Torsion-free connections with prescribed curvature

Efraín Basurto-Arzate

Fakultät für Mathematik

technische universität
dortmund

Symmetry and Shape, September 2024

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Curvature of torsion-free connections Basic background

 \bullet Given a connection on a smooth, connected manifold M, its *curvature tensor* is defined as the tensor field $R^\nabla \in \mathsf{\Gamma}(\bigwedge^2 T^*M \otimes \mathsf{End}(TM))$ defined as

$$
R^{\nabla}(X,Y)Z:=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.
$$

If ∇ is *torsion-free* (i.e. it satisfies $\nabla_{X}Y-\nabla_{Y}X=[X,Y]),$ R^{∇} satisfies the Bianchi identities:

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Curvature of torsion-free connections

Basic background

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Curvature of torsion-free connections

Basic background

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2.

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Algebraic curvature tensors

The Bianchi identities can be algebraically encoded:

Let V a finite real vector space. Let $\mathfrak{h} \subset \mathfrak{g} := \text{End}(V)$ be a Lie subalgebra. The space of algebraic curvature tensors is defined as the subspace

$$
K(\mathfrak{h}) \coloneqq \left\{ R \in \bigwedge^2 V^* \otimes \mathfrak{h} \; \left| \; \sum_{\text{cyc}(x,y,z)} R(x,y)z = 0 \in V \, \forall x,y,z \in V \right. \right\}
$$

$$
\mathcal{K}^1(\mathfrak{h}) \coloneqq \left\{ \phi \in V^* \otimes \mathcal{K}(\mathfrak{h}) \ \left| \ \sum_{\text{cyc}(x,y,z)} \phi(x)(y,z) = 0 \in \mathfrak{h} \ \forall x,y,z \in V \right. \right\}.
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Algebraic curvature tensors

A consequence of the Ambrose-Singer Holonomy Theorem is the fact that the curvature tensor R^∇ satisfies, for every $x\in M$:

> $R_{x}^{\nabla} \in K(\mathfrak{hol}_x(\nabla)),$ $(\nabla R^{\nabla})_{x} \in K^{1}(\mathfrak{hol}_{x}(\nabla)).$

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The problem

These observations naturally lead to the following question:

Under which conditions can it be guaranteed, that a given curvature map S: $U \subset V \longrightarrow K(\mathfrak{a})$ can be induced by the curvature tensor of a torsion-free connection?

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The problem

These observations naturally lead to the following question:

Under which conditions can it be guaranteed, that a given curvature map S: $U \subset V \longrightarrow K(\mathfrak{g})$ can be induced by the curvature tensor of a torsion-free connection?

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Local problem

By choosing a normal coordinate system around a fixed p on the manifold M we can restrict ourselves, without loss of generality, to the case in which $M = U$, where U denotes an open subset of V, which is also star-shaped around 0 such that the exponential map ex $\mathsf{p}_0\colon\mathsf{U}\subset\mathsf{V}\cong\mathsf{T}_0\mathsf{U}\longrightarrow\mathsf{U}$ associated to a torsion-free connection ∇ on $TU = U \times V$ is the identity.

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The principal bundle setting

- Let $\theta \in \Omega^1(\mathcal{F}(U),V)$ denote the tautological form of $\mathcal{F}(U).$
- $\omega^\nabla\in\Omega^1(\mathcal{F}(U),\mathfrak{g})$ denotes the connection form associated to $\nabla.$
- $\mathcal{F}^\nabla\coloneqq \mathrm{d}\omega^\nabla+\omega^\nabla\wedge\omega^\nabla$ denotes the curvature form.
- In this setting, the fact that ∇ is torsion-free is equivalent to

$$
d\theta + \omega^{\nabla} \wedge \theta = 0.
$$

$$
F^{\nabla} \wedge \theta = 0,
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dF^{\nabla} + ad(\omega^{\nabla}) \wedge F^{\nabla} = 0.
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The principal bundle setting

• Let $R \in C^{\infty}(F(U), K(g))^G$ be the unique G-equivariant map associated to R^∇ that satisfies

$$
\mathcal{F}^{\nabla} = R(\theta, \theta).
$$

Parallel translation along radial geodesics defines a smooth section of ${\mathcal{F}}(U)$. Concretely, the map $h\colon U\longrightarrow G$ defined by $h({\mathcal{v}})=P_{{\gamma}_{\mathcal{v}}}$ is a smooth map, and so $\sigma := (v, h) \in \Gamma(F(U)).$

• Along this section we obtain:

where $\Gamma_i = (\Gamma_{ij}^k)_{k,j}$.

$$
\hat{\theta} := \sigma^* \theta = h^{-1} \mathrm{d} v,
$$

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The principal bundle setting

From the relation $\mathcal{F}^\nabla = R(\theta,\theta)$, we thus obtain:

$$
\hat{\mathcal{F}} = \hat{\mathcal{R}}(\hat{\theta}, \hat{\theta}),\tag{1}
$$

where

 $\hat{R} := h^{-1} \cdot R^{\nabla}.$

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An initial value problem

Let $W \subset \bigotimes^m V^* \otimes \mathfrak{g}$ be a subspace.

