Conformal Lorentz geometry of quasi-umbilical timelike surfaces

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(Joint with E. Musso and M. Pember)

Background and motivations

- Let $f : X^2 \to \mathbb{R}^{1,2}$ be a timelike immersion in Minkowski space.
- At umbilical points, $\mathrm{II} \propto \mathrm{I}$ and $H^2 - K = 0$.
- For timelike surfaces, $H^2 - K$ can vanish also at nonumbilics.
  (For timelike surfaces the quantity $H^2 - K$ can take on any real value.)
- In this case, I and II must share a null direction and the shape operator A of f is non-diagonalizable over $\mathbb{C}$.
- Following Clelland [2012], a timelike immersion f is called (totally) quasi-umbilical if A is is non-diagonalizable over $\mathbb{C}$.
- Equivalently, if at every point of X the trace-free part A^0 of the shape operator satisfies $\det(A^0) = 0$, with $A^0 \neq 0$.

Remark (Conformal invariance)

Since the quasi-umbilical property only depends on the conformal class of the induced Lorentz metric, we study it in the more appropriate context of conformal Lorentz geometry.

Related work

- J. Clelland [2012] classified quasi-umbilical surfaces in $\mathbb{R}^{1,2}$ as ruled surfaces whose rulings are all null lines and such that any null curve γ on the surface transversal to the rulings is nondegenerate, i.e., $\gamma' \wedge \gamma'' \neq 0$ at each point.
- D. The [2012] investigated the conformal Lorentz geometry of timelike surfaces, including quasi-umbilical ones, which he called 2-parabolic, in connection with his study of the geometry of second-order Monge-Ampere equations.
- Burstall–Musso–Pember [2022] proved that quasi-umbilical surfaces arise as one of the four classes of orthogonal surfaces of codimension 2 harmonic sphere congruences, which also include
  - S-Willmore surfaces,
  - CMC surfaces in 3-dimensional space forms,
  - surfaces of constant lightcone mean curvature in 3-dim lightcones.

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Quasi-umbilical timelike immersions

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Perspectives and Further Developments

The Einstein universe $\mathcal{E}^{1,2}$

Projective model

- For each nonzero $X \in \mathbb{R}^{2,3}$, let $[X]$ denote the ray of X.
- Let $\mathcal{E}^{1,2}$ be the manifold of null rays of $\mathbb{R}^{2,3}$.
- Endow $\mathcal{E}^{1,2}$ with the Lorentz metric $g_{\mathcal{E}}$ induced from $\mathbb{R}^{2,3}$.
- $(\mathcal{E}^{1,2}, [g_{\mathcal{E}}])$ is a conformally flat Lorentz manifold, oriented and time-oriented. $(\mathcal{E}^{1,2} \cong S^1 \times S^2)$
- $\mathcal{E}^{1,2}$ is a conformal compactification of Minkowski space $\mathbb{R}^{1,2}$.

Pseudo-metric model

- Writing $\mathbb{R}^{2,3} = \mathbb{V}^2_- \oplus \mathbb{V}^3_+$, where $\mathbb{V}^2_-$ is negative definite and $\mathbb{V}^3_+$ is positive-definite, then the orientable, time-orientable submanifold $S^1_- \times S^2_+ \subset \mathbb{R}^{2,3}$, with the induced Lorentzian structure, gives another model of the Einstein universe.

Other models for $\mathcal{E}^{1,2}$:

- as a symmetric R-space;
- as Grassmannian of oriented Lagrangian 2-planes of $\mathbb{R}^4$.

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The restricted conformal group of $\mathcal{E}^{1,2}$

- Let $\mathbb{C}_+^{\uparrow}(\mathcal{E}^{1,2})$ be the group of conformal transformations preserving orientation and time-orientation.
- $\mathbb{C}_+^{\uparrow}(\mathcal{E}^{1,2}) \cong \mathrm{O}_+^{\uparrow}(2,3)$, the identity component of $\mathrm{O}(2,3)$, the pseudo-orthogonal group of $\mathbb{R}^{2,3}$ with scalar product
  $\langle X, Y\rangle = -(X^0Y^4 + X^4Y^0) - X^1Y^1 + X^2Y^2 + X^3Y^3 = {}^tX hY$, $h = (h_{ij})$.
- $\mathrm{O}_+^{\uparrow}(2,3)$ is identified with the space of ligthcone frames, i.e., the bases $(F_0, \dots, F_4)$ of $\mathbb{R}^{2,3}$, such that $\langle F_i, F_j\rangle = h_{ij}$.
- $\mathrm{O}_+^{\uparrow}(2,3)$ acts transitively on $\mathcal{E}^{1,2}$ by $A\cdot[X] = [AX], \forall [X] \in \mathcal{E}^{1,2}$.
- The map $\pi_{\mathcal{E}} : \mathrm{O}_+^{\uparrow}(2,3) \ni A \mapsto [A_0] \in \mathcal{E}^{1,2}$ makes $\mathrm{O}_+^{\uparrow}(2,3)$ into a principal bundle with structure group

$$H_0 = \{A \in \mathrm{O}_+^{\uparrow}(2,3) \mid A\cdot[E_0] = [E_0]\} = \mathrm{O}_+^{\uparrow}(2,3)_{[E_0]}.$$

