Maximal graphs and Calabi-Bernstein's type problems

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Cedeira, octubre 31, 2009

An elliptic PDE arising in Lorentzian Geometry 1/12

$$\left. \begin{aligned} &\left(f(u)^2 - \left(\frac{\partial u}{\partial y}\right)^2 \right) \frac{\partial^2 u}{\partial x^2} + \left(f(u)^2 - \left(\frac{\partial u}{\partial x}\right)^2 \right) \frac{\partial^2 u}{\partial y^2} \\ &\qquad + 2\,\frac{\partial^2 u}{\partial x \partial y}\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} - f(u)\,f'(u)\,|Du|^2 \\ &\qquad + 2\,f(u)\,f'(u)\left(f(u)^2 - |Du|^2 \right) = 0, \end{aligned} \right\} \tag{E.1}$$

$$|Du| < f(u). \tag{E.2}$$

i.e.

$$\mathrm{div}\Big(\frac{Du}{f(u)\sqrt{f(u)^2- \mid Du \mid^2}}\Big) = -\frac{f'(u)}{\sqrt{f(u)^2- \mid Du \mid^2}}\Big(2+\frac{\mid Du \mid^2}{f(u)^2}\Big) \quad \text{(E.1)}$$

$$\mid Du \mid < f(u), \quad \text{(E.2)}$$

An elliptic PDE arising in Lorentzian Geometry 3/12

- We are motivated by the classical Calabi-Bernstein's problem in $\mathbb{L}^3$ (new elliptic problems),
- By the analysis of the behavior of solutions to Calabi-Bernstein's problems in spacetimes close to $\mathbb{L}^3$ (stability).
- This approach can be seen as a mathematical attempt which could light suitable extensions to higher dimensions (extension on noncompact complete Riemannian manifolds).

$f : I \longrightarrow \mathbb{R}$ is a positive smooth function
and $u = u(x, y)$, $(x, y) \in \Omega \subseteq \mathbb{R}^2$.

The constraint (E.2) is the ellipticity condition for equation (E.1).

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The function u satisfies equation (E) (i.e. (E.1) and (E.2) together) if it is extremal, among functions (which satisfy the constraint (E.2)) under interior variation for the action

$$u \longmapsto \int f(u)\,\sqrt{f(u)^2 - |Du|^2}\,dx \wedge dy.$$

This variational problem naturally arises from Lorentzian Geometry.

$M := I \times \mathbb{R}^2$ with the Lorentzian metric

$$\langle \;\; , \;\; \rangle = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_{\mathbb{R}^2}^*(g),$$

where π_I and $\pi_{\mathbb{R}^2}$ denote the projections onto the open interval I of $\mathbb{R}$ and $\mathbb{R}^2$, respectively; g is the usual Riemannian metric of $\mathbb{R}^2$ and $f > 0$ is a smooth function on I.

An elliptic PDE arising in Lorentzian Geometry 5/12

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$(M, \langle \, , \, \rangle)$ is a warped product with base $(I, -dt^2)$, fiber $(\mathbb{R}^2, g)$ and warping function f. We will refer to $(M, \langle \, , \, \rangle)$ as a Robertson-Walker (RW) spacetime.

An elliptic PDE arising in Lorentzian Geometry 6/12

For each $u \in C^\infty(\Omega)$, $u(\Omega) \subset I$, the induced metric on Ω, via the graph $\{(u(x,y), x, y) : (x,y) \in \Omega\} \subset M$, is

$$g_u = -du^2 + f(u)^2 g,$$

which is positive definite, if and only if u satisfies (E.2).

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When g_u is Riemannian, then

$$f(u)\sqrt{f(u)^2 - |Du|^2}\, dx \wedge dy$$

is its area element, and the previous functional is the area functional $\mathcal{A}$.

An elliptic PDE arising in Lorentzian Geometry 7/12

A function u, satisfying (E.2), is a critical point of $\mathcal{A}$ if and only if the spacelike graph has zero mean curvature,

(E) is called the maximal surface equation in M.