Let $\mathcal{L}_{\mathscr{E}}\colon \Omega^k(U,W)\longrightarrow \Omega^k(U,W)$ denote the Lie derivative along the Euler vector field $\mathscr{E} := e_i e^i$.

There exists an integration map $I: \Omega^k(U, W) \longrightarrow \Omega^k(U, W)$ such that for $\eta \in \Omega^k(U,W)$:

$$
l\mathcal{L}_{\mathscr{E}}\eta = \mathcal{L}_{\mathscr{E}}l\eta = \begin{cases} \eta & k \ge 1, \\ \eta - \eta_0 & k = 0. \end{cases}
$$
 (2)

where $\eta_0 \equiv \eta(0) \in C^{\infty}(U, W)$.

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$$

An initial value problem

The main consequence of Proposition [1.2](#page-31-0) is the following:

The pulled-back form $\hat{\theta}$ is a solution to the singular initial value problem

$$
\begin{cases}\n\mathcal{L}_{\mathscr{E}}(\mathcal{L}_{\mathscr{E}} - id)\theta = \hat{R}(\mathscr{E}, \theta)\mathscr{E}, \\
\theta_0 = id, d\theta_0 = 0.\n\end{cases}
$$
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An initial value problem

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Main Theorem

The key observation is the fact that solutions to the singular initial value problem [\(3\)](#page-33-0), for suitable curvature maps $U \longrightarrow K(\mathfrak{g})$, are decisive for the existence of torsion-free connections.

We break down the Main Result in a series of 3 principal statements:

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Main Theorem

Proposition

Let S: $U \longrightarrow K(\mathfrak{g})$ be a real analytic map. Then, the singular initial value problem

$$
\begin{cases}\n\mathcal{L}_{\mathscr{E}}(\mathcal{L}_{\mathscr{E}} - id)\theta = \mathcal{S}(\mathscr{E}, \theta)\mathscr{E}, \\
\theta_0 = id, d\theta_0 = 0\n\end{cases}
$$
\n(4)

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admits a unique real analytic solution.

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Main Theorem

Proposition

Let S: $U \longrightarrow K(\mathfrak{g})$ be a real analytic map. Let θ be the real analytic solution to the singular initial value problem [\(4\)](#page-38-0). Set $\omega = I(S(\mathscr{E}, \theta))$. It then holds:

i)
$$
\theta = d\mathbf{v} + I(\omega \cdot \mathscr{E}),
$$

\nii) $\mathcal{L}_{\mathscr{E}}(d\theta + \omega \wedge \theta) = (d\omega + \omega \wedge \omega - S(\theta, \theta)) \cdot \mathscr{E}.$

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Main Theorem

Theorem

Let $U \subset V$ a star-shaped around 0 open subset, let $S: U \longrightarrow K(\mathfrak{g})$ be a real-analytic map, θ the analytic solution to the singular initial value problem [\(4\)](#page-38-0), and suppose $\omega = I(S(\mathscr{E}, \theta))$ satisfies the consistency relation $d\omega + \omega \wedge \omega = S(\theta, \theta)$. Then there exists a unique torsion-free analytic connection ∇ on a sufficiently small open neighborhood $U'\subset U$ of 0 such that for all $v \in U'$:

$$
S_{\nu} = P_{\gamma_{\nu}}^{-1} \cdot R_{\gamma_{\nu}(1)}^{\nabla},\tag{5}
$$

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Sketch of the Proof

Thanks to the auxiliary results preceeding the Main Theorem, we can provide a short sketch of its Proof:

Let $g: U' \longrightarrow G$ be the map such that $\theta = g \mathrm{d} v$, and let $\Gamma \in \Omega^1 ($ $U', \mathfrak{g})$ be defined by:

$$
\Gamma \coloneqq \mathsf{Ad}(g^{-1}) \circ \omega + g^{-1} \mathrm{d} g.
$$

• This g-valued 1-Form satisfies $\Gamma \wedge dv = 0$ (ii) in Proposition [2.2\)](#page-39-0). This implies that, for any i, j, k ,

$$
\Gamma_{ij}^k = \Gamma_{ji}^k,
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where $\Gamma=\Gamma_{ij}^ke^i\otimes e^j\otimes e_k.$

