvature. $F(\gamma)$ has a physical interpretation as the "elastic" energy stored in a thin rod shaped as γ. The *elastic curves* are the critical points of this functional.

   a) A less known elastica: the Cornu spiral.
6. Membranes over planar elasticae.

Look the *Calculus of Variations and Geometry* topic in [6].

2 Another point of view: Elasticae in the Sphere

The variational problem associated to functionals of the type $F(\gamma) = \int_\gamma P(\kappa)$, with P a smooth function and κ the curvature of the curve γ, has a long tradition in Mathematics. If $P(\kappa) = const$ is constant we were talking about geodesics and its study can be traced back to the very beginning of the calculus of variations. By using a less elementary choice of $P(\kappa)$, D. Bernoulli proposed around 1740 a simple geometric model according to which an elastic curve or elastica is a minimum of the bending energy functional. Planar Elastic curves were classified by L. Euler in 1743. The elastica problem in real space forms has been recently considered under different points of view, More generally, the geometric importance of minimizing a curvature energy functional of the type $F(\gamma) = \int_\gamma P(\kappa)$, defined on a certain space of curves in the 3-dimensional Euclidean space $\mathbf{R}^3$, was pointed out by W. Blaschke in his book on Differential Geometry.

Critical points of $F(\gamma) = \int_\gamma P(\kappa)$ in real space forms must lie fully in a 3-dimensional space. The corresponding Euler-Lagrange equations can be expressed in terms of the first and second Frenet curvatures, that is in terms of the curvature and torsion of γ. By using the symmetries of the problem, one can obtain first integrals and show that κ and τ can be integrated by quadratures. Curvature and torsion determines the corresponding critical curve up to rigid motions of ambient space, but even if we were able to obtain explicit expressions for κ and τ, explicit integration of the Frenet equations are rarely possible.

Fortunately, we can choose a special coordinate system where the coordinates of a critical point are also integrable by quadratures. On the other hand, boundary and initial conditions of the variational problem have to be precised. Due to its geometric importance we want to study closed critical points. We notice that even if we obtain periodic solutions of the EL equations the corresponding periodic curve is not necessarily closed. Again, we use the special coordinate system we talked about before, to give closure conditions for a curve which corresponds to periodic solutions of EL equations to close up. This is as far as one can go with a general Lagrangian $F(\gamma) = \int_\gamma P(\kappa)$. The point now is to choose $P(\kappa)$ so that above program can be further developed in order to complete the whole integration process and obtain the classification of closed critical points in terms of the integration parameters.

In order to show this point of view, we consider the elastic energy functional $F(\gamma) = \int_\gamma \kappa^2$ defined in the Sphere, because it offers many options such as:

- searching for closed elastic curves in $\mathbf{S}^2$,
- searching for closed elastic curves in $\mathbf{S}^3$,
- Obtaining Willmore vesicles in $\mathbf{R}^3$ from elastic curves in $\mathbf{S}^2$.

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