An important case of equation (E) is $I = \mathbb{R}$ and $f \equiv 1$. Then M is the Lorentz-Minkowski spacetime $\mathbb{L}^3$ and equation (E) possesses a well-known Calabi-Bernstein property, namely

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The only entire (i.e. defined on all $\mathbb{R}^2$) solutions to maximal surface equation (E) in $\mathbb{L}^3$ are the affine functions

$$u(x, y) = ax + by + c$$

such that $a^2 + b^2 < 1$.

An elliptic PDE arising in Lorentzian Geometry 8/12

- This relevant fact is a special case of more general theorems obtained by Calabi[1] and by Cheng-Yau[2].

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[1] E. Calabi, Examples of Bernstein problems for some non-linear equations, *Proc. Sympos. Pure Math.* **15** (1970), 223–230.

[2] S.T. Cheng, S.T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski space, *Ann. of Math.* **104** (1976), 407–419.

[3] F.J.M. Estudillo, A. Romero, Generalized maximal surfaces in Lorentz-Minkowski space $\mathbb{L}^3$, *Math. Proc. Camb. Phil. Soc.* **111** (1992), 515–524.

[4] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, *Tokyo J. Math.* **6** (1983), 297–309.

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- It can be also stated in terms of the local complex representation of the surface[3,4].

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An elliptic PDE arising in Lorentzian Geometry 9/12

• A direct simple proof of that result using only Liouville's theorem on harmonic functions on $\mathbb{R}^2$ was given by the author[5].

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[5] A. Romero, Simple proof of Calabi-Bernstein's theorem, *Proc. Amer. Math. Soc.* **124** (1996), 1315–1317

[6] F.J.M. Estudillo, A. Romero, On the Gauss curvature of maximal surfaces in the 3-dimensional Lorentz-Minkowski space, *Comment. Math. Helvetici* **69** (1994), 1–4.

[7] L.J. Alías, B. Palmer, On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem, *Bull. London Math. Soc.* **33** (2001), 454–458.

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- A local integral estimate of the Gauss curvature was given by Alías-Palmer[7] which also proves Calabi-Bernstein's theorem.

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When these solutions are the only entire solutions to equation (E)*?*

When equation (E) *has no entire solution?*

We will answer these questions using the assumptions:

- f is not locally constant, (i.e. there is no flat open subset of M) in this case, the RW spacetime M is said proper, and
- M satisfies a natural curvature condition, the Null Convergence Condition (which is defined later).

Strategy

(1) On any maximal surface of a proper RW spacetime obeying NCC, there exists a positive superharmonic function which is constant if and only if the surface is an open portion of a spacelike slice $t = t_0$ with $f'(t_0) = 0$.

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(2) Given a spacelike graph S such that $\sup(f(t)|_S) < \infty$, ($t := \pi_I \circ x$), its metric is conformally related to a metric g^* which is complete when the graph is entire and $\inf(f(t)|_S) > 0$.

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(3) On any maximal graph S such that $\sup(f(t)|_S) < \infty$, g^* has non-negative Gauss curvature.

Null Convergence Condition 1/3

A RW spacetime obeys the null convergence condition (NCC), when its Ricci tensor, $\overline{\mathrm{Ric}}$, satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for any null tangent vector Z, i.e. $Z \neq 0$ satisfies $\langle Z, Z \rangle = 0$.

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NCC arises from Physics. In fact, NCC on a spacetime is a necessary condition in order that the spacetime obeys Einstein's equation. NCC is an energy condition which must satisfy realistic spacetimes[8]

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Null Convergence Condition 2/3

Taking into account that the fiber of M is flat, we have[9]

$$\overline{\mathrm{Ric}}(X, Y) = \left(\frac{f''}{f} + \frac{(f')^2}{f^2} \right) \langle X^F, Y^F \rangle - \frac{2f''}{f} \langle X, \partial_t \rangle \langle Y, \partial_t \rangle,$$

for any tangent vectors X, Y to M, where

$$X^F := X + \langle X, \partial_t \rangle \, \partial_t \quad \text{and} \quad Y^F := Y + \langle Y, \partial_t \rangle \, \partial_t$$

are the components of X and Y on the fiber $\mathbb{R}^2$ of M.

[9] See for instance B. O'Neill, *Semi-Riemannian Geometry with applications to Relativity*, Academic Press, 1983 (Corollary 7.43).