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Sketch of the Proof

Thanks to the auxiliary results preceeding the Main Theorem, we can provide a short sketch of its Proof:

Let $g: U' \longrightarrow G$ be the map such that $\theta = g \mathrm{d} v$, and let $\mathsf{\Gamma} \in \Omega^1 (\mathit{U}', \mathfrak{g})$ be defined by:

$$
\Gamma \coloneqq \mathsf{Ad}(g^{-1}) \circ \omega + g^{-1} \mathrm{d} g.
$$

• This g-valued 1-Form satisfies $\Gamma \wedge dV = 0$ (ii) in Proposition [2.2\)](#page-39-0). This implies that, for any i, j, k ,

$$
\Gamma_{ij}^k=\Gamma_{ji}^k,
$$

where $\mathsf{\Gamma} = \mathsf{\Gamma}^k_{ij} e^i \otimes e^j \otimes e_k.$

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Sketch of the Proof

Γ thus defines a torsion-free connection on U' by the formula

$$
\nabla_X s \coloneqq \mathrm{d} s(X) + \Gamma(X) s.
$$

• The curvature
$$
R^{\nabla} = d\Gamma + \Gamma \wedge \Gamma
$$
 satisfies

$$
R^{\nabla}=g^{-1}\cdot S.
$$

That is:

$$
S = g \cdot R^{\nabla} \Longrightarrow S_{v} = g(v) \cdot R_{v}^{\nabla} = P_{\gamma_{v}}^{-1} \cdot R_{\gamma_{v}(1)}^{\nabla}.
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Corollary

In the context of Theorem [1,](#page-40-0) one notices that

$$
(\mathrm{d}S + \rho_*(\omega) \wedge S)(g^{-1}) = g \cdot \nabla R^{\nabla}.
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In other words: The map $({\rm d}S+\rho_*(\omega)\wedge S)(g^{-1})\colon\thinspace U'\longrightarrow V^*\otimes K(\mathfrak{g})$ actually takes values in the subspace $\mathcal{K}^1(\mathfrak{g}).$

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Outline

- [Statement of the problem](#page-2-0)
- [The convenient setting](#page-14-0)

[Main Results](#page-35-0)

- **[Main Results](#page-35-0)**
- **[Basic Ideas for the Proof of the Main Theorem](#page-41-0)**

[Applications](#page-50-0)

• [Holonomy of torsion-free connections](#page-50-0)

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Holonomy of torsion-free connections

Theorem [1](#page-40-0) is well-suited for the study of holonomy theory. Indeed one has the following:

Let S: $U\longrightarrow K(\mathfrak{g}),\ \theta\in\Omega^1(U,V),\ \omega\in\Omega^1(U,\mathfrak{g})$ $U\longrightarrow K(\mathfrak{g}),\ \theta\in\Omega^1(U,V),\ \omega\in\Omega^1(U,\mathfrak{g})$ $U\longrightarrow K(\mathfrak{g}),\ \theta\in\Omega^1(U,V),\ \omega\in\Omega^1(U,\mathfrak{g})$ as in Theorem 1. Let ∇ be the torsion-free connection on TU' which satisfies $S_{\mathsf{v}}=P_{\gamma_{\mathsf{v}}}^{-1}\cdot R_{\gamma_{\mathsf{v}}(1)}^{\nabla}$ on U'. It holds:

$$
\mathfrak{hol}_0(\nabla)=\text{span}\{S_v(x,y)\mid v\in U', x,y\in V\}.
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Holonomy of torsion-free connections

An immediate consequence of the last Theorem is the following:

Corollary

In the situation of Theorem [1](#page-40-0). The holonomy algebra of the torsion-free connection induced by the analytic map S is contained in the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ if, and only if, the map S takes values in the subspace $K(\mathfrak{h})$.

• The main takeaway is the fact solutions to the singular IVP

$$
\begin{cases}\n\mathcal{L}_{\mathscr{E}}(\mathcal{L}_{\mathscr{E}} - id)\theta = \mathcal{S}(\mathscr{E}, \theta)\mathscr{E}, \\
\theta_0 = id, d\theta_0 = 0\n\end{cases}
$$

are key in the problem of prescribing torsion-free connections.

- In contrast to the classical Ambrose-Singer Holonomy Theorem, Theorem [2](#page-51-0) offers a significant simplification of that statement.
- Outlook
	- How big is the set

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\{\mathsf{S}\colon\mathsf{U}\longrightarrow\mathsf{K}(\mathfrak{g})|\;\mathrm{d}\omega+\omega\wedge\omega=\mathsf{S}(\theta,\theta)\}
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For Further Reading I

E. Basurto-Arzate.

Torsion-free Connections with prescribed Curvature. <https://doi.org/10.48550/arXiv.2406.01530>

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