$\mathcal{E}^{1,2}$ as symmetric R-space

- As a homogeneous space, $\mathcal{E}^{1,2} \cong \mathrm{O}_+^\uparrow(2,3)/H_0$.
- Let $\mathfrak{h}_0 \leq \mathfrak{o}(2,3)$ be the Lie algebra of H_0.
- The polar $\mathfrak{h}_0^\perp$ is an abelian subalgebra of $\mathfrak{o}(2,3)$.
- Thus $\mathfrak{h}_0$ is a parabolic subalgebra of $\mathfrak{o}(2,3)$ of height 1.
- $\forall\ [V] \in \mathcal{E}^{1,2}$, $\mathfrak{h}_{[V]}$ is a height 1 parabolic subalgebra of $\mathfrak{o}(2,3)$.
- The adjoint representation of $\mathrm{O}_+^\uparrow(2,3)$ induces a transitive action of $\mathrm{O}_+^\uparrow(2,3)$ on the orbit $\mathcal{O}$ through $\mathfrak{h}_0 = \mathfrak{h}_{[E_0]}$.
- Accordingly, the orbit $\mathcal{O}$ acted upon transitively by $\mathrm{O}_+^\uparrow(2,3)$ is a symmetric R-space [Tits, Gindikin, Kaneyuki, Nagano, ...].
- The map $\mathcal{E}^{1,2} \ni [V] \mapsto \mathfrak{h}_{[V]} \in \mathcal{O}$ is an equivariant smooth diffeomorphism.
- This describes $\mathcal{E}^{1,2}$ as the symmetric R-space given by the conjugacy class $\mathcal{O}$ of height 1 parabolic subalgebras of $\mathfrak{o}(2,3)$.

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Perspectives and Further Developments

The AdS chambers and the AdS walls of $\mathcal{E}^{1,2}$

- Let $S \in \mathbb{R}^{2,3}$ be a unit spacelike vector: $\langle S, S\rangle = 1$.
- The positive and negative anti-de Sitter (AdS) chambers of $\mathcal{E}^{1,2}$ determined by S are the open sets

$$\mathcal{A}_S^{\varepsilon} := \{[V] \in \mathcal{E}^{1,2} \mid \mathrm{sgn}\langle V, S\rangle = \varepsilon\} \subset \mathcal{E}^{1,2} \quad (\varepsilon = \pm 1).$$

- The common boundary of the AdS chambers $\mathcal{A}_S^{\varepsilon}$

$$\partial\mathcal{A}_S = \{[V] \in \mathcal{E}^{1,2} \mid \langle V, S\rangle = 0\} \cong S^1 \times S^1,$$

  is called the AdS wall determined by S.
- The AdS walls are totally umbilical timelike tori of $\mathcal{E}^{1,2}$.
- Contracting $dV_{\mathcal{E}}$ with S defines a volume form on $\partial\mathcal{A}_S$.
- $\mathcal{E}^{1,2}$ is the disjoint union of the $\mathcal{A}_S^{\varepsilon}$ and $\partial\mathcal{A}_S$.

The neutral space form $S^{2,2}$

- Accordingly, the smooth hyperquadric

$$S^{2,2} = \{S \in \mathbb{R}^{2,3} \mid \langle S, S \rangle = 1\} \cong \mathbb{R}^2 \times S^2 \subset \mathbb{R}^{2,3}$$

  can be viewed as the set of all (oriented) AdS walls of $\mathcal{E}^{1,2}$, i.e., the set of all (oriented) totally umbilical timelike tori of $\mathcal{E}^{1,2}$.
- $S^{2,2}$ inherits from $\mathbb{R}^{2,3}$ a $(2,2)$ neutral pseudo-metric g_S.
- $\mathrm{O}_+^\uparrow(2,3)$ acts transitively on $S^{2,2}$ on the left preserving g_S.
- $S^{2,2} \cong \mathrm{O}_+^\uparrow(2,3)/\mathrm{O}_+^\uparrow(2,2)$.

