Ricci solitons as submanifolds of complex hyperbolic spaces

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CENTRO DE INVESTIGACIÓN E TECNOLOXÍA MATEMÁTICA DE GALICIA

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Classes

- $c > 0$, shrinking.
- $c = 0$, steady.
- $c < 0$, expanding.

Homogeneous setting

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Definition

An algebraic Ricci soliton is a Lie group G with left-invariant metric such that

$$\mathrm{Ric} = c\,\mathrm{id} + \mathcal{D},$$

with $\mathcal{D}\colon \mathfrak{g} \to \mathfrak{g}$ a derivation.

Solvsoliton: Solvable algebraic Ricci soliton.

Nilsoliton: Nilpotent algebraic Ricci soliton.

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Any homogeneous Ricci soliton with $c < 0$ is isometric to a Lie subgroup of the solvable model of a symmetric space of non-compact type.

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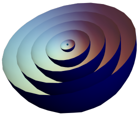
Iwasawa decomposition and solvable model

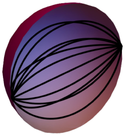
$\widetilde{M} \cong G/K$ symmetric space of non-compact type $\implies$ G real semisimple.

Iwasawa decomposition theorem

$G = KAN$, K compact, A abelian, N nilpotent.



$K \curvearrowright \mathbb{R}H^3$



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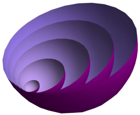

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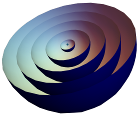

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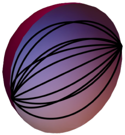

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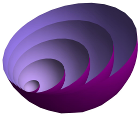

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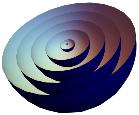

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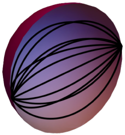

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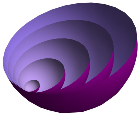

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Solvable model

$\widetilde{M}$ is isometric to the Lie group AN with a left-invariant metric.

Subgroups of AN and Einstein and Ricci soliton metrics

- Tamaru (2011): Parabolic subgroups of semisimple Lie groups and Einstein solvmanifolds.
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Objective today: Classify Lie subgroups of $AN \cong \mathbb{C}H^n$ that are algebraic Ricci solitons with the induced metric.

$\rightsquigarrow$ $\mathbb{C}H^n$: Simplest symmetric space of non-compact type where the classification is open.

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Structure of solvsolitons [J. Lauret, 2011]

S solvmanifold with metric Lie algebra $(\mathfrak{s}, \langle\cdot,\cdot\rangle)$. Consider the orthogonal decomposition $\mathfrak{s} = \mathfrak{b} \oplus \mathrm{Nil}(\mathfrak{s})$. Then S is a solvsoliton iff

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$\updownarrow$

Nilsolitons $L < N$ and their possible rank-one non-nilpotent extensions in AN.

Kähler angle

Let V be a real subspace of $\mathbb{C}^n$.

Definition

V constant Kähler angle $\varphi \in [0, \frac{\pi}{2}]$ if $\angle(V, \mathrm{i}\, v) = \varphi$ for all $v \in V$, $v \neq 0$.

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Theorem [Díaz-Ramos, Kollross, Domínguez-Vázquez, 2017]

Let $V \subset \mathbb{C}^n$ be any real subspace. Then $V = V_1 \oplus \ldots \oplus V_r$ such that:

- V_k has constant Kähler angle φ_k and $\varphi_k \neq \varphi_l$ if $k \neq l$.
- $\mathbb{C}V_k \perp \mathbb{C}V_l$ for every $k \neq l$.

Kähler angle and algebraic Ricci soliton condition

$AN \cong \mathbb{C}H^n$ solvable model, $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{a} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

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Let $\mathfrak{h} \subset \mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ ($\mathfrak{g}_1 \simeq \mathbb{C}^{n-1}$, $\mathfrak{g}_2 \cong \mathbb{R}$) be a Lie subalgebra.

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Is $\mathcal{D}(c) := c\,\mathrm{id} + \mathrm{Ric}^H$ a derivation of $\mathfrak{h}$ for some $c \in \mathbb{R}$?

$$\mathcal{D}(c)[X, Y] \stackrel{?}{=} [\mathcal{D}(c)X, Y] + [X, \mathcal{D}(c)Y], \; X, Y \in \mathfrak{h}.$$

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$\mathfrak{m} \subset \mathfrak{g}_1$,

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$c(\varphi_i) \neq c(\varphi_j)$ if $i \neq j \implies \mathfrak{m} = \mathfrak{m}_{\varphi_1} \oplus \mathfrak{m}_{\frac{\pi}{2}}$.

Theorem (— , Sanmartín-López)

A Lie subgroup $H < AN \cong \mathbb{C}H^n$ of dim ≥ 2 is an algebraic Ricci soliton iff

Subalgebra	Isometric to	Einstein?

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Corollary 1

$S < AN \cong \mathbb{C}H^n$, considered with the induced metric. Then:
S Einstein $\iff$ S is a symmetric space.

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Corollary 3

Let $S < AN \cong \mathbb{C}H^n$. Suppose that S is an algebraic Ricci soliton with the induced metric.
$\mathrm{Nil}(S)$ is non-flat $\iff$ $\mathrm{Nil}(S)$ is a minimal submanifold of N.

Corollaries II

$\mathbb{R}^n$ [A. Di Scala, 2002], $\mathbb{R}H^n$ [A. Di Scala, C. Olmos, 2001]

In $\mathbb{R}^n$ and in $\mathbb{R}H^n$ minimal homogeneous submanifolds are totally geodesic.

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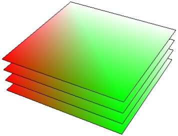

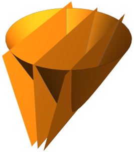

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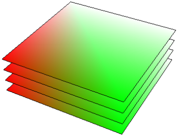

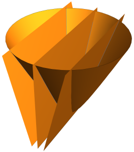

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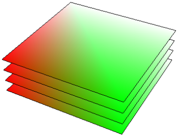

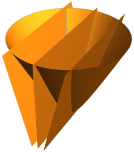

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Corollary 4

Let $S < AN \cong \mathbb{C}H^n$.
S Einstein and minimal in $AN \iff S$ totally geodesic in AN.