Null Convergence Condition 3/3

Consequently, for a null tangent vector Z, it reduces to

$$\overline{\mathrm{Ric}}(Z, Z) = -(\log f)'' \langle Z, \partial_t \rangle^2.$$

Therefore, a RW space M obeys NCC if and only if its warping function satisfies

$$(\log f)'' \leq 0.$$

Timelike conformal symmetry

The technique we will use here is based on the existence on M of the vector field

$$\xi := f(\pi_I)\,\partial_t.$$

which is timelike and satisfies[10]

$$\overline{\nabla}_X \xi = f'(\pi_I)\,X,$$

for any X tangent to M. Thus, ξ is conformal with

$$\mathcal{L}_\xi \langle\, ,\, \rangle = 2\,f'(\pi_I)\,\langle\, ,\, \rangle$$

and its metrically equivalent 1-form is closed.

[10] See for instance B. O'Neill, (Corollary 7.35).

Let $(M, \langle\ ,\ \rangle)$ be a RW spacetime and let $x : S \longrightarrow M$ be a (connected) immersed spacelike surface in M.

The unitary timelike vector field $\partial_t := \frac{\partial}{\partial t} \in \mathfrak{X}(M)$ determines a time-orientation on M. It allows us to construct $N \in \mathfrak{X}^{\perp}(S)$ as the only, globally defined, unitary timelike normal vector field on S in the same time-orientation of $-\partial_t$.

[11] See for instance B. O'Neill, (Proposition 5.30).

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Thus, from the wrong way Cauchy-Schwarz inequality,[11] we have

$$\langle N, \partial_t \rangle \geq 1$$

and $\langle N, \partial_t \rangle = 1$ holds at a point p if and only if $N(p) = -\partial_t(p)$.

[11] See for instance B. O'Neill, (Proposition 5.30).

A spacelike slice is a spacelike surface x such that $\pi_I \circ x$ is a constant. A spacelike surface is a spacelike slice if and only if it is orthogonal to ∂_t or, equivalently, orthogonal to ξ.
Denote by $\partial_t^T := \partial_t + \langle N, \partial_t \rangle N$ the tangential component of ∂_t on S. It is not difficult to see

$$\nabla t = -\partial_t^T$$

where ∇t is the gradient of $t := \pi_I \circ x$.

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From the Gauss formula, taking into account $\xi^T = f(t)\partial_t^T$, and previous expression, the Laplacian of t satisfies

$$\Delta t = -\frac{f'(t)}{f(t)}\left\{2 + |\nabla t|^2\right\} + \langle N, \partial_t \rangle \operatorname{trace}(A)$$

where $f(t) := f \circ t$, $f'(t) := f' \circ t$ and A is the shape operator associated to N.

The function $H := -\frac{1}{2}\,\mathrm{trace}(A)$ is called the mean curvature of S relative to N. A spacelike surface S with $H = 0$ is called maximal. In fact, $H = 0$ if and only if S is (locally) a critical point of the area functional. Note that, with our choice of N, the shape operator of $t = t_0$ is $A = \frac{f'(t_0)}{f(t_0)}\, I$ and $H = -\frac{f'(t_0)}{f(t_0)}$.

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If S is a maximal surface, we get

$$\Delta t = -\frac{f'(t)}{f(t)}\left\{2 + \mid \nabla t \mid^2\right\}.$$

and t is harmonic if and if $f'(t) = 0$. Assume f is not locally constant, in this case t is harmonic if and if $t = t_0$, with $f'(t_0) = 0$.

• This contrasts with the case of maximal surfaces in Lorentz-Minkowski spacetime $\mathbb{L}^3$ (and, of course, of minimal surfaces in Euclidean space $\mathbb{R}^3$), where the coordinates of the immersion are harmonic functions.

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• This fact is crucial to introduce the (local) conformal Weierstrass representation of the surface, which allows to express in terms of conformal data the geometry of the surface.

A function on a maximal surface induced from f

When S is a maximal surface, using previous formula for the Laplacian of t, we get

$$\Delta f(t) = -2\,\frac{f'(t)^2}{f(t)} + f(t)(\log f)''(t)\mid \nabla t \mid^2$$

Thus, if it is assumed that M satisfies NCC, then

$$\Delta f(t) \leq 0,$$

that is, $f(t)$ is a positive superharmonic function on S, and note that if f is not locally constant, then $f(t)$ is constant if and only if t is constant.