Remark

Consider as a model for AdS_3 the hyperquadric $\mathcal{A} \subset \mathbb{R}^4$, $\mathcal{A} : -(x^1)^2 - (x^2)^2 + (x^3)^2 + (x^4)^2 = -1$, with the Lorentzian structure induced by the neutral scalar product $-x^1y^1 - x^1y^2 + x^3y^3 + x^4y^4$. The AdS chambers $\mathcal{A}_S^\varepsilon$ are conformal embeddings of AdS_3 in $\mathcal{E}^{1,2}$.

Visualization of the Einstein universe

- Let $\mathrm{T} \subset \mathbb{R}^3$ be the open solid torus obtained by rotating around the z-axis the (open) unit xz-plane disk at $(2, 0, 0)$.
- The points of T can be suitably parametrized: $P(r, \phi, \theta) = \big((r\cos\phi + 2)\cos\theta, (r\cos\phi + 2)\sin\theta, r\sin\phi\big)$
- There are smooth diffeomorphisms $\mathcal{T}_\varepsilon : \mathrm{T} \to \mathcal{A}_S^\varepsilon$, which extend to a diffeomorphism $\partial\mathrm{T} \cong \partial\mathcal{A}_S$ between the boundary $\partial\mathrm{T}$ of T and the AdS wall $\partial\mathcal{A}_S$.
- Thus, $\mathcal{E}^{1,2}$ is identified with the disjoint union $\overline{\mathrm{T}} \times \{-1, 1\}$ of two copies of the closed solid torus modulo the equivalence relation: $\begin{cases} [(P, \epsilon)]_\sim = \{(P, \epsilon)\}, & P \in \mathrm{Int}(\mathrm{T}) \\ [(P, \epsilon)]_\sim = \{(P, 1)(P, -1)\}, & P \in \partial\mathrm{T} \end{cases}$
- We will use the "toroidal" projections $\mathcal{T}_\varepsilon^{-1} : \overline{\mathcal{A}}_S^\varepsilon \to \overline{\mathrm{T}}$ to visualize the geometry of curves and surfaces in $\mathcal{E}^{1,2}$.

Visualization of the Einstein universe

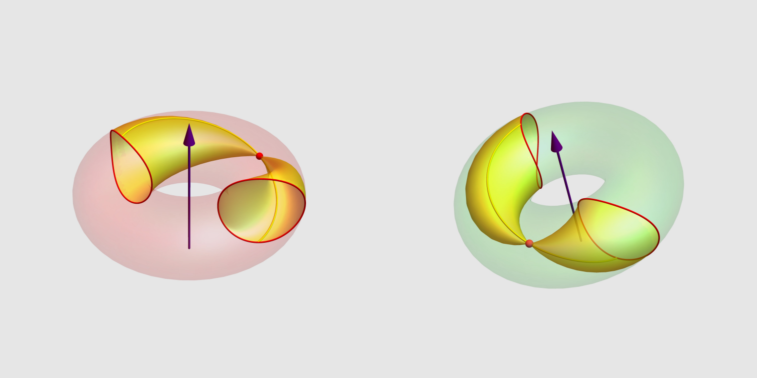

Figure: The negative (left) and positive (rigth) AdS chambers. The Einstein universe $\mathcal{E}^{1,2}$ can be thought of as two closed solid tori of revolution of $\mathbb{R}^3$ glued along the boundary. The AdS wall, i.e., the surface obtained by identifying the two boundaries, is a totally umbilical torus of signature $(1, 1)$. The yellow singular surface is a lightcone of $\mathcal{E}^{1,2}$.

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Moving frames and timelike immersions

- Let $f : X \to \mathcal{E}^{1,2}$ be a timelike immersion.
- Let $\pi_0 : \mathcal{F}_0(f) \to X$, where

$$\mathcal{F}_0(f) = \{(x, F) \in X \times \mathrm{O}^{\uparrow}_{+}(2,3) \mid f(x) = \pi_0(F) = [F_0]\},$$

  be the pullback on X of the H_0-bundle $\pi_{\mathcal{E}} : \mathrm{O}^{\uparrow}_{+}(2,3) \to \mathcal{E}^{1,2}$.
- A conformal frame field along f is a local section of $\mathcal{F}_0(f)$, i.e., a smooth map $F : U \to \mathrm{O}^{\uparrow}_{+}(2,3)$, such that $f = \pi_0 \circ F$.
- For any F we put $\phi = F^*\varphi = (\phi^i_j)$. Any other frame on U is given by $\tilde{F} = FR$, where $R : U \to H_0$ is a smooth map.
- If $\tilde{\phi} = (\tilde{F})^*\varphi$, then $\tilde{\phi} = R^{-1}\phi R + R^{-1}dR$.