The Gauss curvature K of a maximal surface S in M, taking into account the Gauss equation and the expression for the Ricci tensor of M previously shown, satisfies

$$K = \frac{f'(t)^2}{f(t)^2} - (\log f)''(t) \mid \partial_t^T \mid^2 + \frac{1}{2}\,\mathrm{trace}(A^2),$$

where

$$\frac{f'(t)^2}{f(t)^2} - (\log f)''(t) \mid \partial_t^T \mid^2$$

is, at any point $p \in S$, the sectional curvature in M of the tangent plane $dx_p T_p S$.

Therefore,

For any maximal surface in a RW spacetime, we have

$$K \geq \frac{f'(t)^2}{f(t)^2} - (\log f)''(t) \mid \partial_t^T \mid^2,$$

with equality if and only if the surface is totally geodesic. In particular, if the RW spacetime satisfies NCC we always have

$$K \geq 0.$$

The normal component of the timelike conf. symmetry 1/2

Now, consider the function $\langle N, \xi \rangle$ on S, where ξ is the timelike conformal and closed vector field distinguished on M.

$$\nabla \langle N, \xi \rangle = -A\xi^T,$$

$\xi^T := \xi + \langle N, \xi \rangle N$ is the tangential component on S. Therefore,

$$| \nabla \langle N, \xi \rangle|^2 = \frac{1}{2}\,\mathrm{trace}(A^2)\big\{\langle N, \xi \rangle^2 - f(t)^2\big\}$$

A direct computation, using the Codazzi equation, gives

$$\Delta \langle N, \xi \rangle = \overline{\mathrm{Ric}}(N, \xi^T) + \mathrm{trace}(A^2)\langle N, \xi \rangle$$

The normal component of the timelike conf. symmetry 2/2

From the expression for the Ricci tensor of M previously shown, we have

$$\overline{\mathrm{Ric}}(N, \xi^T) = -(\log f)''(t) \mid \partial_t^T \mid^2 \langle N, \xi \rangle$$

Thus

$$\Delta \langle N, \xi \rangle = \left\{ K - \frac{f'(t)^2}{f(t)^2} + \frac{1}{2} \mathrm{trace}(A^2) \right\} \langle N, \xi \rangle .$$

First of all, note that if a non-locally constant positive smooth function $f : I \longrightarrow \mathbb{R}$ satisfies

$$(\log f)'' \leq 0$$

and has a critical point t_0 then it is unique.

In fact, $f(t_0)$ must be the global maximum value of f. Therefore

$$\sup(f) = f(t_0).$$

Now, consider an entire spacelike graph

$$\{(u(x,y),x,y) : (x,y)\in\mathbb{R}^2\}\subset M,$$

so that u satisfies (A.2) everywhere on $\mathbb{R}^2$.

Note that $t\Big(u(x,y),x,y\Big)=u(x,y)$ for any $(x,y)\in\mathbb{R}^2$, and thus t and u can be naturally identified on the spacelike graph.

It is not difficult to see that the unitary timelike normal vector field in the same time-orientation of $-\partial_t$ is

$$N=\frac{-f(u)}{\sqrt{f(u)^2-|Du|^2}}\left(1,\,\frac{1}{f(u)^2}\frac{\partial u}{\partial x},\,\frac{1}{f(u)^2}\frac{\partial u}{\partial y}\right).$$

So, we obtain

$$\langle N, \xi \rangle^2 = \frac{f(u)^4}{f(u)^2 - |Du|^2}.$$

Define the following Riemannian metric

$$g' := \langle N, \xi \rangle^2 \, g_u,$$

where $g_u = -du^2 + f(u)^2 g$ is the induced metric on $\mathbb{R}^2$.

Entire maximal graphs 4/6

Assume $\epsilon := \inf(f) > 0$. Given a smooth curve on $\mathbb{R}^2$, denote by L' and L_0 its lengths with respect to g' and the usual metric g of $\mathbb{R}^2$, respectively. It easily follows

$$L' \geq \epsilon^2 L_0,$$

which implies that divergent curves have infinite g'-length. Therefore, g' is complete.

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which implies that divergent curves have infinite g'-length. Therefore, g' is complete.