Second order frames and the Conformal Gauss Map

- A conformal frame field $F : U \to \mathrm{O}_+^\uparrow(2,3)$ is of second order if
$$\phi_0^3 = 0, \quad \phi_0^1 \wedge \phi_0^2 > 0, \quad \phi_1^3 \wedge \phi_0^2 + \phi_2^3 \wedge \phi_0^1 = 0.$$
- If F is of second order, any other on U is given by $\tilde{F} = FR$, for a smooth map $R : U \to H_2$, where H_2 is a subgroup of H_0.
- The totality of second order frames along f gives rise to an H_2-bundle $\pi_2 : \mathcal{F}_2(f) \to X$, $\mathcal{F}_2(f) = \{(x, F) \in X \times \mathrm{O}_+^\uparrow(2,3)\}$, being F any local second order frame field along f.

Proposition

Second order frame fields exist near any point of X. The second order frame fields along f are the local sections of a reduced subbundle $\pi_2 : \mathcal{F}_2(f) \to X$ of π_0, with structure group H_2.

Remark (Definition)

Since $\mathcal{F}_2(f) \ni (x, F) \mapsto F_3 \in S^{2,2}$ is constant along the fibers of $\pi_2 : \mathcal{F}_2(f) \to X$, there exists a unique smooth map $\mathcal{N}_f : X \to S^{2,2}$, the conformal Gauss map of f, such that $F_3 = \mathcal{N}_f \circ \pi_2 = \pi_2^*(\mathcal{N}_f)$.

The Conformal Gauss Map

Remark

For $x \in X$, the AdS wall $\partial \mathcal{A}_{\mathcal{N}_f(x)}$ is characterized by having second order analytic contact with f at $f(x)$ and is called the central torus of the surface at $f(x)$. The order of contact is strictly bigger than two if and only if x is an umbilical point of f.

Remark

On $\mathcal{F}_2(f)$, $\begin{cases} \phi_1^3 = h_{11}\phi_0^1 + h_{12}\phi_0^2 \\ \phi_2^3 = h_{12}\phi_0^1 + h_{22}\phi_0^2 \end{cases}$, with $h_{11} = h_{22}$, and we compute

$$(\star) \quad \langle dF_3, dF_3 \rangle = (h_{11}^2 - h_{12}^2)\left(-(\phi_0^1)^2 + (\phi_0^2)^2\right).$$

Since $\mathcal{N}_f = F_3$ on $\mathcal{F}_2(f)$ and $-(\phi_0^1)^2 + (\phi_0^2)^2$ defines the induced conformal Lorentz structure on X, it follows from $(\star)$ that the conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ is weakly conformal.

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Isothermic maps

- Classically, a Riemannian surface is called isothermic if there is an isothermal coordinate system for which the shape operator is diagonalized (a conformally invariant condition).
- The classical theory of isothermic surfaces in $\mathbb{R}^3$ (in the conformal 3-sphere S^3) continues to hold when the conformal 3-sphere S^3 is replaced by an arbitrary symmetric R-space [Burstall, Donaldson, Pedit, Pinkall, 2011].
- A Lorentzian surface in $\mathbb{R}^{1,2}$ (in the Einstein universe $\mathcal{E}^{1,2}$) is isothermic if there is an isothermal coordinate system for which the shape operator has one of the 3 possible canonical forms for a self-adjoint operator on a Lorentz vector space.

Isothermic maps in $\mathcal{E}^{1,2}$

- Consider $\mathcal{E}^{1,2}$ as a symmetric R-space, i.e., as the orbit $\mathcal{O}$ of the height 1 parabolic subalgebra $\mathfrak{h}_0 \leq \mathfrak{o}(2,3)$.
- Let $f : X \to \mathcal{E}^{1,2} \cong \mathcal{O}$ be a timelike immersion. Consider the vector bundle

$$\mathfrak{H}_f^{\perp} = \left\{ (x, \mathbf{h}) \in X \times \mathfrak{o}(2,3) \mid \mathbf{h} \in \mathfrak{h}_x^{\perp} \right\},$$

where, $\forall\, x \in X$, $\mathfrak{h}_x$ is the Lie algebra of the stabilizer of $f(x)$.

Definition

A timelike immersion $f : X \to \mathcal{E}^{1,2}$ is isothermic if there is a nonzero closed 1-form $\delta \in \Omega^1(X) \otimes \mathfrak{H}_f^{\perp}$. Thus δ takes values in $\mathfrak{H}_f^{\perp}$ and is closed when viewed as an $\mathfrak{o}(2,3)$-valued form.
The 1-form δ is called an infinitesimal deformation of f.

Isothermic maps and deformation

The notion of isothermic map is related to the deformation pbm of submanifolds in homogeneous spaces [Cartan, Griffiths, Jensen,...].

- Two maps $f : M \to G/H$, $\hat{f} : \hat{M} \to G/H$ are kth order deformations of each other (with unfixed parameters) if there exist a diffeomorphism $\Phi : M \to \hat{M}$, the change of parameters, and a nonconstant map $\Delta : M \to G$, the deformation, such that f and $\Delta(p) \cdot \hat{f} \circ \Phi$ have kth order contact at p, for every $p \in M$.
- If $M = \hat{M}$ and $\Phi = \mathrm{Id}_M$, then f and $\hat{f}$ are kth order deformations of each other (with fixed parameters).

Proposition (Burstall et al)

If G/H is a symmetric R-space and $f : M \to G/H$ is isothermic, then f has second order deformations.

Isothermic maps and deformation

- In our case, let $f : X \to \mathcal{E}^{1,2}$ be an isothermic timelike immersion, let $p : \tilde{X} \to X$ be a simply connected covering of X, and δ be an infinitesimal deformation of f.
- Since $\mathfrak{h}_x^\perp$ is abelian, δ satisfies the Maurer–Cartan equation. So, there exists a smooth map $\Delta : \tilde{X} \to \mathrm{O}_+^\uparrow(2,3)$, such that $\Delta^{-1} d\Delta = p^*\delta$. Then,

$$\hat{f} : \tilde{X} \ni \tilde{x} \longmapsto \Delta(\tilde{x}) \cdot f|_{p(\tilde{x})} \in \mathcal{E}^{1,2}$$

  is a second order deformation (with fixed parameters) of $f \circ p$.
- The deformability of f depends on the existence of an infinitesimal deformation originating a map $\Delta : \tilde{X} \to \mathrm{O}_+^\uparrow(2,3)$ which is invariant under the deck transformations of p.

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Quasi-umbilical timelike immersions

Definition

If $f : X \to \mathcal{E}^{1,2}$ is a timelike immersion, $\langle d\mathcal{N}_f, d\mathcal{N}_f \rangle = \varrho_f f^*(g_{\mathcal{E}})$, for a smooth real-valued function ϱ_f. A point $x \in X$ is called

- a 2-elliptic point if $\varrho_f(x) < 0$;
- a 2-hyperbolic point if $\varrho_f(x) > 0$;
- a quasi-umbilical point (or a 2-parabolic point) if $\varrho_f(x) = 0$, with $f(x) \wedge d\mathcal{N}_f|_x \neq 0$;
- an umbilical point if $f(x) \wedge d\mathcal{N}_f|_x = 0$. If x is umbilical, $\varrho_f(x) = 0$.

If all points of X are of a fixed type, f is said to be, respectively, elliptic, hyperbolic, quasi-umbilical, and totally umbilical.

- x is quasi-umbilical $\iff$ the shape operator of f at x is non-diagonalizable over $\mathbb{C}$.
- A timelike immersion f is totally umbilical $\iff$ $f(X)$ is an open set of an AdS wall (a totally umbilical timelike torus).

Regular quasi-umbilical immersions

Lemma

Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical immersion. Then

- *The conformal Gauss map $\mathcal{N}_f$ has rank 1.*
- $\mathrm{Im}\, d\mathcal{N}_f$ *is a null line bundle such that* $\mathrm{Im}\, d\mathcal{N}_f = \mathrm{Ker}\, d\mathcal{N}_f$*.*
  ($d\mathcal{N}_f|_x$ is a nilpotent self-adjoint endomorphism of T_xX.)

Definition

- A quasi-umbilical immersion f is regular if the leaf space Λ of the distribution $\mathrm{Ker}\, d\mathcal{N}_f$ is a connected 1-dim manifold.
- Thus, $\mathcal{N}_f$ factors on Λ through a submersion $\pi_\Lambda : X \to \Lambda$ with connected fibers and a null curve $\Gamma : \Lambda \to S^{2,2}$, i.e. $\mathcal{N}_f = \Gamma \circ \pi$
- $\Gamma : \Lambda \to S^{2,2}$ is an immersed curve, the directrix curve of f.
- Locally, every quasi-umbilical immersion is regular.

General properties of quasi-umbilical immersions

Theorem (Musso, -, Pember 2024)

Let $f : X \to \mathcal{E}^{1,2}$ be a quasi-umbilical immersion.