Put $\lambda := \sup(f)(< \infty)$ and consider the Riemannian metric

$$g^* = (\langle N, \xi \rangle + \lambda)^2 \, g_u$$

on $\mathbb{R}^2$. The completeness of g' easily gives that g^* is also complete.

The advantage of g^* over g' is that we can control its Gauss curvature. In fact, we will see that g^* has non-negative Gauss curvature.

If K^* and K denote the Gauss curvatures of g^* and g_u, respectively, then

$$K - (\langle N, \xi \rangle + \lambda)^2 K^* = \Delta \log(\langle N, \xi \rangle + \lambda)$$

$$\Delta \log(\langle N, \xi\rangle + \lambda) = \frac{\Delta\langle N, \xi\rangle}{\langle N, \xi\rangle + \lambda} - \frac{|\nabla\langle N, \xi\rangle|^2}{(\langle N, \xi\rangle + \lambda)^2}$$

$$\leq \frac{1}{\langle N, \xi\rangle + \lambda}\left\{\left(K - \frac{f'(u)^2}{f(u)^2}\right)\langle N, \xi\rangle + \frac{1}{2}\,\mathrm{trace}(A^2)\lambda\right\}$$

$$\leq \frac{1}{\langle N, \xi\rangle + \lambda} K\left(\langle N, \xi\rangle + \lambda\right) = K,$$

which gives $K^* \geq 0$.

Theorem 1.-[12] *If f is not locally constant, has* $\inf(f) > 0$*, satisfies* $(\log f)'' \leq 0$ *and there exists* $t_0 \in I$ *such that* $f'(t_0) = 0$*, then the only entire solution to*

$$\left.\begin{aligned} &\left(f(u)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + \left(f(u)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} \\ &\qquad + 2\,\frac{\partial^2 u}{\partial x \partial y}\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} - f(u)\,f'(u)\,|Du|^2 \\ &\qquad + 2\,f(u)\,f'(u)\left(f(u)^2 - |Du|^2\right) = 0, \end{aligned}\right\} \tag{E.1}$$

$$|Du| < f(u). \tag{E.2}$$

is $u = t_0$*.*

[12] J.M. Latorre, A. Romero, New examples of Calabi-Bernstein problems for some nonlinear equations, *Diff. Geom. Appl.* **15** (2001), 153–163.

Proof of Theorem 1. From the conformal invariance of superharmonic functions, we have that $f(t)$ is a positive superharmonic function of $(\mathbb{R}^2, g^*)$ where g^* is the Riemannian metric previously defined.

From a classical result by Ahlfors and Blanc-Fiala-Huber[13], we know that a complete 2-dimensional Riemannian manifold with non-negative Gauss curvature is parabolic. Therefore, $(\mathbb{R}^2, g^*)$ is the parabolic and $f(t)$ must be constant. Thus, $u(x, y)$ equals to the constant t_0 for all $(x, y) \in \mathbb{R}^2$, with $f'(t_0) = 0$.

[13] See for instance J.L. Kazdan, Parabolicity and the Liouville property on complete Riemannian manifolds, *Aspects of Math.* vol. **E10**, Edited by A.J. Tromba, Friedr. Vieweg and Sohn, Bonn 1987, 153–166.

Theorem 2.-(parametric case)[14] *The only complete maximal surfaces S in a proper RW spacetime with fiber $\mathbb{R}^2$, which satisfies NCC, are the spacelike slices $t = t_0$ with $f'(t_0) = 0$.*

Proof. Again from the result by Ahlfors and Blanc-Fiala-Huber, we know that S is parabolic. Therefore, $f(t)$ is constant and $t = t_0$ with $f'(t_0) = 0$.

[14] J.M. Latorre, A. Romero, and R.M. Rubio, A. Romero, On maximal surfaces in certain non-flat 3-dimensional Robertson-Walker spacetimes, (preprint 2008) where a local upper integral estimate of $|\nabla f(t)|^2$ is obtained.

[15] L.J. Alías, A. Romero and M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes, *General Relativity and Grav.* **27** (1995), 71–84. (A sufficient condition is given in Proposition 3.3).

Theorem 2.-(parametric case)[14] *The only complete maximal surfaces S in a proper RW spacetime with fiber* $\mathbb{R}^2$*, which satisfies NCC, are the spacelike slices* $t = t_0$ *with* $f'(t_0) = 0$.