1. *The conformal Gauss map $\mathcal{N}_f : X \to S^{2,2}$ of f is harmonic.*
2. *If f is regular, then f is isothermic and its infinitesimal deformations depend on one arbitrary function in one variable.*
3. *In particular, a quasi-umbilical immersion is locally isothermic.*

The dual map of a quasi-umbilical surface

Let $f : X \to \mathcal{E}^{1,2}$ be quasi-umbilical immersion. A second order frame field along f is said to be adapted if
$h_{11} = h_{22} = \pm 1, \quad \phi^3_4 = \phi^0_0 - 2\phi^2_1 = 0, \quad (\phi^1_4 + \phi^2_4) \wedge (\phi^1_0 + \phi^2_0) = 0.$

Proposition

Adapted frame fields exist near any point of X. If $F, \hat{F}$ are adapted frame fields, then $\hat{F} = F\,R$, where $R : U \to H_*$ is a smooth map into an abelian 1-dimensional closed subgroup H_* of H_0. The adapted frame fields along f are the local sections of a reduced subbundle $\mathcal{F}_*(f)$ of $\mathcal{F}_2(f)$, with structure group H_*.

Remark (Definition)

The map $\mathcal{F}_*(f) \ni (x, F) \mapsto [F_4] \in \mathcal{E}^{1,2}$ is constant along the fibers. It descends to a map $f^\sharp : X \to \mathcal{E}^{1,2}$, called the dual of f.

Remark

$f^\sharp$ is the second envelope of the 1-parameter family of central tori of f, i.e., $f^\sharp(x) \in \partial\mathcal{A}|_x$, $df^\sharp|_x(T_xX) \subset T_{f^\sharp(x)}(\partial\mathcal{A}|_x)$, $\forall\, x \in X$, where $\partial\mathcal{A}|_x$ is the AdS wall of $\mathcal{N}_f(x)$.

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Exceptional and general quasi-umbilical surfaces

Definition

Let $f : X \to \mathcal{E}^{1,2}$ be quasi-umbilical and let $f^\sharp$ be its dual map:

- f is called exceptional if the form $\langle d\mathcal{N}_f, df^\sharp\rangle \equiv 0$;
- f is called general if the form $\langle d\mathcal{N}_f, df^\sharp\rangle \neq 0$.

Let Λ be a connected 1-dimensional manifold and ζ be a nonzero 1-form (line element) on Λ. For a smooth map $G : \Lambda \to \mathbb{R}^k$, let G', G'', etc., denote the derivatives with respect to ζ.

Definition

An immersed curve $\Gamma : \Lambda \to S^{2,2}$ is called biisotropic if $\langle \Gamma', \Gamma'\rangle = \langle \Gamma'', \Gamma''\rangle = 0$ and $(\Gamma' \wedge \Gamma'')|_\tau \neq 0$, for every $\tau \in \Lambda$.

Proposition

Let $f : X \to \mathcal{E}^{1,2}$ be a regular quasi-umbilical immersion. Then, f is exceptional $\iff$ its directrix curve Γ is biisotropic.

Exceptional quasi-umbilical surfaces

Lemma

Let $\Gamma : \Lambda \to S^{2,2}$ be biisotropic. There exist an null 2-dim subspace $\mathbb{V}_\Gamma \subset \mathbb{R}^{2,3}$ and a 3-dim subspace $\mathbb{W}_\Gamma \subset \mathbb{R}^{2,3}$, such that

$$[(\Gamma' \wedge \Gamma'')|_\tau] = \mathbb{V}_\Gamma \subset \mathbb{W}_\Gamma = [(\Gamma \wedge \Gamma' \wedge \Gamma'')|_\tau], \quad \forall \tau \in \Lambda.$$

- $S^{2,2} \cap \mathbb{W}_\Gamma$, the set of unit vectors in $\mathbb{W}_\Gamma$, is the union of two disjoint affine (oriented) planes parallel to $\mathbb{V}_\Gamma$.
- Thus $\Gamma(\Lambda)$ is contained in either plane, say $\mathcal{V}_\Gamma$, and Γ can be viewed as an affine plane curve in $\mathcal{V}_\Gamma$.
- Let dA be a positive element of $\Lambda^2(\mathbb{V}_\Gamma)$.
- Since $\Gamma' \wedge \Gamma'' \neq 0$, there exists a (unique) affine line element of Γ, such that $dA(\Gamma', \Gamma'') = 1$.
- Fixing the affine element, $\Gamma \wedge \Gamma' \wedge \Gamma''$ and $\Gamma' \wedge \Gamma''$ are constant.
- Then, $\Gamma''' = h\Gamma'$, where $h : \Lambda \to \mathbb{R}$ is a smooth function, the affine curvature of Γ relative to dA.

Exceptional quasi-umbilical surfaces

Definition

A lightcone basis $\mathfrak{C} = (C_0, \ldots, C_4)$ is calibrated to Γ and ζ if
$\Gamma' \wedge \Gamma'' = C_4 \wedge (C_1 - C_2), \quad \Gamma \wedge \Gamma' \wedge \Gamma'' = C_3 \wedge C_4 \wedge (C_1 - C_2)$.

- If $\mathfrak{C}$ is calibrated, $\Gamma = C_3 + x(C_1 - C_2) + yC_4$, where $(x, y) : \Lambda \to \mathbb{R}^2$ is an affine plane curve, the affine reduction of Γ relative to $\mathfrak{C}$.
- Lettting $\upsilon = \sqrt{(x')^2 + (y')^2}$ and $\mu = xy' - x'y$, the vector field

$$\mathrm{R} = -x' C_0 + \frac{1}{2} y'(C_1 + C_2) + \frac{\mu^2}{2\upsilon^2} y'(C_1 - C_2) + \mu C_3 - \frac{\mu^2}{2\upsilon^2} x' \mathrm{C}_4$$

  is an null normal vector field along Γ relative to $\mathfrak{C}$.
- By construction, $\langle \mathrm{R}, \mathrm{R} \rangle = \langle \mathrm{R}, \Gamma \rangle = \langle \mathrm{R}, \Gamma' \rangle = 0$, $\langle \mathrm{R}, \Gamma'' \rangle = 1$.
- The null plane $[(\mathrm{R} \wedge \Gamma')|_\tau$ is independent of the choice of R.

Exceptional quasi-umbilical surfaces

- The normal tube of Γ is the circle bundle over Λ given by $\pi_\Lambda : \mathbb{T}_\Gamma = \{(\tau, [V]) \in \Lambda \times \mathcal{E}^{1,2} \mid V \in [(\mathrm{R} \wedge \Gamma')|_\tau\} \longrightarrow \Lambda$.
- If R is a null normal vector, the elements of the fiber $\pi_\Lambda^{-1}(\tau)$ are described by $[V(\tau,\theta)] = [\cos(\theta)\mathrm{R}|_\tau + \frac{1}{2}\sin(\theta)\Gamma'|_\tau]$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.
- $\mathbb{T}_\Gamma$ can be identified with $\Lambda \times S^1$ by the mapping

$$\Lambda \times S^1 \ni (\tau,\theta) \mapsto (\tau, [V(\tau,\theta)]) \in \mathbb{T}_\Gamma.$$

- Consider $\mathrm{C}_\pm = \{(\tau, [V(\tau, \pm\pi/2)]) \mid \tau \in \Lambda\} \subset \mathbb{T}_\Gamma$, and call them the umbilical curves of $\mathbb{T}_\Gamma$.
- The complement of $\mathrm{C}_+ \cup \mathrm{C}_-$ are the two connected open sets

$$\mathbb{T}_\Gamma^\pm = \{(\tau, [V(\tau,\theta)]) \mid \tau \in \Lambda, \ \mathrm{sgn}(\cos\theta) = \pm 1\},$$

called the positive and negative parabolic components of $\mathbb{T}_\Gamma$.

Exceptional quasi-umbilical surfaces

Definition

Call $f_\Gamma : \mathbb{T}_\Gamma \to \mathcal{E}^{1,2}$, $(\tau, [V]) \longmapsto [V]$, the tautological map of Γ.
Denote by $f_\Gamma^\pm$ the restrictions of f_Γ to $\mathbb{T}_\Gamma^\pm$, respectively.

Theorem (Musso, -, Pember 2024)

Let $\Gamma : \Lambda \to S^{2,2}$ be a biisotropic curve.

- *The tautological map f_Γ of Γ is a timelike immersion with conformal Gauss map $\Gamma \circ \pi_\Lambda$.*
- *The umbilic locus of f_Γ is the disjoint union of the two null curves $\mathrm{C}_\pm \subset \mathbb{T}_\Gamma$ that disconnect $\mathbb{T}_\Gamma$ into the open sets $\mathbb{T}_\Gamma^\pm$.*
- *The restrictions $f_\Gamma^\pm$ of f_Γ to $\mathbb{T}_\Gamma^\pm$ are exceptional quasi-umbilical immersions.*

Conversely, if $f : X \to \mathcal{E}^{1,2}$ is a regular quasi-umbilical immersion of exceptional type and $\Gamma : \Lambda \to S^{2,2}$ is its directrix curve, then $f(X)$ is contained in either $f_\Gamma(\mathbb{T}_\Gamma^+)$ or $f_\Gamma(\mathbb{T}_\Gamma^-)$.