Proof. Again from the result by Ahlfors and Blanc-Fiala-Huber, we know that S is parabolic. Therefore, $f(t)$ is constant and $t = t_0$ with $f'(t_0) = 0$.

Note that an entire spacelike graph in M is not necessarily complete. On the other hand, A complete spacelike surface is not necessarily a graph[15]

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[14] J.M. Latorre, A. Romero, and R.M. Rubio, A. Romero, On maximal surfaces in certain non-flat 3-dimensional Robertson-Walker spacetimes, (preprint 2008) where a local upper integral estimate of $|\nabla f(t)|^2$ is obtained.

[15] L.J. Alías, A. Romero and M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes, *General Relativity and Grav.* **27** (1995), 71–84. (A sufficient condition is given in Proposition 3.3).

Theorem 3.- *If f has* $\inf(f) > 0$ *and* $\sup(f) < \infty$, *satisfies* $(\log f)'' \leq 0$ *and f' has no zero, then there exists no entire solution to equation*

$$\left.\begin{aligned} &\left(f(u)^2 - \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + \left(f(u)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial y^2} \\ &\qquad + 2\,\frac{\partial^2 u}{\partial x \partial y}\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} - f(u)\,f'(u)\,|Du|^2 \\ &\qquad + 2\,f(u)\,f'(u)\left(f(u)^2 - |Du|^2\right) = 0, \end{aligned}\right\} \tag{E.1}$$

$$|Du| < f(u). \tag{E.2}$$

Some comments on stability

Intuitively, the RW spacetime $(I \times \mathbb{R}^2, -dt^2 + f(t)^2 g)$ can be thought as obtained from a perturbation of the flat metric of $\mathbb{L}^3$, close to $\mathbb{L}^3$ if f is near the constant function 1.

For a natural topology in the subset of RW spacetimes with fiber $\mathbb{R}^2$, warping function $f : I \longrightarrow \mathbb{R}$, such that $\inf(f) > 0$ and satisfying NCC, $\mathcal{M}$, we have:

• There exist RW spacetimes in $\mathcal{M}$ close to $\mathbb{L}^3$ where equation (A) has no solution.

• There exist RW spacetimes $\mathcal{M}$ close to $\mathbb{L}^3$ where equation (A) has only one solution.

Extensions to the non-flat case 1/2

Theorem 4.-[16] *Let (F, g) be a 2-dimensional complete Riemannian manifold and let $f : I \to \mathbb{R}$ be a smooth positive function such that $\mathrm{Inf}(K^F) \geq -(f')^2$. If f is non locally constant, $\mathrm{Inf}(f) > 0$, $(\log f)'' \leq 0$ and there exists t_0 such that $f'(t_0) = 0$, then the only entire solution to equation*

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\left(2 + \frac{\mid Du \mid^2}{f(u)^2}\right) \quad \text{(E.1)}$$

$$\mid Du \mid < f(u), \quad \text{(E.2)}$$

is $u = t_0$.

[16] M. Caballero, A. Romero and R.M. Rubio, Uniqueness of maximal surfaces in Generalized Robertson-Walker spacetimes and Calabi-Bernstein's type problems, J. Geom Phys. (to appear).

Theorem 5.-[17] *Let (F, g) be a 2-dimensional complete Riemannian manifold and let $f : I \to \mathbb{R}$ be a smooth positive function such that $\mathrm{Inf}(K^F) \geq -(f')^2$. If f is non locally constant, $\mathrm{Inf}(f) > 0$, $\sup(f) < \infty$, $(\log f)'' \leq 0$ and f' has no zero, then there exists no entire solution to equation*

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\left(2 + \frac{\mid Du \mid^2}{f(u)^2}\right) \quad \text{(E.1)}$$

$$\mid Du \mid < f(u), \quad \text{(E.2)}$$

[17] M. Caballero, A. Romero and R.M. Rubio, Uniqueness of maximal surfaces in Generalized Robertson-Walker spacetimes and Calabi-Bernstein's type problems, J. Geom Phys. (to appear).

Moitas grazas polo vosa

amable escoita!

Muchas gracias por vuestra

amable atención!