Exceptional quasi-umbilical surfaces

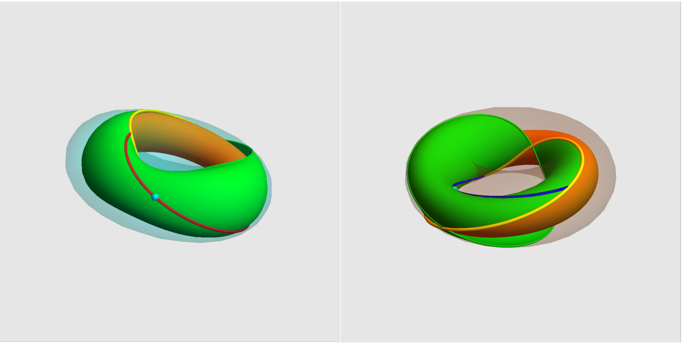

Figure: The positive (left) and negative (right) parabolic cylinders $f_\Gamma^+(\mathbb{T}_\Gamma^+)$ and $f_\Gamma^-(\mathbb{T}_\Gamma^-)$ and the central torus of $f_\Gamma(\mathbb{T}_\Gamma)$ for the tube $\mathbb{T}_\Gamma$ along the biisotropic "circle" $\Gamma = E_3 + \cos(s)(E_1 - E_2) + \sin(s)E_4$. On the left, the positive part of the central torus; on the right, its negative part. The red and blue curves are the positive and negative parts of the line of tangency of the osculating torus with $f_\Gamma(\mathbb{T}_\Gamma)$. The yellow line is the umbilical line.

General quasi-umbilical surfaces

- The directrix curve Γ of a general quasi-umbilical immersion is a generic null curve, i.e., Γ'' can be either timelike or spacelike.
- For a generic Γ we introduce a preferred parameter (proper time) and construct a canonical moving frame along the curve, from which one determines the two fundamental differential invariants of Γ: the left and right curvatures κ_λ and κ_ϱ.
- The canonical moving frame is used to build two normal tubes $\mathbb{T}_\lambda$ and $\mathbb{T}_\varrho$ along Γ, the left and right normal tubes of Γ, naturally identified by a diffeomorphism J_λ^ϱ.
- The left and right tubes intersect along two null curves $\mathrm{C}_\pm$, whose complementary set has two connected components, $\mathbb{T}_\lambda^\pm$ and $\mathbb{T}_\varrho^\pm$, the parabolic components of the normal tubes.
- The canonical projections $\pi_\lambda : \mathbb{T}_\lambda \to \Lambda$ and $\pi_\varrho : \mathbb{T}_\varrho \to \Lambda$ make $\mathbb{T}_\lambda$ and $\mathbb{T}_\varrho$ into circle bundles over Λ. We then construct two canonical maps, $f_\lambda : \mathbb{T}_\lambda \to \mathcal{E}^{1,2}$ and $f_\varrho : \mathbb{T}_\varrho \to \mathcal{E}^{1,2}$, the left and right tautological maps of Γ.

General quasi-umbilical surfaces

Theorem (Musso, -, Pember 2024)

Let $\Gamma : \Lambda \to S^{2,2}$ *be a generic null curve.*

- *The tautological map* f_λ *is a timelike immersion with umbilic locus* $\mathrm{C}_+ \cup \mathrm{C}_-$ *and conformal Gauss map* $\mathcal{N}_{f_\lambda} = \Gamma \circ \pi_\lambda$*. The restrictions* $f^\pm_\lambda$ *of* f_λ *to the parabolic components* $\mathbb{T}^\pm_\lambda$ *of the left normal tube* $\mathbb{T}_\lambda$ *are quasi-umbilical immersions of general type with dual maps* $(f^\pm_\lambda)^\sharp = f^\pm_\varrho \circ J^\varrho_\lambda$*.*
- *A similar statement holds for* λ *replaced by* ϱ*.*

Conversely, let $f : X \to \mathcal{E}^{1,2}$ *be a regular quasi-umbilical immersion of general type. Then* $f(X)$ *is contained in either* $f_\lambda(\mathbb{T}^+_\lambda)$ *or* $f_\lambda(\mathbb{T}^-_\lambda)$*, where* f_λ *is the left tautological immersion originated by the directrix curve* Γ *of* f*.*

Further developments and comments

- Study the Exterior Differential System defining second order deformable timelike surfaces in $\mathcal{E}^{1,2}$.
- Are deformable timelike surfaces necessarily isothermic?
- Are quasi-umbilical surfaces singular solutions of such an EDS?
- Study elliptic and hyperbolic timelike surfaces.